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Working Draft

Even-Length Cycles Through Length 36 and Per-Length Finiteness for a Second-Order Collatz Map

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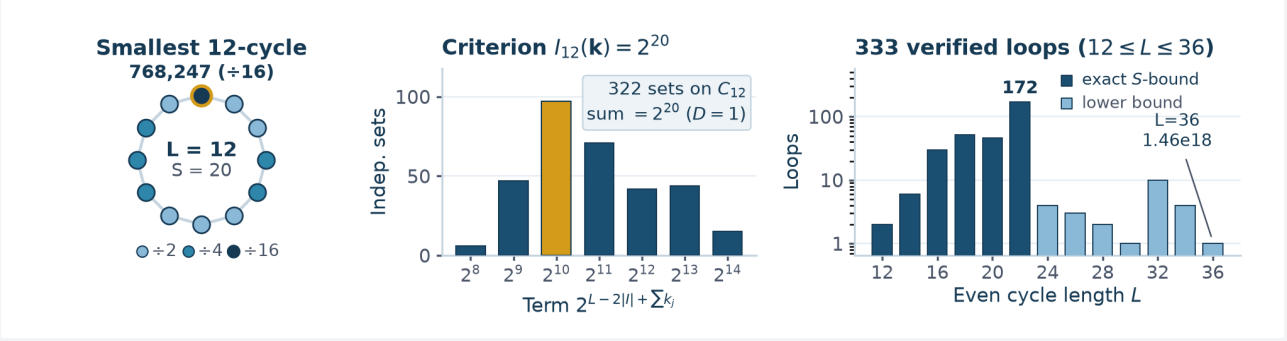
Ⓢ *Supporting Information:* `recompute.py` *(standalone exact verifier for* $L \le 36$ *and complete unimodular census* $L \le 12$*)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Define $T_c(x_{n-1}, x_n) = (x_n, x_{n+1})$ on odd pairs by $x_{n+1} = \mathrm{odd}(2x_n + x_{n-1} + c)$ $(c \in \{+1, -1\})$. Sweeps to 200,001 find four T_{+1} cycles (lengths $1, 2, 54, 29$). For $\mathbf{k} \in \mathbb{Z}_{\ge 1}^L$, $S = \sum k_j$, and $Q = N_{k_L} \cdots N_{k_1}$ (N_k with rows $(0, 2^k)$, $(1, 2)$), we prove $\mathrm{tr}(Q) = I_L(\mathbf{k}) = \sum_{I \in \mathcal{I}(C_L)} 2^{L-2|I|+\sum_{j \in I} k_j}$ over independent sets of C_L, and that every primitive unimodular tuple $(I_L(\mathbf{k}) = 2^S)$ of even length $L \ge 2$ yields a positive odd T_{+1} cycle. We prove $D > 0$ for T_{+1} ($D < 0$ for T_{-1}), per-length finiteness for T_{+1}, $S \le L + L_L - 1$, and infinitely many 5-cycles for T_{-1}. We verify the smallest 12-cycle from $(768{,}247, 3{,}121{,}313)$ ($S = 20$, 322 sets summing to 2^{20}), complete unimodular counts for even $L \le 12$ (0 for $L \le 10$; 2 at $L = 12$), and 333 loops across even $L = 12..36$ (counts $2, 6, 30, 52, 46, 172$ for $L \le 22$; min 1,460,685,166,063,904,419 at $L = 36$). Even $L \ge 38$, infinitude, and boundedness stay open ($52 \to 46$ dip; T_{-1} gap at $L = 15$); no Lean 4 proof is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)
Abstract claims: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 11B37, 37P99, 11Y55 **Keywords:** Collatz recurrences, second-order maps, cycle equations, transfer matrices, independent sets, Lucas numbers

1. INTRODUCTION AND MAIN RESULTS

In the classical Syracuse form of the Collatz map, $x_{n+1} = \mathrm{odd}(3x_n + 1)$, each iterate depends only on the single preceding odd integer, and the only known cycle is the fixed point 1. A natural two-term variant retains the same linear weight sum $2 + 1 = 3$ while distributing it across the two most recent odd terms:

$$x_{n+1} = \mathrm{odd}(2x_n + x_{n-1} + c), \qquad c \in \{+1, -1\}, \qquad (1)$$

acting on states $(x_{n-1}, x_n) \in (2\mathbb{Z}_{\ge 0} + 1)^2$ via $T_c(x_{n-1}, x_n) = (x_n, x_{n+1})$, where $\mathrm{odd}(m) = m/2^{v_2(m)}$ removes all factors of 2. Because $2x_n + x_{n-1} + c$ is always a positive even integer when x_{n-1}, x_n are positive odd integers, every step divides by 2^{k_j} for an exact halving exponent $k_j = v_2(2x_n + x_{n-1} + c) \ge 1$. On long orbits, the unchecked growth factor per step (approximately 2.414, the larger root of $\lambda^2 - 2\lambda - 1 = 0$) is nearly balanced by the expected halving factor, yielding an empirical logarithmic drift of roughly -0.117 nats per step.

Direct forward iteration from small starting pairs suggests a sparse cycle landscape for T_{+1}: every pair with both entries up to 16,000 (256,000,000 pairs) and every odd pair with both entries up to 200,001 (10,000,200,001 pairs) enters one of just four small cycles, with minimal pairs $(1, 1)$ (length 1), $(1, 3)$ (length 2), $(109, 615)$ (length 54), and $(2{,}199, 6{,}667)$ (length 29). Yet this small-start picture is misleading. Solving the linearized cycle equations directly for prescribed halving patterns $\mathbf{k} = (k_1, \ldots, k_L)$ uncovers hundreds of larger cycles whose entries all exceed 760,000 and are therefore invisible to small-start censuses.

The algebraic mechanism behind every one of these larger cycles is a combinatorial power-of-2 identity on the L-vertex cycle graph C_L. For $k \ge 1$, set

$$N_k := \begin{pmatrix} 0 & 2^k \\ 1 & 2 \end{pmatrix}, \qquad Q(\mathbf{k}) := N_{k_L} N_{k_{L-1}} \cdots N_{k_1},$$

with total halving count $S(\mathbf{k}) := \sum_{j=1}^L k_j$ and odd cycle-equation denominator

$$D(\mathbf{k}) := 2^{S(\mathbf{k})} - \mathrm{tr}(Q(\mathbf{k})) + (-1)^L.$$

Call a tuple $\mathbf{k} \in \mathbb{Z}_{\ge 1}^L$ *unimodular* if $\mathrm{tr}(Q(\mathbf{k})) = 2^{S(\mathbf{k})}$ (equiv-

alently $D(\mathbf{k}) = (-1)^L$), and *primitive* if $\mathbf{k}$ is not a nontrivial repetition of a shorter block. Let $\mathcal{I}(C_L)$ denote the family of independent sets of the cycle graph C_L (subsets of $\{1,\dots,L\}$ containing no two cyclically adjacent indices), whose cardinality for $L \geq 2$ is the L-th Lucas number L_L ($L_2 = 3, L_3 = 4, L_4 = 7, \dots, L_{12} = 322$).

Theorem 1 (Ring Independent-Set Trace Formula and Unimodular Cycle Criterion). *Let $L \geq 2$ and $\mathbf{k} = (k_1,\dots,k_L) \in \mathbb{Z}^L_{\geq 1}$, with $S = S(\mathbf{k})$ and $Q = Q(\mathbf{k})$.*

1. (Independent-set expansion.) *The trace of Q satisfies*
$$\operatorname{tr}(Q) = I_L(\mathbf{k}) := \sum_{I \in \mathcal{I}(C_L)} 2^{L-2|I|+\sum_{j\in I} k_j}. \quad (2)$$
Hence $2^{\lceil L/2 \rceil} \mid \operatorname{tr}(Q)$, the congruence $D(\mathbf{k}) \equiv (-1)^L \pmod{2^{\lceil L/2 \rceil}}$ holds, and $I_L(\mathbf{k})$ is invariant under cyclic shifts and reversal.
2. (Unimodular cycles.) *Suppose $\mathbf{k}$ is primitive and unimodular, meaning $I_L(\mathbf{k}) = 2^S$. If L is even, then $\mathbf{k}$ is the exact halving pattern of a unique positive odd cycle of T_{+1} of minimal period L. If $L \geq 3$ is odd, then $\mathbf{k}$ is the exact halving pattern of a unique positive odd cycle of the mirror map T_{-1} of minimal period L (and gives a negative solution for T_{+1}).*

Theorem 1 reduces the search for even-length cycles of T_{+1} to a purely combinatorial question: finding primitive exponent tuples $\mathbf{k} \in \mathbb{Z}^L_{\geq 1}$ whose L_L powers of 2 in (2) sum to 2^S. It also explains why unimodular cycles appear in reversal pairs and why parity separates T_{+1} from its mirror T_{-1}. Our second structural theorem establishes a global sign law, proves that T_{+1} has only finitely many cycles of each fixed length (making the per-length cycle count well defined), and gives an explicit finite search cap for unimodular patterns.

Theorem 2 (Sign Law, Per-Length Finiteness, and Unimodular Exponent Bound). *Let T_{+1} and T_{-1} be defined by (1).*

1. (Sign law and Jensen floor.) *Every cycle of T_{+1} of length $L \geq 1$ with halving pattern $\mathbf{k}$ satisfies $D(\mathbf{k}) > 0$ and $2^{S(\mathbf{k})} > 3^L$. Every cycle of T_{-1} satisfies $D(\mathbf{k}) < 0$.*
2. (Per-length finiteness.) *For every integer $L \geq 1$, the map T_{+1} has only finitely many cycles of length L.*
3. (Explicit unimodular cap.) *For every $L \geq 2$, every unimodular tuple $\mathbf{k} \in \mathbb{Z}^L_{\geq 1}$ satisfies $S(\mathbf{k}) \leq L + L_L - 1$.*

The per-length finiteness of Theorem 2(2) is specific to the $+1$ sign and fails for the mirror map T_{-1}, which admits an explicit non-unimodular infinite family at length 5.

Proposition 3 (Failure of Per-Length Finiteness for the Mirror Map). *For every integer $K \geq 11$ with $K \equiv 1 \pmod{10}$, set $x = (2^{K+1} + 7)/11$. Then x is an integer satisfying $x \equiv 5 \pmod 8$, and the five odd integers $(2x-1,\ 1,\ x,\ x,\ (3x-1)/2)$ form a primitive 5-cycle of T_{-1} with halving pattern $(1,1,1,1,K)$ and $D = -121$. In particular, T_{-1} has infinitely many cycles of length 5.*

On the computational side, combining Theorem 1, Theorem 2(3), and deterministic integer verification establishes the exact unimodular onset at $L = 12$ and verifies explicit cycles at every even length through $L = 36$.

Proposition 4 (Smallest Unimodular 12-Cycle and Complete Unimodular Census for Even $L \leq 12$). *Starting from (768,247, 3,121,313), the map T_{+1} traces a primitive 12-cycle with halving pattern $(1,2,2,1,1,1,2,2,2,1,4,1)$, total halvings $S = 20$, and $L_{12} = 322$ ring independent-set terms summing to $2^{20} = 1{,}048{,}576$ ($D = 1$). Complete monotone-pruned search up to the Theorem 2(3) cap $S \leq L + L_L - 1$ certifies zero unimodular cycles of T_{+1} at even lengths $L \in \{2,4,6,8,10\}$ ($S \leq 4, 10, 23, 54, 132$) and exactly two at $L = 12$ ($S \leq 333$): this cycle and its reversal twin with least pair (1,102,001, 4,297,121).*

Proposition 5 (Verified Even-Length Cycles Through $L = 36$ and Bounded-Exponent Counts). *The archived witness table contains 333 distinct primitive unimodular ($D = 1$) cycles of T_{+1} covering every even length $L \in \{12, 14, \dots, 36\}$, every one independently solved and walked in exact integer arithmetic by `recompute.py`—including the length-36 witness with least term 1,460,685,166,063,904,419 and $S = 59$. Within the exhaustive S-bounds of Table 1 (agreed upon by two independent C/GMP search codes), the exact counts of T_{+1} cycles at even lengths $L = 12, 14, 16, 18, 20, 22$ are $2, 6, 30, 52, 46$, and 172, alongside lower bounds $4, 3, 2, 1, 10, 4$, and 1 at $L = 24, 26, 28, 30, 32, 34, 36$.*

All verified cycles at lengths 12 through 36 motivate the governing open conjecture of this manuscript.

Conjecture 6 (Even-Length Cycles for All $L \geq 12$ and Orbit Boundedness). *The following two questions remain unsettled for T_{+1}:*

1. *For every even integer $L \geq 12$, the map T_{+1} has at least one cycle of exact length L (sufficient combinatorial form via Theorem 1(2): there exists a primitive tuple $\mathbf{k} \in \mathbb{Z}^L_{\geq 1}$ with $I_L(\mathbf{k}) = 2^{S(\mathbf{k})}$). Consequently, T_{+1} has infinitely many distinct cycles.*
2. *Every orbit of T_{+1} starting from positive odd integers is eventually periodic.*

We emphasize two empirical reasons for caution regarding Conjecture 6(1): the bounded-S cycle counts are nonmonotonic (52 at $L = 18$ dropping to 46 at $L = 20$), and for the mirror map T_{-1}, whose odd-length unimodular cycles satisfy the exact same equation $I_L(\mathbf{k}) = 2^S$, the exhaustive counts exhibit a zero gap after onset ($2, 6, 0, 38, 44, 118$ at odd lengths $L = 11, 13, 15, 17, 19, 21$). Figure 1 summarizes the full discovery workflow and verification boundary.

Instant Witness Check: Smallest 12-Cycle of T_{+1}

```
a, b = 768247, 3121313
ks = []
for _ in range(12):
    m = 2 * b + a + 1
    k = (m & -m).bit_length() - 1
    a, b = b, m >> k
    ks.append(k)
assert (a, b) == (768247, 3121313) and sum(ks) == 20
```

Roadmap. Section 2 fixes notation, sets up the forward and backward affine cycle equations, and works through small-start orbits and the 12-cycle linear solve. Section 3 gives complete self-contained proofs of Theorems 1 and 2 and Proposition 3. Section 4 presents the exact 12-step binary cascade, the 322-term independent-set decomposition, the complete unimodular census for $L \leq 12$, the 333-loop verification table

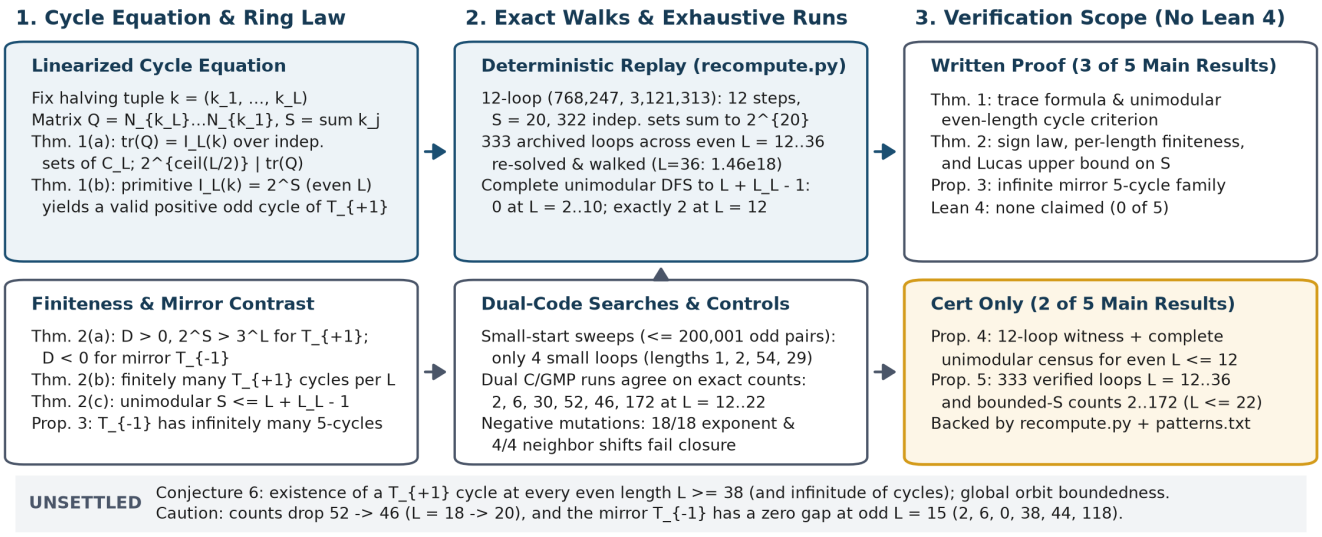


Figure 1. Discovery and verification architecture for Theorems 1–2, Propositions 3–5, and Conjecture 6. No Lean 4 formalization is claimed (0 of 5 proved main results in Lean 4; 3 of 5 by written proof; 2 of 5 by archived deterministic certificate `recompute.py` over `patterns.txt`). Existence for even $L \geq 38$, infinitude of T_{+1} cycles, and global orbit boundedness remain unsettled.

through $L = 36$, and negative-mutation controls. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 reviews prior-work searches, unread items (`[to check]`), and open questions, and Section 7 provides data availability and AI disclosure.

2. DEFINITIONS, CYCLE EQUATIONS, AND WORKED EXAMPLES

Let $\mathbb{O} := 2\mathbb{Z}_{\geq 0} + 1 = \{1, 3, 5, \dots\}$ be the set of positive odd integers. For any integer $m \geq 1$, write $v_2(m)$ for the 2-adic valuation of m (the largest integer $k \geq 0$ such that $2^k \mid m$) and $\mathrm{odd}(m) := m/2^{v_2(m)} \in \mathbb{O}$. For $c \in \{+1, -1\}$, define $T_c : \mathbb{O}^2 \to \mathbb{O}^2$ by

$$T_c(a, b) := (b,\ \mathrm{odd}(2b + a + c)).$$

Since $a, b \in \mathbb{O}$, the sum $2b + a + c$ is even and satisfies $2b + a + c \geq 2(1) + 1 - 1 = 2$, so $k = v_2(2b + a + c) \geq 1$ and T_c is well defined on $\mathbb{O}^2$. A *cycle* (or *loop*) of T_c of *length* $L \geq 1$ is a periodic orbit $v_0, v_1, \dots, v_{L-1} \in \mathbb{O}^2$ with $v_j = (x_j, x_{j+1})$, $T_c(v_{j-1}) = v_j$ for $1 \leq j \leq L$, and $v_L = v_0$ (equivalently $x_L = x_0$ and $x_{L+1} = x_1$), where L is minimal unless explicitly stated otherwise. Its *halving pattern* is the tuple $\mathbf{k} = (k_1, \dots, k_L) \in \mathbb{Z}_{\geq 1}^L$ given by $k_j = v_2(2x_j + x_{j-1} + c)$, with total halving sum $S(\mathbf{k}) = \sum_{j=1}^{L} k_j$. Two cycles are identified if one is a cyclic shift of the other; we canonically index a cycle to begin at its unique smallest element $x_0 = \min_j x_j$ (breaking any tie lexicographically on $\mathbf{k}$).

Forward and backward affine operators. Write states as column vectors $v_j = (x_j, x_{j+1})^\top$. Fixing an exponent $k \geq 1$, one step with k halvings is the affine map $v_j = F_{k,c}(v_{j-1})$ where

$$F_{k,c}(v) := 2^{-k} N_k v + c\, 2^{-k} e_2,$$
$$N_k := \begin{pmatrix} 0 & 2^k \\ 1 & 2 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that $\det N_k = -2^k$, so N_k is invertible over $\mathbb{Q}$, and the inverse (backward) step $G_{k,c} := F_{k,c}^{-1}$ is the *integral* affine map

$$G_{k,c}\begin{pmatrix} x_j \\ x_{j+1} \end{pmatrix} = \begin{pmatrix} 2^k x_{j+1} - 2x_j - c \\ x_j \end{pmatrix} = M_k \begin{pmatrix} x_j \\ x_{j+1} \end{pmatrix} - c \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3}$$

where $M_k := \left(\begin{smallmatrix} -2 & 2^k \\ 1 & 0 \end{smallmatrix}\right) = (2^{-k} N_k)^{-1} \in M_2(\mathbb{Z})$.

Composing L forward steps along $\mathbf{k} = (k_1, \dots, k_L)$ yields

$$v_L = F_{\mathbf{k},c}(v_0) = P v_0 + c\, t, \tag{4}$$

where $Q := N_{k_L} N_{k_{L-1}} \cdots N_{k_1}$, $P := 2^{-S} Q$, and

$$t := 2^{-S} \sum_{j=1}^{L} 2^{S_{j-1}} (N_{k_L} \cdots N_{k_{j+1}}) e_2 \tag{5}$$

with $S_0 = 0$, $S_r = \sum_{i=1}^{r} k_i$, and the empty product equal to I_2. Since $\det P = (-1)^L 2^{-S}$, expanding $\det(I - P) = 1 - \mathrm{tr}(P) + \det(P)$ yields

$$\det(I - P) = \frac{2^S - \mathrm{tr}(Q) + (-1)^L}{2^S} = \frac{D(\mathbf{k})}{2^S}.$$

Whenever $D(\mathbf{k}) \neq 0$, the cycle equation $(I - P)v_0 = c\, t$ has a unique rational solution $v_0 \in \mathbb{Q}^2$. That rational pair generates a genuine cycle of T_c with pattern $\mathbf{k}$ if and only if both coordinates are positive odd integers and walking L steps of T_c reproduces $\mathbf{k}$ exactly.

Worked example 1: small starts and length-2 patterns. Starting T_{+1} at $(3, 5)$ produces $2(5) + 3 + 1 = 14 = 2 \cdot 7$, then $2(7) + 5 + 1 = 20 = 4 \cdot 5$, followed by $9, 3, 1, 3, 1, 3, \dots$ (halving exponents $1, 2, 1, 3, 4, 1, 3, 1$), entering the 2-cycle $(1, 3)$ after five preperiodic terms. Across all 63,001 odd starting pairs with entries at most 501 (verified directly by `recompute.py`), every orbit enters either $(1, 1)$ (13,787 pairs) or $(1, 3)$ (49,214 pairs); upstream exhaustive pair censuses up to 16,000 (256,000,000 pairs) and odd pairs up to 200,001 (10,000,200,001 pairs) find only two

further small loops: a 54-cycle from $(109,615)$ ($S = 91$, $D = 16{,}638{,}719{,}030{,}521{,}076{,}910{,}129{,}153$) and a 29-cycle from $(2{,}199, 6{,}667)$ ($S = 48$, $D = 245{,}020{,}229{,}631$).

At length $L = 2$, solving (4) for all 16 tuples $(k_1, k_2) \in \{1,2,3,4\}^2$ shows how rarely rational solutions are positive integers: only $(1,3)$ and $(3,1)$ give the 2-cycle $(1,3)$ (with $D = 3$) and $(2,2)$ repeats the fixed point $(1,1)$, whereas typical tuples yield fractions such as $(17/11, 3/11)$ for $(1,4)$ or negative pairs such as $(-5,-3)$ for $(1,2)$.

Worked example 2: the 12-cycle linear solve. For $\mathbf{k}_{12} = (1,2,2,1,1,1,2,2,2,1,4,1)$ ($S = 20$), propagating $x_n = (A_n a + B_n b + C_n)/2^{e_n}$ from $(a,b) = (x_0, x_1)$ gives at closure:

$$x_{12} = (44{,}800\,a + 118{,}016\,b + 342{,}528)/2^{19} = a,$$
$$x_{13} = (364{,}032\,a + 958{,}976\,b + 3{,}152{,}896)/2^{20} = b,$$

or equivalently $-479{,}488\,a + 118{,}016\,b = -342{,}528$ and $364{,}032\,a - 89{,}600\,b = -3{,}152{,}896$. The integer coefficient determinant is $524{,}288 = 2^{19} \cdot 1$ (so $D(\mathbf{k}_{12}) = 1$), yielding the exact positive odd solution $(a,b) = (768{,}247,\ 3{,}121{,}313)$. By contrast, shifting a by just -2 to $(768{,}245,\ 3{,}121{,}313)$ collapses to the fixed point $(1,1)$ at term 63; all four odd neighbors $(768{,}247 \pm 2, 3{,}121{,}313)$ and $(768{,}247, 3{,}121{,}313 \pm 2)$ fall into $(1,1)$ or $(1,3)$ within 225 steps (`recompute.out`).

3. PROOFS OF THE STRUCTURAL THEOREMS

We now prove Theorems 1 and 2 and Proposition 3 from first principles.

Proof of Theorem 1. **Part (1).** Expand $\mathrm{tr}(Q) = \mathrm{tr}(N_{k_L} \cdots N_{k_1})$ as a sum over closed walks $(s_0, \dots, s_L) \in \{1,2\}^{L+1}$ with $s_L = s_0$:

$$\mathrm{tr}(Q) = \sum_{\substack{(s_0,\dots,s_L)\in\{1,2\}^{L+1} \\ s_L = s_0}} \prod_{j=1}^{L} (N_{k_j})_{s_j, s_{j-1}}.$$

The entries of $N_k = \left(\begin{smallmatrix} 0 & 2^k \\ 1 & 2 \end{smallmatrix}\right)$ are $(N_k)_{1,1} = 0$, $(N_k)_{1,2} = 2^k$, $(N_k)_{2,1} = 1$, and $(N_k)_{2,2} = 2$. Because $(N_k)_{1,1} = 0$, a closed walk contributes a nonzero product if and only if no two consecutive states (including $s_0 = s_L$ and s_1) are both equal to 1. Identifying a walk with the set $I := \{j \in \{1,\dots,L\} : s_j = 1\}$, this non-adjacency condition states precisely that $I \in \mathcal{J}(C_L)$ (for $L \geq 2$, the vertices 1 and L are adjacent in C_L, forbidding $s_1 = s_L = 1$).

Fix any $I \in \mathcal{J}(C_L)$. Each index $j \in I$ has $s_j = 1$ and $s_{j-1} = 2$, contributing factor $(N_{k_j})_{1,2} = 2^{k_j}$. The cyclically following index $j^+ \equiv j + 1 \pmod L$ has $s_{j^+} = 2$ and $s_j = 1$, contributing factor $(N_{k_{j^+}})_{2,1} = 1$; since I is independent, the $2|I|$ indices $I \cup \{j^+ : j \in I\}$ are pairwise distinct. Each of the remaining $L - 2|I|$ indices r satisfies $s_r = s_{r-1} = 2$ and contributes $(N_{k_r})_{2,2} = 2$. Hence the walk weight of I is $2^{L-2|I|+\sum_{j\in I} k_j}$, proving (2).

Because $k_j \geq 1$ for every j and $|I| \leq \lfloor L/2 \rfloor$ for every $I \in \mathcal{J}(C_L)$, each exponent in (2) obeys

$$L - 2|I| + \sum_{j\in I} k_j \geq L - |I| \geq L - \lfloor L/2 \rfloor = \lceil L/2 \rceil.$$

Thus $2^{\lceil L/2 \rceil} \mid \mathrm{tr}(Q)$. Since $S \geq L \geq \lceil L/2 \rceil$, we also have $2^{\lceil L/2 \rceil} \mid 2^S$, giving $D(\mathbf{k}) = 2^S - \mathrm{tr}(Q) + (-1)^L \equiv (-1)^L \pmod{2^{\lceil L/2 \rceil}}$. Finally, cyclic shifts and the reversal map $j \mapsto L + 1 - j$ are graph automorphisms of C_L, preserving $\mathcal{J}(C_L)$ and $|I|$, so $I_L(\mathbf{k})$ is invariant under both operations.

Part (2). Assume $\mathbf{k}$ is primitive and unimodular, so $\mathrm{tr}(Q) = 2^S$, $\mathrm{tr}(P) = 1$, and $D(\mathbf{k}) = (-1)^L$. Let $c := (-1)^L \in \{+1, -1\}$ (so $c = +1$ for even L, and $c = -1$ for odd L). We verify integrality, oddness with exact exponents, positivity, and minimal period in four short steps.

Step 1 (Exact integrality via the backward map). Compose the L integral backward maps (3): $G := G_{k_1,c} \circ \cdots \circ G_{k_L,c}$ has the form $G(v) = Mv + w$ with $M = M_{k_1} \cdots M_{k_L} = P^{-1} \in M_2(\mathbb{Z})$ and $w \in \mathbb{Z}^2$. Because $G = F^{-1}_{\mathbf{k},c}$, a vector $v_0 \in \mathbb{Q}^2$ satisfies $F_{\mathbf{k},c}(v_0) = v_0$ if and only if $(I - M)v_0 = w$. Using $I - P^{-1} = -P^{-1}(I - P)$ and $\det(-P^{-1}) = (\det P)^{-1}$, we obtain

$$\det(I - M) = \frac{\det(I - P)}{\det(P)} = \frac{D(\mathbf{k})/2^S}{(-1)^L/2^S} = (-1)^L D(\mathbf{k}) = (-1)^{2L} = 1.$$

Hence $I - M \in \mathrm{SL}_2(\mathbb{Z})$, so $(I - M)^{-1} \in \mathrm{SL}_2(\mathbb{Z})$ and the unique fixed point $v_0 = (I - M)^{-1} w$ lies in $\mathbb{Z}^2$. Applying the integral backward maps $G_{k_j,c}$ recursively shows that every state $v_j = (x_j, x_{j+1})$ around the cycle lies in $\mathbb{Z}^2$.

Step 2 (Oddness and exact 2-adic valuations). For each $j \in \{1,\dots,L\}$, the backward step $v_{j-1} = G_{k_j,c}(v_j)$ reads

$$x_{j-1} = 2^{k_j} x_{j+1} - 2x_j - c.$$

Since $k_j \geq 1$ and $c \in \{+1, -1\}$ is odd, $x_{j-1} \equiv -c \equiv 1 \pmod 2$. Thus every integer $x_0, x_1, \dots, x_{L-1}$ is odd. Rewriting the identity as $2x_j + x_{j-1} + c = 2^{k_j} x_{j+1}$ shows that $v_2(2x_j + x_{j-1} + c) = k_j$ exactly (no extra factors of 2 can divide x_{j+1}).

Step 3 (Strict positivity via Cayley–Hamilton). Because P is a 2×2 matrix with $\mathrm{tr}(P) = 1$ and $\det(P) = (-1)^L 2^{-S}$, the Cayley–Hamilton theorem gives $P^2 - P + (-1)^L 2^{-S} I = 0$, or equivalently

$$(I - P) \cdot ((-1)^L Q) = (-1)^L 2^S (P - P^2) = I.$$

Therefore $(I - P)^{-1} = (-1)^L Q$, and the fixed point of $v_0 = Pv_0 + c\,t$ is simply

$$v_0 = (I - P)^{-1}(c\,t) = (-1)^L c\,Qt.$$

For $L \geq 2$, the product of any two matrices $N_{k'} N_k = \left(\begin{smallmatrix} 2^{k'} & 2^{k'+1} \\ 2 & 2^k + 4 \end{smallmatrix}\right)$ is strictly positive entrywise, so $Q > 0$ entrywise. In (5), the $j = L - 1$ summand is $2^{-k_L - k_{L-1}} N_{k_L} e_2 = 2^{-k_L - k_{L-1}} (2^{k_L}, 2)^\top > 0$, while all other summands are nonnegative, so $t > 0$ entrywise. Hence $Qt > 0$ strictly. For $c = (-1)^L$, we have $(-1)^L c = +1$, so $v_0 = Qt > 0$ strictly; since every cyclic shift of $\mathbf{k}$ is likewise unimodular of length L, every $x_j > 0$. (Conversely, for odd L and $c = +1$, $(-1)^L c = -1$ gives $v_0 = -Qt < 0$, so no positive cycle of T_{+1} occurs.)

Step 4 (Exact minimal period L). Combining Steps 1–3, $(x_0, \dots, x_{L-1})$ is a positive odd cycle of T_c with exact halving pattern $\mathbf{k}$. If its minimal period d satisfied $d < L$, then $d \mid L$ and the sequence of step valuations $k_j = v_2(2x_j + x_{j-1} + c)$ would repeat with period d, contradicting primitivity of $\mathbf{k}$. □

Before proving Theorem 2, recall why $|\mathcal{I}(C_L)| = L_L$ for $L \geq 2$: the path P_m on $m \geq 1$ vertices has F_{m+2} independent sets ($F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$). Independent sets of C_L omitting vertex L correspond to $\mathcal{I}(P_{L-1})$ (count F_{L+1}); those containing vertex L omit vertices 1 and $L-1$, giving $1 = F_1$ set at $L = 2$, $1 = F_2$ set at $L = 3$, and $\mathcal{I}(P_{L-3})$ (count F_{L-1}) for $L \geq 4$. Hence $|\mathcal{I}(C_L)| = F_{L+1} + F_{L-1} = L_L$ for all $L \geq 2$.

Proof of Theorem 2. **Part (1) (Sign law and Jensen floor).** Suppose $x_0, \dots, x_{L-1} \in \mathbb{O}$ form a cycle of T_c with pattern $\mathbf{k}$ and sum S. If $L = 1$, then $2^{k_1} x_0 = 3x_0 + c$ gives $(2^{k_1} - 3)x_0 = c$: for $c = +1$ this forces $k_1 = 2, x_0 = 1, D(2) = 1 > 0, 2^2 > 3^1$; for $c = -1$ it forces $k_1 = 1, x_0 = 1, D(1) = -1 < 0$.

Now assume $L \geq 2$. Then $P = 2^{-S}Q > 0$ entrywise and $t > 0$ entrywise (as shown in Theorem 1). By the Perron–Frobenius theorem for strictly positive 2×2 matrices, the spectral radius $\rho = \rho(P) > 0$ is a simple eigenvalue with a strictly positive left eigenvector $u^\top > 0$ ($u^\top P = \rho u^\top$), and the second eigenvalue λ_2 is real with $|\lambda_2| < \rho$. Multiplying $(I - P)v_0 = c\,t$ on the left by $u^\top$ gives

$$(1 - \rho)(u^\top v_0) = c\,(u^\top t).$$

Since $u^\top v_0 > 0$ and $u^\top t > 0$:

- If $c = +1$, then $1 - \rho > 0$, so $|\lambda_2| < \rho < 1$, whence $1 - \lambda_2 > 0$ and $\det(I - P) = (1 - \rho)(1 - \lambda_2) > 0$, giving $D(\mathbf{k}) = 2^S \det(I - P) > 0$.
- If $c = -1$, then $1 - \rho < 0$, so $\rho > 1$. Since $|\lambda_2| = |\det P|/\rho = 2^{-S}/\rho < 1$, we still have $1 - \lambda_2 > 0$, so $\det(I - P) = (1 - \rho)(1 - \lambda_2) < 0$, giving $D(\mathbf{k}) < 0$.

For the Jensen floor on T_{+1}, each step obeys $2^{k_j} x_{j+1} = 2x_j + x_{j-1} + 1 > 2x_j + x_{j-1}$ (indices modulo L). Putting $r_j := x_{j-1}/x_j > 0$, we have $\prod_{j=1}^{L} r_j = 1$, and by the arithmetic–geometric mean inequality $2 + r_j = 1 + 1 + r_j \geq 3r_j^{1/3}$:

$$2^S = \prod_{j=1}^{L} \frac{2^{k_j} x_{j+1}}{x_j} > \prod_{j=1}^{L} (2 + r_j) \geq \prod_{j=1}^{L} 3 r_j^{1/3} = 3^L.$$

Part (2) (Per-length finiteness for T_{+1}). For $L = 1$, only $(1, 1)$ exists. Fix $L \geq 2$. By Part (1), every pattern $\mathbf{k}$ realizing a cycle of T_{+1} has $D(\mathbf{k}) > 0$, so $\det(I - P) \neq 0$ and $\mathbf{k}$ determines at most one cycle of T_{+1}.

Suppose for contradiction that T_{+1} has infinitely many distinct cycles of length L. Then there exists an infinite sequence of pairwise distinct halving patterns $\mathbf{k}^{(n)} = (k_1^{(n)}, \dots, k_L^{(n)}) \in \mathbb{Z}_{\geq 1}^L$ realizing positive odd cycles $(x_0^{(n)}, \dots, x_{L-1}^{(n)})$ of T_{+1}. Because only finitely many tuples in $\mathbb{Z}_{\geq 1}^L$ have bounded sum, $S^{(n)} := S(\mathbf{k}^{(n)}) \to \infty$. Passing to a subsequence coordinate by coordinate, we may assume that for each $j \in \{1, \dots, L\}$, the sequence $(k_j^{(n)})_{n \geq 1}$ is either eventually constant or tends to $+\infty$. The diverging set $J := \{j \in \{1, \dots, L\} : k_j^{(n)} \to \infty\}$ is nonempty since $S^{(n)} \to \infty$.

As $k \to \infty$, the step matrices $A_k := 2^{-k} N_k$ converge to the rank-one limit A_∞ with rows $(0, 1)$ and $(0, 0)$, while $2^{-k} e_2 \to 0$. Consequently, the products $P^{(n)} = A_{k_L^{(n)}} \cdots A_{k_1^{(n)}}$ converge entrywise to a matrix P_∞ with determinant $\det P_\infty = \lim_{n \to \infty} (-1)^L 2^{-S^{(n)}} = 0$, and the translations $t^{(n)}$ converge to a finite nonnegative vector $t_\infty \in \mathbb{R}^2_{\geq 0}$.

Now divide Theorem 1(1) by 2^S to write the normalized trace $Z(\mathbf{k}) := \operatorname{tr}(P) = \operatorname{tr}(Q)/2^S$ as

$$Z(\mathbf{k}) = \sum_{I \in \mathcal{I}(C_L)} 2^{L - 2|I| - \sum_{j \notin I} k_j} = \sum_{I \in \mathcal{I}(C_L)} 2^{-|I|} \prod_{j \notin I} 2^{1 - k_j}. \quad (6)$$

Split (6) into terms with $J \subseteq I$ and terms with $J \not\subseteq I$. For every $I \in \mathcal{I}(C_L)$ with $J \subseteq I$, all $j \notin I$ lie outside J, so $k_j^{(n)}$ is eventually constant and the sum of these terms equals a constant $Z_\infty \geq 0$ for all large n (with $Z_\infty = 0$ if $J \notin \mathcal{I}(C_L)$). For every I with $J \not\subseteq I$, at least one $j \in J \setminus I$ diverges, so its term is strictly positive and tends to 0. Since $J \neq \emptyset$, the empty set $I = \emptyset$ always satisfies $J \not\subseteq \emptyset$, so the remainder $R^{(n)} := Z(\mathbf{k}^{(n)}) - Z_\infty$ satisfies $R^{(n)} > 0$ for all n and $R^{(n)} \to 0$. In particular, $\operatorname{tr}(P_\infty) = Z_\infty$ and $Z(\mathbf{k}^{(n)}) = Z_\infty + R^{(n)} > Z_\infty$ for all large n.

Case 1: $Z_\infty < 1$. Since $\det(P_\infty) = 0$ and $\operatorname{tr}(P_\infty) = Z_\infty$, we have $\det(I - P_\infty) = 1 - Z_\infty > 0$. By continuity of matrix inversion, $v_0^{(n)} = (I - P^{(n)})^{-1} t^{(n)} \to (I - P_\infty)^{-1} t_\infty$ as $n \to \infty$, so the starting pairs $(x_0^{(n)}, x_1^{(n)})$ stay uniformly bounded. Because every step operator A_k and translation $2^{-k} e_2$ has entries in $[0, 1]$, all L cycle elements $x_0^{(n)}, \dots, x_{L-1}^{(n)}$ are uniformly bounded by a finite constant M independent of n. Pick any $j \in J$; since $x_{j+1}^{(n)} \geq 1$ (indices modulo L),

$$2^{k_j^{(n)}} \leq 2^{k_j^{(n)}} x_{j+1}^{(n)} = 2x_j^{(n)} + x_{j-1}^{(n)} + 1 \leq 3M + 1,$$

contradicting $k_j^{(n)} \to \infty$.

Case 2: $Z_\infty \geq 1$. Then $Z(\mathbf{k}^{(n)}) = Z_\infty + R^{(n)} > 1$ for all sufficiently large n, meaning $\operatorname{tr}(Q^{(n)}) > 2^{S^{(n)}}$. Since both quantities are integers, $\operatorname{tr}(Q^{(n)}) \geq 2^{S^{(n)}} + 1$, whence

$$D(\mathbf{k}^{(n)}) = 2^{S^{(n)}} - \operatorname{tr}(Q^{(n)}) + (-1)^L \leq -1 + (-1)^L \leq 0,$$

contradicting the Sign Law ($D(\mathbf{k}^{(n)}) > 0$) of Part (1).

Both cases yield a contradiction, proving that T_{+1} has only finitely many cycles of length L.

Part (3) (Carry lemma and explicit unimodular cap). We first note an elementary *carry lemma*: if $N \geq 1$ powers of 2 with integer exponents $e_1 \leq e_2 \leq \dots \leq e_N$ satisfy $\sum_{r=1}^{N} 2^{e_r} = 1$, then $e_1 \geq -(N - 1)$. Indeed, for $N = 1$ we have $e_1 = 0 = -(1 - 1)$. For $N \geq 2$, every term is positive so $e_1 \leq -1$; multiplying by 2^{-e_1} yields $\sum_{r=1}^{N} 2^{e_r - e_1} = 2^{-e_1}$, where the right-hand side is even and every term with $e_r > e_1$ is even while every term with $e_r = e_1$ equals 1. Hence the minimum exponent e_1 occurs an even number of times (at least twice), so $e_1 = e_2$. Merging $2^{e_1} + 2^{e_2} = 2^{e_1 + 1}$ produces $N - 1$ powers of 2 summing to 1; by induction their minimum exponent is at least $-(N - 2)$, so $e_1 + 1 \geq -(N - 2)$, i.e., $e_1 \geq -(N - 1)$.

Now let $\mathbf{k} \in \mathbb{Z}_{\geq 1}^L$ ($L \geq 2$) be unimodular. Dividing (2) by 2^S writes 1 as a sum of $N = |\mathcal{I}(C_L)| = L_L$ powers of 2 with integer exponents $T_I = L - 2|I| - \sum_{j \notin I} k_j$. Applying the carry lemma to the empty-set exponent $T_\emptyset = L - S$ gives $L - S \geq -(L_L - 1)$, that is, $S \leq L + L_L - 1$. □

Notice that the proof of Theorem 2(2) is non-effective (a compactness argument): it bounds S for non-unimodular cycles at fixed L only implicitly, whereas Part (3) gives an

A. 12-step binary halving cascade from (768,247, 3,121,313)

Step	$2x_n + x_{n-1} + 1$	Binary (shaded = trailing 0s)	$\div 2^k$	Next x_{n+1}
1	7,010,874	11010101111101000111010	÷2	3,505,437
2	10,132,188	100110101001101011011100	÷4	2,533,047
3	8,571,532	10000010110010101000 1100	÷4	2,142,883
4	6,818,814	11010000000101111111110	÷2	3,409,407
5	8,961,698	10001000101111101010001 0	÷2	4,480,849
6	12,371,106	10111100110001001010001 0	÷2	6,185,553
7	16,851,956	100000001001000111111010 0	÷4	4,212,989
8	14,611,532	11011110111101000100 1100	÷4	3,652,883
9	11,518,756	10101111110000110010 0100	÷4	2,879,689
10	9,412,262	10001111100111101010011 0	÷2	4,706,131
11	12,291,952	10111011100011110111 0000	÷16	768,247
12	6,242,626	10111110100000101000010	÷2	3,121,313

Closure: steps 11 & 12 return (768,247, 3,121,313); total shaded zeros S = 20.

B. Weight of 322 independent sets

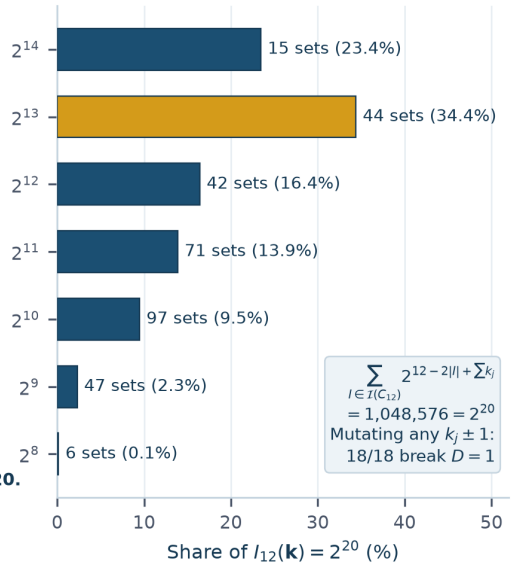


Figure 2. Exact verification of the smallest 12-cycle of T_{+1} (Proposition 4). Panel A: all 12 steps from (768,247, 3,121,313) with sums $2x_n + x_{n-1} + 1$ in decimal and right-aligned binary; shaded trailing zeros record the halving exponents $(1, 2, 2, 1, 1, 1, 2, 2, 2, 1, 4, 1)$ totaling $S = 20$, and steps 11–12 return (768,247, 3,121,313). Panel B: exact decomposition of the $L_{12} = 322$ independent sets of C_{12} by power-of-2 term, summing to $2^{20} = 1{,}048{,}576$ ($D = 1$).

explicit effective cap for all unimodular patterns. Moreover, the proof reveals why infinite families at fixed length can arise only for T_{-1}: they require $Z_\infty = 1$, forcing $Z(\mathbf{k}^{(n)}) > 1$ and $D(\mathbf{k}^{(n)}) < 0$. Proposition 3 realizes this exact mechanism at $L = 5$ with $J = \{5\}$.

Proof of Proposition 3. Write $K = 10m + 1$ with integer $m \geq 1$. Since $2^{10} = 1{,}024 \equiv 1 \pmod{11}$, we have $2^{K+1} = 2^{10m+2} \equiv 4 \pmod{11}$, so $2^{K+1}+7 \equiv 0 \pmod{11}$ and $x = (2^{K+1}+7)/11$ is a positive integer. Because $K \geq 11$, $2^{K+1} \equiv 0 \pmod 8$, so $11x \equiv 7 \pmod 8$, giving $3x \equiv 7 \pmod 8$ and $x \equiv 5 \pmod 8$ (in particular x is odd, and $(3x-1)/2 \equiv 7 \equiv 3 \pmod 4$ is odd).

Starting T_{-1} from the odd pair $(x_0, x_1) = (2x-1, 1)$, we compute five exact steps:

$$\begin{aligned} 2(1) + (2x-1) - 1 &= 2x = 2^1 \cdot x, \\ 2x + 1 - 1 &= 2x = 2^1 \cdot x, \\ 2x + x - 1 &= 3x - 1 = 2^1 \cdot \tfrac{3x-1}{2}, \\ 2 \cdot \tfrac{3x-1}{2} + x - 1 &= 4x - 2 = 2^1 \cdot (2x-1), \\ 2(2x-1) + \tfrac{3x-1}{2} - 1 &= \tfrac{11x-7}{2} = 2^K \cdot 1. \end{aligned}$$

Every quotient $x, x, (3x-1)/2, 2x-1, 1$ is a positive odd integer, so the exact halving pattern is $(1, 1, 1, 1, K)$ (primitive since $K > 1$), and after five steps the pair $(x_5, x_6) = (2x-1, 1)$ closes the cycle. Direct multiplication gives $N_1 = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 2 \end{smallmatrix}\right)$, $N_1^2 = \left(\begin{smallmatrix} 2 & 4 \\ 2 & 6 \end{smallmatrix}\right)$, $N_1^3 = \left(\begin{smallmatrix} 4 & 12 \\ 6 & 16 \end{smallmatrix}\right)$, and $N_1^4 = \left(\begin{smallmatrix} 12 & 32 \\ 16 & 44 \end{smallmatrix}\right)$, whence $Q = N_K N_1^4 = \left(\begin{smallmatrix} 16\cdot 2^K & 44\cdot 2^K \\ 44 & 120 \end{smallmatrix}\right)$. Thus $\mathrm{tr}(Q) = 2^{K+4} + 120$ and $D = 2^{K+4} - (2^{K+4} + 120) - 1 = -121$. Distinct K yield strictly increasing x, giving infinitely many distinct 5-cycles of T_{-1}. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

The Smallest 12-Cycle and Its 322-Term Sum. Figure 2 displays all 12 steps of the smallest unimodular cycle of T_{+1} starting from (768,247, 3,121,313) alongside the exact 322-term independent-set weight distribution. In binary, dividing out the maximal power of 2 strips the shaded trailing zeros from each sum $2x_n + x_{n-1} + 1$: the counts are $(1, 2, 2, 1, 1, 1, 2, 2, 2, 1, 4, 1)$, totaling $S = 20$ halvings across one lap, with every cycle element lying between 20 and 23 binary digits (least term 768,247 created by the single ÷16 step, largest term 6,185,553).

Enumerating all $L_{12} = 322$ independent sets of C_{12} directly in `recompute.py` yields the exact multiset of powers of 2 in (2):

$$\begin{aligned} &6 \cdot 2^8 + 47 \cdot 2^9 + 97 \cdot 2^{10} + 71 \cdot 2^{11} \\ &\quad + 42 \cdot 2^{12} + 44 \cdot 2^{13} + 15 \cdot 2^{14} = 1{,}048{,}576 = 2^{20}. \end{aligned}$$

The cancellation is brittle: in `recompute.py`, perturbing any single coordinate $k_j \mapsto k_j + 1$ (12/12 coordinates) or $k_j \mapsto k_j - 1$ whenever $k_j > 1$ (6/6 coordinates) destroys $D = 1$ and breaks cycle closure in all 18/18 mutations. By Theorem 1(1), reversing $\mathbf{k}_{12}$ yields a second distinct unimodular 12-cycle with canonical pattern $(2, 2, 2, 1, 1, 1, 2, 2, 1, 1, 4, 1)$ and least pair (1,102,001, 4,297,121).

Complete Unimodular Search Through $L = 12$ and All 333 Loops Through $L = 36$. From (6), every summand of $Z(\mathbf{k}) = \mathrm{tr}(Q)/2^S$ is a product of factors 2^{1-k_j} ($k_j \geq 1$), so $Z(\mathbf{k})$ is monotonically non-increasing in every exponent k_j. Consequently, given any prefix $(k_1, \ldots, k_m)$, replacing all remaining $L - m$ entries by 1 yields an exact upper bound

on $Z(\mathbf{k})$ over all completions; whenever this upper bound falls strictly below 1, neither the current value of k_m nor any larger value can complete to a unimodular pattern. At depth $L-1$, the equation $\mathrm{tr}(N_{k_L}Q') = 2^{s+k_L}$ simplifies to $q'_{12} + 2q'_{22} = 2^{k_L}(2^s - q'_{21})$, solving for k_L in $O(1)$ integer arithmetic.

`recompute.py` runs this monotone-pruned exact search up to the full Theorem 2(3) cap $S \le L + L_L - 1$: across all rotations for even $L \in \{2,4,6,8,10\}$ ($S \le 4, 10, 23, 54, 132$; 3, 26, 409, 11,533, and 726,606 nodes respectively, finding 0 unimodular patterns) and with exact rotational symmetry breaking $k_L = \max_j k_j$ for $L = 12$ ($S \le 333$; 20,469,258 nodes, finding exactly the 2 primitive cycles above). This independently reproduces the archived C/GMP and Python full-rotation runs and certifies Proposition 4 in under 25 seconds on one core.

Table 1 and Figure 3 synthesize all 334 archived even-length loops (1 small 2-loop plus 333 unimodular loops for $12 \le L \le 36$). Every one of the 333 loops at $L = 12..36$ is independently verified in `recompute.py` to be primitive, satisfy $D = 1$ and $2^S > 3^L$, and close in exact integer walks; for all 308 loops in the exhaustive range $L = 12..22$, `recompute.py` also confirms closure under pattern reversal. At $L = 36$, the verified halving tuple

$$\mathbf{k}_{36} = (1,1,2,1,2,2,2,1,1,1,1,3,1,1,2,1,1,2,\\ 1,1,1,2,2,1,1,2,3,2,1,3,1,4,3,2,2,1)$$

($S = 59$, $D = 1$) yields least pair (1,460,685,166,063,904,419, 2,328,020,849,736,368,435).

Heuristic growth and empirical caution. Theorem 1(1) forces the congruence $D(\mathbf{k}) \equiv (-1)^L$ modulo $2^{\lceil L/2 \rceil}$. Heuristically, this divisibility boosts the chance of hitting $D = 1$ at even L by roughly $2^{L/2}$ over an unstructured odd integer of size 2^S, outpacing the empirical large-deviation decay along near-unit-drift patterns (estimated near $e^{-0.022L}$ from drift -0.117 and step variance 0.312). However, as Figure 3(A) shows, counts are non-monotonic ($52 \to 46$ from $L = 18$ to $L = 20$), and patterns cluster in reversal pairs and matrix-relation families such as

$$N_k N_3 N_1 N_j = N_{k-1} N_1 N_3 N_{j+1} \qquad (j \ge 1,\ k \ge 2),$$

which preserves both Q and S at fixed L. For T_{-1}, whose odd-length unimodular cycles satisfy the identical equation $I_L(\mathbf{k}) = 2^S$, the bounded-S counts at odd lengths $L = 11, 13, 15, 17, 19, 21$ are $2, 6, 0, 38, 44, 118$, showing an actual zero gap at $L = 15$ (searched to $S \le 75$). Thus an isolated gap at some even length $L \ge 38$ cannot be ruled out by heuristic averages alone.

Table 1. Summary of even-length cycles of T_{+1} for $2 \le L \le 36$. Every loop counted in the third column is archived in `patterns.txt` and independently solved and walked in exact integers by `recompute.py` (333 loops across $12 \le L \le 36$, all unimodular with $D = 1$; the single small 2-loop has $D = 3$). Exhaustive S-bounds for all D (columns 4–5) come from dual independent C/GMP searches (using the spectral-radius filter tied to the 200,001 odd-pair census); unimodular completeness for $L \le 12$ up to $S \le L + L_L - 1$ is independently replayed in `recompute.py`.

L	L_L	Verified loops	Count status	All-D search $S_{\max}$	Observed S range	Least element min x_i	D values
2	3	1	Small loop	600 (no others)	4	1	{3}
4	7	0	Complete uni. / all-D	600 ($L + L_L - 1 = 10$)	—	—	—
6	18	0	Complete uni. / all-D	600 ($L + L_L - 1 = 23$)	—	—	—
8	47	0	Complete uni. / all-D	400 ($L + L_L - 1 = 54$)	—	—	—
10	123	0	Complete uni. / all-D	300 ($L + L_L - 1 = 132$)	—	—	—
12	322	2	Complete uni. / all-D	160 ($L + L_L - 1 = 333$)	20	768,247	{1}
14	843	6	Exact ($S \le S_{\max}$)	90 (uni. 42)	23–25	7,051,559	{1}
16	2,207	30	Exact ($S \le S_{\max}$)	66 (uni. 40)	26–28	73,102,551	{1}
18	5,778	52	Exact ($S \le S_{\max}$)	40	29–34	729,551,789	{1}
20	15,127	46	Exact ($S \le S_{\max}$)	40	33–38	7,177,273,775	{1}
22	39,603	172	Exact ($S \le S_{\max}$)	42	36–41	62,398,364,703	{1}
24	103,682	4	Lower bound	targeted	39	918,725,920,319	{1}
26	271,443	3	Lower bound	targeted	42–44	10,991,570,958,047	{1}
28	710,647	2	Lower bound	targeted	46–47	148,176,299,432,591	{1}
30	1,860,498	1	Lower bound	targeted	50	2,042,530,046,742,771	{1}
32	4,870,847	10	Lower bound	targeted	52–53	8,934,629,109,185,893	{1}
34	12,752,043	4	Lower bound	targeted	56	57,655,742,904,218,627	{1}
36	33,385,282	1	Lower bound	targeted	59	1,460,685,166,063,904,419	{1}

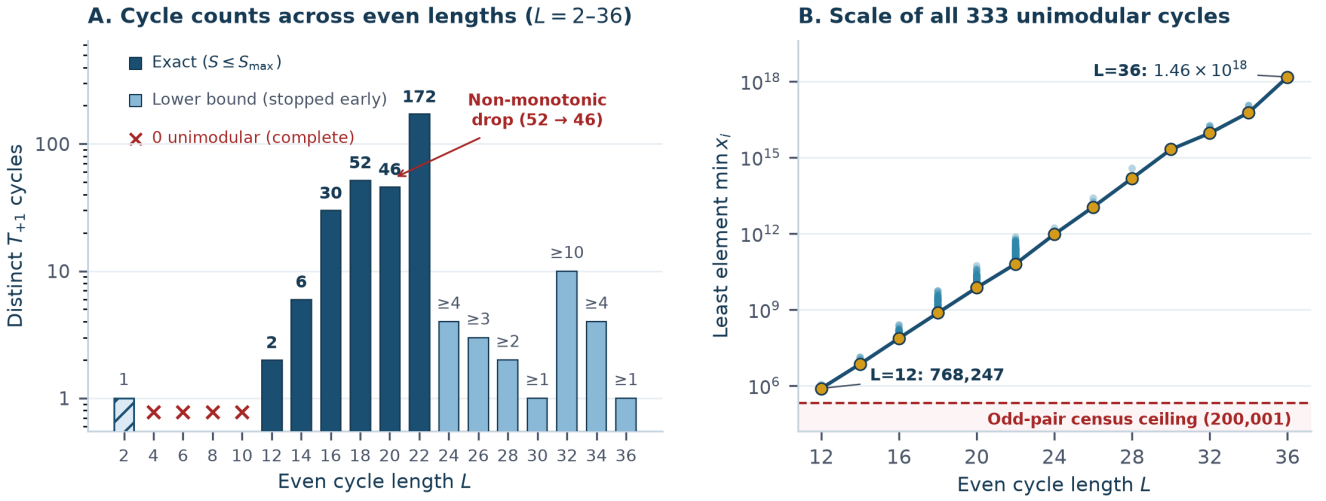


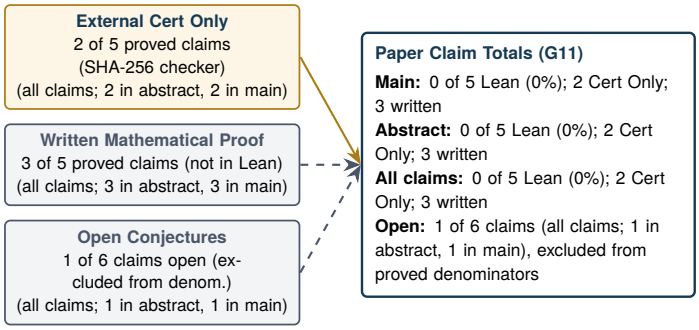
Figure 3. Empirical cycle census and scale growth across even lengths $2 \le L \le 36$ (Proposition 5). Panel A: exact S-bounded counts $2, 6, 30, 52, 46, 172$ at $L = 12..22$ (dark navy), lower bounds from stopped-early targeted searches at $L = 24..36$ (light blue), zero crosses at $L = 4..10$ (complete unimodular exclusion), and the single small 2-loop (hatched); note the non-monotonic dip from 52 at $L = 18$ to 46 at $L = 20$. Panel B: least cycle element $\min x_i$ for all 333 verified unimodular loops (768,247 at $L = 12$ to 1.46×10^{18} at $L = 36$), lying entirely above the 200,001 odd-pair census ceiling.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 2 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 2. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Trace formula tr(Q) = I_L(k) over C_L independent sets, divisibility 2^{ceil(L/2)} | tr(Q), and unimodular criterion (primitive I_L(k) = 2^S yields a valid positive odd cycle of T_{+1} for even L >= 2 and of T_{-1} for odd L >= 3)	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via transfer-matrix closed-walk expansion on C_L, integral backward maps in SL_2(Z), and Cayley–Hamilton positivity (Section 3); also spot-checked on 200 random patterns (L = 2..12) in recompute.py. No Lean 4 formalization.
Thm. 2: Sign law (D > 0 and 2^S > 3^L for T_{+1}; D < 0 for T_{-1}), per-length finiteness of T_{+1} cycles for every L >= 1, and explicit unimodular cap S <= L + L_L - 1 for L >= 2	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via 2x 2 Perron–Frobenius, AM–GM, compactness on diverging exponent coordinates (non-effective for non-unimodular cycles), and the carry lemma on Lucas(L) powers of 2 (Section 3). No Lean 4 formalization.
Prop. 3: Explicit infinite family of non-unimodular 5-cycles of T_{-1} with x = (2^{K+1} + 7)/11 for K = 1 (mod 10), K >= 11, pattern (1, 1, 1, 1, K), and D = -121	Written	Written proof	Lean: none; fidelity: n/a	Elementary written congruence and 5-step algebraic proof (Section 3); K = 11..201 also walked directly in recompute.py. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 4: Smallest unimodular 12-cycle from (768,247, 3,121,313) (S = 20, 322 independent sets summing to 2^20 = 1,048,576, D = 1) and complete unimodular census for even L <= 12 (0 for L <= 10; exactly 2 reversal-twin cycles at L = 12)	Cert Only	recompute.py (c25240093bf5) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Certified directly by recompute.py (exact 12-step walk, 322-term independent-set enumeration, and complete monotone-pruned DFS up to Theorem 2(3) bound S <= L + L_L - 1: 3 to 726,606 nodes all-rotation for L <= 10; 20,469,258 nodes k_L-max rot-broken for L = 12).
Prop. 5: 333 verified distinct primitive unimodular cycles of T_{+1} across every even length L = 12..36 (least term 1,460,685,166,063,904,419 at L = 36, S = 59) and exact S-bounded counts 2, 6, 30, 52, 46, 172 at L = 12..22	Cert Only	recompute.py (c25240093bf5) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. All 333 loops in patterns.txt independently re-solved and walked in exact integers by recompute.py; exactness of the S-bounded counts 2, 6, 30, 52, 46, 172 at L = 12..22 rests on archived dual-code C/GMP runs tied to the 200,001 odd-pair census (not replayed inside recompute.py).
Conj. 6: Existence of a T_{+1} cycle of every even length L >= 12 (hence infinitely many cycles) and eventual periodicity of every orbit of T_{+1}	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open conjectures (excluded from G11 denominators); even lengths L >= 38, infinitude of T_{+1} cycles, and global orbit boundedness remain open, tempered by the 52 -> 46 count dip and mirror zero gap at L = 15.

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: fc190c83cb73 | Fidelity SHA-256: 81062c8bcb9b.

6. PRIOR WORK AND OPEN QUESTIONS

Second-order Collatz-type difference equations have been investigated in several related directions, though under different transition rules. Clark [3] (`[to check]`) and Clark and Lewis (1995, *Congressus Numerantium* `[to check]`) introduced parity-branched second-order Collatz analogues, systematically expanded into sixteen sign variants by Amleh, Grove, Kent, and Ladas [1] (`[to check]`) and later automated by Zeilberger [10] (`[to check]`). Kent [7] (`[to check]`) studied a modified second-order Collatz difference equation from this same parity-branched family. Clark [4] (`[to check]`) constructed a ceiling-based second-order integer iteration admitting periodic solutions of arbitrarily large period, and Simons [9] (`[to check]`) analyzed cycles of higher-order Collatz sequences of the form $x_{n+1} = (x_n + 2x_{n-1} + q)/2$ for odd steps (closest in coefficients, but dividing by a single factor of 2 rather than taking the full odd part).

In the odd-part and p-free Fibonacci direction, Avila and Khovanova [2] (`[to check]`) studied n-free Fibonacci sequences (for $n = 2$, $x_{n+1} = \mathrm{odd}(x_n + x_{n-1})$, OEIS `A248797`, where terms are bounded above by the initial pair), while Guy, Khovanova, and Salazar [6] (`[to check]`) analyzed Conway's subprime Fibonacci sequences (OEIS `A214897`). Without the odd-part reduction, $2x_n + x_{n-1} + 1$ relates to the Pell-type sequence OEIS `A048739`; OEIS `A053521`–`A053523` record single periodic orbits of real-coefficient two-term odd-part maps without cycle-count laws. The numbers $x = (2^{K+1} + 7)/11$ in Proposition 3 match OEIS `A171216` (listed for a congruence property, without dynamical interpretation). Annotated bibliographies of $3x + 1$ generalizations appear in Lagarias [8] (`[to check]`) and Guy [5] (`[to check]`).

Prior-work searches logged. Logged searches covered OEIS term lookups (on the cycle counts 2, 6, 30, 52, 46, 172, the mirror counts 2, 6, 0, 38, 44, 118, small-loop elements, and the 12- and 14-loop least terms 768,247, 1,102,001, 7,051,559), OEIS word queries, web and arXiv abstract queries, 8 keyed CORE API queries (2026-11-18), and a 4-query Google Scholar sweep (34 records inspected, 2026-11-22). Neither the rule $x_{n+1} = \mathrm{odd}(2x_n + x_{n-1} + 1)$, the ring independent-set criterion of Theorem 1, per-length finiteness (Theorem 2), nor Conjecture 6 was found in the searches logged.

Unread and unsearched items (`[to check]`). The following sources remain unread in full or unsearched and are marked `[to check]` in `references-checked.md`: (1) Guy's *Unsolved Problems in Number Theory* [5] (sections E16 and E17); (2) Keith Matthews' generalized $3x + 1$ web pages and MathWorld; (3) full texts of Clark [3], Clark and Lewis (1995), Amleh, Grove,

Kent, and Ladas [1], Clark [4], Kent [7], Kent (2012, Springer chapter on piecewise-defined difference equations), and Simons [9] (inspected via abstracts/landing pages); (4) full PDFs beyond abstracts/excerpts of Zeilberger [10], Avila and Khovanova [2], Guy, Khovanova, and Salazar [6], and Lagarias [8]; and (5) direct expert consultation.

Concrete open questions. Beyond Conjecture 6 itself (even lengths $L \geq 38$, infinitude of T_{+1} cycles, and orbit boundedness), three targeted questions stand out: (i) Can an effective upper bound on $S(\mathbf{k})$ be proved for *non-unimodular* cycles of T_{+1} at fixed L, turning Theorem 2(2) into an effective finiteness theorem and settling unconditionally whether any cycle exists at even $L \in \{4, 6, 8, 10\}$ above the Table 1 S-bounds? (ii) Is every cycle of T_{+1} other than the four small loops $(1, 1)$, $(1, 3)$, 54-cycle, and 29-cycle unimodular ($D = 1$, hence of even length)? (iii) Does a constructive splicing operation exist that builds unimodular tuples of length $L + 2$ (or $L + 2m$) from shorter tuples, overcoming the non-multiplicativity of matrix traces?

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, and the standalone verifier live in `papers/c04-loops-of-every-even-length/draft1/`, reading the primary archived witness list `docs/confirmed-strong-and-novel/c04-loops-of-every-even-length/patterns.txt`. Running `python3 papers/c04-loops-of-every-even-length/draft1/recompute.py` finishes in roughly 24 seconds on a single core and deterministically verifies the worked examples, the 63,001-pair small-start census (≤ 501), the 12-cycle exact solve and walk, the 322-term independent-set sum $= 2^{20}$, 200 random Theorem 1(1) tests, all 333 archived loops across even $L = 12..36$, reversal-pair closure for $L = 12..22$, the complete unimodular DFS up to $S \leq L + L_L - 1$ for even $L \leq 12$, 20 mirror 5-cycles of Proposition 3, 18/18 exponent-mutation and 4/4 neighbor-perturbation negative controls, and the Section 1 Python snippet. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes discovered the unimodular cycle mechanism, formulated and checked the written proofs, ran the independent exhaustive searches and exact verifications, and drafted this manuscript. No human mathematician has yet refereed the proof.

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REFERENCES

(1) Amleh, A. M.; Grove, E. A.; Kent, C. M.; Ladas, G. On some difference equations with eventually periodic solutions. *Journal of Mathematical Analysis and Applications* **1998**, *223* (1), 196–215. DOI: 10.1006/jmaa.1998.5971.

(2) Avila, B.; Khovanova, T. Free Fibonacci sequences. Preprint, **2014**, arXiv:1403.4614.

(3) Clark, D. Second order difference equations related to the Collatz $3n + 1$ conjecture. *Journal of Difference Equations and Applications* **1995**, *1* (1), 73–85. DOI: 10.1080/10236199508808008.

(4) Clark, D. Periodic solutions of arbitrary length in a simple integer iteration. *Advances in Difference Equations* **2006**, *2006*, 1–10. DOI: 10.1155/ade/2006/35847.

(5) Guy, R. K. *Unsolved Problems in Number Theory*, 3rd ed.; Problem Books in Mathematics; Springer: New York, **2004**. DOI: 10.1007/978-0-387-26677-0.

(6) Guy, R. K.; Khovanova, T.; Salazar, J. Conway's subprime Fibonacci sequences. *Mathematics Magazine* **2014**, *87* (5), 323–337. DOI: 10.4169/math.mag.87.5.323. arXiv:1207.5099.

(7) Kent, C. M. A modified second-order Collatz equation as a mathematical model of bipolar disorder. *Springer Proceedings in Mathematics & Statistics* **2020**, 153–166. DOI: 10.1007/978-3-030-56323-3_13.

(8) Lagarias, J. C. The $3x + 1$ problem: An annotated bibliography, II (2000–2009). Preprint, **2006**, arXiv:math/0608208.

(9) Simons, J. L. Cycles of higher-order Collatz sequences. *Notes on Number Theory and Discrete Mathematics* **2022**, *28* (1), 48–63. DOI: 10.7546/nntdm.2022.28.1.48-63.

(10) Zeilberger, D. Teaching the computer how to discover(!) and then prove(!!) (all by itself(!!!)) analogs of Collatz's notorious $3x + 1$ conjecture. Preprint, **2009**, arXiv:0903.4050.