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Working Draft

The Most Unit Cubes a Plane Can Cut: Exact Law, Pure-Kernel Bounds, and Certificate Soundness

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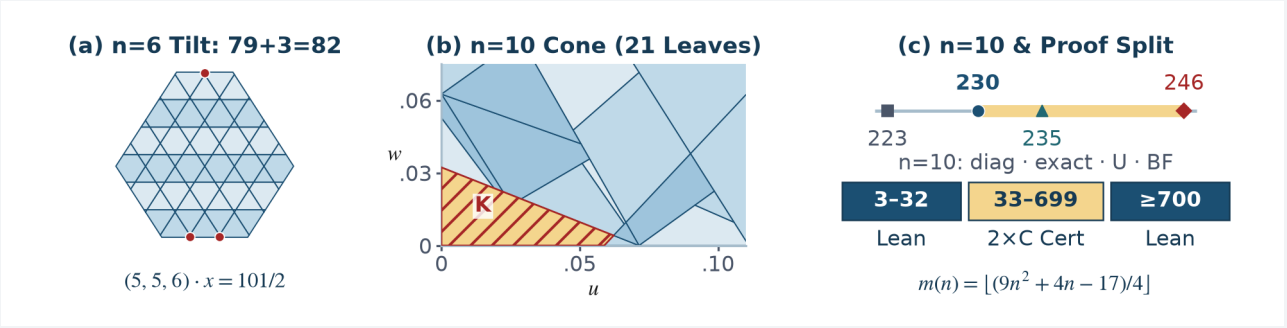
Supporting Information: Reproducible checker (`recompute.py`), Lean 4 library (`lean/CheeseCutM/`), Palomar record, and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Subdivide $[0,n]^3$ into n^3 unit cubes and cut it once with a real plane, counting every cube whose open interior is met. Bárány and Frankl proved $m(n) \le 9n^2/4+2n+1$, built a tilted plane cutting $F(n) = \lfloor (9n^2+4n-17)/4 \rfloor$ cubes ($n \ge 3$), left $230 \le m(10) \le 246$ open, and remarked that $F(n)$ could be improved by 2 for large n. We prove that $m(n) = F(n)$ and $m(n) < F(n)+2$ for every $n \ge 3$ ($m(1) = 1, m(2) = 7$), answering $m(10) = 230$. Inside the pure Lean 4 kernel (zero `sorry`, zero `native_decide`, standard axioms), we prove over $\mathbb{R}$ the tilted-plane lower bound for all $n \ge 3$, the universal upper bound $U(n) = \lfloor (9n^2+6n-20)/4 \rfloor$ for all $n \ge 2$, the exact law $m(n) = F(n)$ for $3 \le n \le 32$ via Farkas-BSP trees, $m(n) = F(n)$ for all $n \ge 700$, the $+2$ refutation for all $n \ge 700$, and vertex-certificate soundness for all $n \ge 16$. Two independent exact-integer C runs discharge $33 \le n \le 699$. Classification of all maximizing planes and the optimal four-dimensional normal family remain open.

Verification Scope (G11): WARN: Human referee: pending
Main results: 7 of 8 in Lean Kernel (88%) · 1 Lean + Cert · 1 open (not counted)
Abstract claims: 7 of 8 in Lean Kernel (88%) · 1 Lean + Cert · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 52C07, 52C35, 11H06 **Keywords:** lattice cubes, plane sections, digital planes, Farkas certificates, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

In an $n \times n \times n$ block of unit cubes, a plane perpendicular to the body diagonal $(1,1,1)$ cuts 223 open cubes at $n = 10$. Tilting its normal slightly to $(9,9,10)$ preserves all 223 cubes and picks up 7 additional cubes, reaching 230. Bárány and Frankl [6] bounded $m(10)$ between 230 and 246 (writing that the exact value was unknown), established $m(n) \le 9n^2/4 + 2n + 1$, constructed a tilted plane meeting $F(n) = \lfloor (9n^2 + 4n - 17)/4 \rfloor$ cubes for $n \ge 3$, and remarked without proof in Section 5 of their author manuscript that $F(n)$ could be improved by 2 for sufficiently large n.

Theorem 1 (Exact Law for All $n \ge 3$ — Lean + Certificate). *Let $m(n)$ be the maximum number of open unit cubes of $[0,n]^3$ met by a single nonzero real plane in $\mathbb{R}^3$, and define*

$$F(n) := \left\lfloor \frac{9n^2 + 4n - 17}{4} \right\rfloor. \tag{1}$$

Subject to the externally verified vertex-arrangement certificate condition `ArrVerifyPass` on the finite window $33 \le n \le 699$ (discharged by two independent exact-integer C implementations), every integer $n \ge 3$ satisfies

$$m(n) = F(n) \qquad \text{and} \qquad m(n) < F(n) + 2.$$

At the boundary sizes, Lean's pure kernel proves $m(1) = 1 \ne F(1) = 0$ and $m(2) = 7 \ne F(2) = 6$, confirming that the threshold $n \ge 3$ is sharp. The exact verification boundary splits cleanly into seven unconditional Lean 4 pure-kernel theorems and the single mixed theorem above:

Theorem 2 (Unconditional Small-Box Law, $3 \le n \le 32$). *For ev-*

ery integer $3 \le n \le 32$, $m(n) = F(n)$ *unconditionally over* $\mathbb{R}$.

Corollary 3 (Bárány–Frankl Open Case $n = 10$). $m(10) = 230$ *unconditionally over* $\mathbb{R}$.

Theorem 4 (Unconditional Asymptotic-Range Law, $n \ge 700$). *For every integer* $n \ge 700$, $m(n) = F(n)$ *unconditionally over* $\mathbb{R}$.

Corollary 5 (Refutation of the +2 Remark for $n \ge 700$). *For every integer* $n \ge 700$, $m(n) < F(n) + 2$ *unconditionally over* $\mathbb{R}$.

The lower bound belongs to Bárány and Frankl [6]; our formalization verifies it for both parities of $n \ge 3$, together with an improved universal upper bound and the real-geometric soundness of the finite vertex-arrangement checker:

Proposition 6 (Bárány–Frankl Tilted-Plane Lower Bound). *For every integer* $n \ge 3$, *there exists a nonzero real plane in* $\mathbb{R}^3$ *cutting exactly* $F(n)$ *open unit cubes of* $[0, n]^3$.

Proposition 7 (Universal Upper Bound for All $n \ge 2$). *For every integer* $n \ge 2$ *and every nonzero real plane* $ax + by + cz = t$, *the cut count satisfies*

$$\mathrm{cutCount}(n, a, b, c, t) \le U(n) := \left\lfloor \frac{9n^2 + 6n - 20}{4} \right\rfloor.$$

In particular, $U(10) = 235$, tightening the published upper bound 246 unconditionally before any case split.

Proposition 8 (Vertex-Principle Certificate Soundness). *For every integer* $n \ge 16$ *and arrangement configuration* cfg, *if* ArrVerifyPass(n, cfg) *holds* (*the covering cone radius satisfies* $R_1 \ge 3r(n)$ *and every nonzero arrangement vertex in* $Z(R_1)$ *attains truncated pairing or even-n coupling value at least* $D(n)$), *then every nonzero real plane cuts at most* $F(n)$ *open unit cubes of* $[0, n]^3$.

At $n = 10$, the integral normal $(9, 9, 10)$ at half-integer height 234.5 cuts exactly 230 open unit cubes. The seven-line Python check below verifies all 1,000 cubes in exact integer arithmetic (executed verbatim by `recompute.py`).

Instant Witness Check (7 lines, verified verbatim by recompute.py)

```
from itertools import product
n = 10
h, s = n - 1, (3 * n - 4) // 2
t2 = 2 * h * (s + 3) + 1
lows = (h*(i+j+k) + k for i, j, k in product(range(n), repeat=3))
assert sum(2*q < t2 < 2*(q + 3*h + 1) for q in lows) == 230
assert (9 * n * n + 4 * n - 17) // 4 == 230
```

Figure 1 illustrates the $[0, 6]^3$ tilted extremizer ($79 + 3 = 82$ cubes) alongside the three-stage discovery and proof architecture. Section 2 sets conventions and proves Proposition 6. Section 3 develops the diagonal-chain reduction, opposite-row pairing, even-n height coupling, Proposition 7, and the slope localization feeding Theorem 4 (proved in full in Appendix A). Section 4 establishes the pure-kernel Farkas-BSP trees closing Theorem 2 and Corollary 3, proves Proposition 8, and documents the two independent exact-integer C sweeps discharging $33 \le n \le 699$ in Theorem 1. Section 5 presents the machine-generated Lean 4 Reader's Trust Boundary. Section 6 covers dimensional extensions, prior work, and open questions, and Section 7 records reproducibility and AI disclosure.

2. SETUP, CONVENTIONS, AND THE TILTED LOWER BOUND

Set $h = n - 1$. Index the n^3 unit cubes of $[0, n]^3$ by lower corners $z = (i, j, k) \in \{0, \dots, h\}^3$, with open cell $\mathrm{OpenCube}(z) = z + (0, 1)^3$. A nonzero normal $v = (a, b, c) \in \mathbb{R}^3$ ($v \ne 0$) and height $t \in \mathbb{R}$ define the plane $v \cdot x = t$, which *meets* z if $(v \cdot x = t) \cap \mathrm{OpenCube}(z) \ne \emptyset$. Reflecting coordinate axes, permuting coordinates, and scaling positively preserve $\mathrm{cutCount}(n, a, b, c, t)$ and reduce any nonzero plane to $0 \le a \le b \le c$ with $W = a + b + c > 0$. For nonnegative v, the linear functional $v \cdot x$ maps $\mathrm{OpenCube}(z)$ onto the open interval $(v \cdot z,\ v \cdot z + W)$, so z is cut precisely when $v \cdot z < t < v \cdot z + W$. Boundary touches contribute zero, and a sufficiently small perturbation of (v, t) preserves every cut cell.

Proof of Proposition 6. Fix $n \ge 3$ and put $h = n - 1$, $s = \lfloor (3n - 4)/2 \rfloor$. Consider the tilted plane

$$hx + hy + nz = h(s + 3) + \tfrac{1}{2}. \tag{2}$$

For $z = (i, j, k) \in \{0, \dots, h\}^3$ and $\sigma = i + j + k$, the cell interval has integer endpoints $h\sigma + k$ and $h(\sigma + 3) + k + 1$. Since $0 \le k \le h$, comparing with $h(s + 3) + \frac{1}{2}$ shows that (2) cuts every cube with $\sigma \in \{s, s+1, s+2\}$, together with two disjoint boundary strips: $\sigma = s - 1$ with $k = h$, and $\sigma = s + 3$ with $k = 0$.

Let $H(r) = [z^r](1 + z + \cdots + z^h)^3$ count cubes with $\sigma = r$. By unimodality and symmetry of $(1 + z + \cdots + z^h)^3$, the three consecutive central slices $s, s + 1, s + 2$ achieve the best diagonal-plane count $m_{\mathrm{diag}}(n) = H(s) + H(s + 1) + H(s + 2)$. Writing $\binom{r}{2}_+ = r(r - 1)/2$ for $r \ge 2$ and 0 otherwise, expanding $(1 - z^n)^3/(1 - z)^3$ gives $H(r) = \sum_{j=0}^{3} (-1)^j \binom{3}{j} \binom{r - jn + 2}{2}_+$. The number of pairs $(i, j) \in \{0, \dots, h\}^2$ with sum r is $\max(0, n - |r - h|)$, which sums across $r = s - 1 - h$ and $r = s + 3$ to exactly $n - 3$ extra cubes in both parities:

Parity	$m_{\mathrm{diag}}(n)$	Extra strips	Total $F(n)$
$n = 2q$	$9q^2 - 2$	$2q - 3$	$9q^2 + 2q - 5$
$n = 2q + 1$	$9q^2 + 9q + 1$	$2q - 2$	$9q^2 + 11q - 1$

The last column matches (1) for every $n \ge 3$. □

3. DIAGONAL CHAINS, ROW PAIRING, AND LOCALIZATION

Normalize $W = 1$ with $0 \le x_1 \le x_2 \le x_3$ and $x_1 + x_2 + x_3 = 1$. Following Bárány and Frankl [6, §6], partition $\{0, \dots, h\}^3$ into diagonal chains running parallel to $(1, 1, 1)$: each chain starts at a cell with minimum coordinate 0 and steps by $(1, 1, 1)$ until its maximum coordinate reaches h. The number of chains and the required missed-chain deficit are

$$C(n) = n^3 - (n - 1)^3 = 3n^2 - 3n + 1, \quad D(n) = C(n) - F(n). \tag{3}$$

Successive lower endpoints along a chain increase by $x_1 + x_2 + x_3 = 1$, so the open interval $(t - 1, t)$ contains at most one of them; hence every plane cuts at most $C(n)$ cubes, and proving $\mathrm{cutCount} \le F(n)$ reduces to proving at least $D(n)$ missed chains.

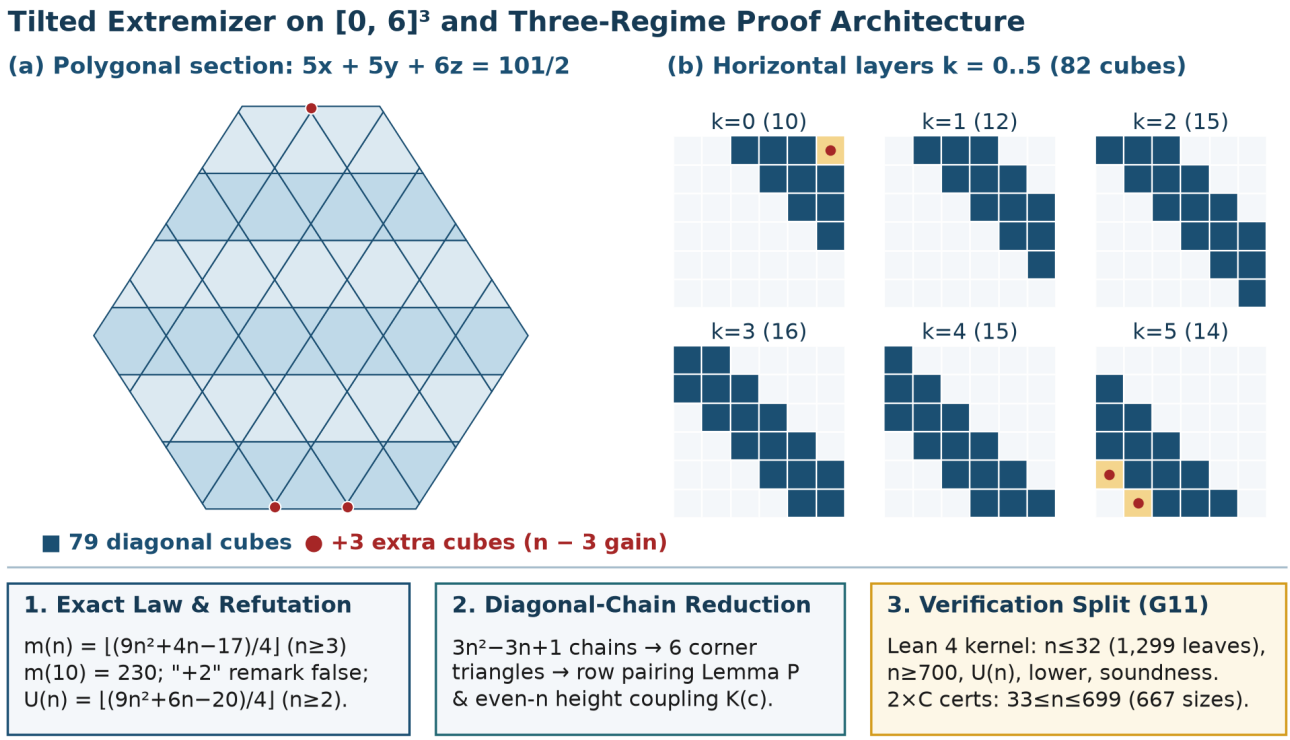


Figure 1. Exact tilted section of $[0,6]^3$ by $5x+5y+6z=101/2$ and three-regime proof architecture. Top left: all 82 polygonal cell intersections (79 diagonal cubes in blue, 3 additional tilted cubes marked by red dots). Top right: the same 82 cubes across horizontal layers $k=0,\dots,5$ with counts $10,12,15,16,15,14$. Bottom: quantitative separation of results, diagonal-chain reduction, and the G11 verification split between the Lean 4 pure kernel ($3\le n\le 32$, $n\ge 700$, $U(n)$, lower bound, soundness) and the two independent exact-integer C certificate runs ($33\le n\le 699$).

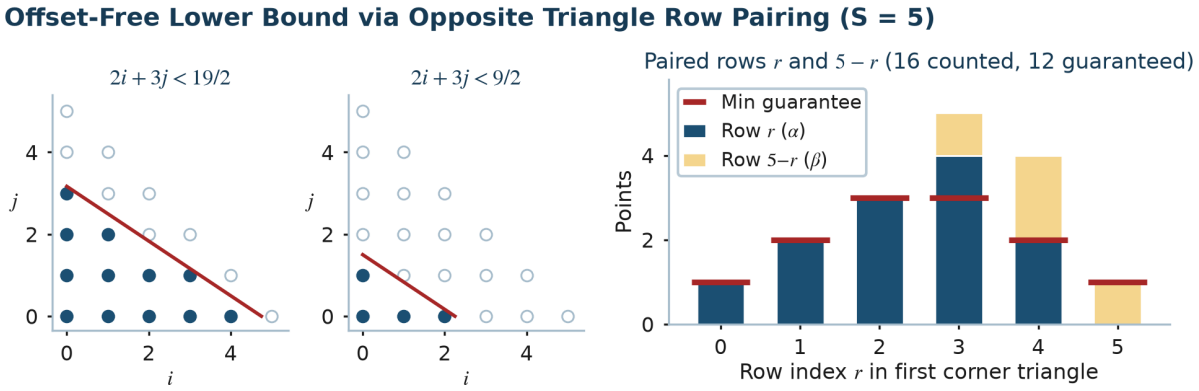


Figure 2. Opposite-row pairing (Lemma 9) for $p=2$, $q=3$, $\alpha=19/2$, $\beta=9/2$ ($\gamma=14$), and $S=5$. Left and center: lattice points strictly below each threshold line (row counts $1,2,3,4,2,0$ and $1,2,1,0,0,0$). Right: pairing row r with row $5-r$ gives 16 points while every stacked bar clears the red offset-free guarantee $1,2,3,3,2,1$ (sum $12=T(6)$).

This immediately settles $n\le 4$: at $n=1$, $m(1)=1$; at $n=2$, $C(2)=7$ is attained by $x+y+z=7/2$ (whereas the unperturbed central plane $x+y+z=3$ passes through $(1,1,1)$ and meets only 6 open cubes); at $n=3$, $C(3)=19=F(3)$ (the 1995 AHSME Problem 30 bisector [1]); at $n=4$, the four single-cell chains $(0,3,0),(0,3,3)$ and $(3,0,0),(3,0,3)$ have lower-endpoint gaps $3x_3\ge 1$ within each pair, forcing at least 2 missed chains out of $C(4)=37$ and giving $m(4)=35=F(4)$.

Assume now $n\ge 5$. If $x_3\ge 1/2$, each vertical column $(i,j,\cdot)$ contains at most 2 cut cubes, giving at most $2n^2\le F(n)$. Hence we may assume $x_3<1/2$, which forces $x_1=1-x_2-x_3\ge 1-2x_3>0$. By small perturbation preserving all strict cell inequalities, assume $0<x_1<x_2<x_3$ and $x_2-x_1\ne x_3-x_2$. Reflecting the box gives $t\le n/2$; if $t<hx_3$, no cube with $k=h$ is cut, so shifting $k\mapsto k+1$ and $t\mapsto t+x_3$

preserves all cuts until t enters the central band

$$hx_3 \le t \le n - hx_3. \tag{4}$$

For each coordinate $k \in \{1,2,3\}$, let $p_k \le q_k$ be the other two weights, $s_k = p_k + q_k = 1 - x_k$, and $d_k = q_k - p_k$. Define the lattice triangle count and corner thresholds

$$\Delta(p,q;b) = \#\{(i,j) \in \mathbb{N}_0^2 : pi + qj < b\},$$

$$\alpha_k = hs_k - t, \qquad \beta_k = t - hx_k - 1.$$

A chain starting on $z_k = 0$ is missed whenever its lower endpoint exceeds t; measuring inward from the far corner of that face yields $p_k i + q_k j < \alpha_k$. Because (4) gives $\alpha_k \le hp_k$, the triangle stays strictly inside the face and chains from distinct faces are disjoint. Symmetrically, chains ending with lower endpoint below $t-1$ give $\Delta(p_k, q_k; \beta_k)$, disjoint from the start misses. Thus

$$C(n) - \mathrm{cutCount} \ge \sum_{k=1}^{3} (\Delta(p_k,q_k;\alpha_k) + \Delta(p_k,q_k;\beta_k)). \tag{5}$$

Crucially, $\gamma_k := \alpha_k + \beta_k = 2hs_k - n$ is independent of the plane height t.

Opposite-row pairing (Lemma P). Set $T(m) = \lfloor (m+1)^2/4 \rfloor$ for $m \ge 1$ and $T(m) = 0$ for $m \le 0$. Summing $\min(r+1, S-r+1)$ over $0 \le r \le S$ gives $T(S+1)$, and capping each term at height $L \le \lfloor S/2 \rfloor + 1$ gives $T(S+1) - T(S+1-2L)$.

Lemma 9 (Opposite-Row Pairing). *Fix $0 < p \le q$, thresholds $\alpha, \beta \in \mathbb{R}$, and an integer $S \ge 0$. Set $d = q - p$, $\gamma = \alpha + \beta$, and*

$$L = \begin{cases} \min(\lfloor S/2 \rfloor + 1,\ \max(0, \lceil (\gamma - pS)/d \rceil)) & \text{if } d > 0, \\ (\lfloor S/2 \rfloor + 1)\, \mathbf{1}_{\{\gamma > pS\}} & \text{if } d = 0. \end{cases}$$

Then $\Delta(p,q;\alpha) + \Delta(p,q;\beta) \ge G(S) := T(S+1) - T(S+1-2L)$.

Proof. On the antidiagonal $i+j = r$ of length $r+1$, the weights are $pr, pr+d, \dots, pr+rd$. For $d > 0$, the accepted count is $\lceil (\alpha - pr)/d \rceil$ clipped to $[0, r+1]$. Pair row r of the α-triangle with row $S-r$ of the β-triangle. Before clipping and rounding, the two quotients sum to $(\gamma - pS)/d$. If either row is full, that row alone contributes at least $\min(r+1, S-r+1, L)$; if neither is full, ceiling rounding and nonnegativity only raise the sum above L. Summing over $0 \le r \le S$ yields $G(S)$. When $d = 0$ and $\gamma > pS$, at least one of $\alpha > pr$ or $\beta > p(S-r)$ holds for every r, giving a full row. □

Figure 2 illustrates Lemma 9 on $p = 2, q = 3, \alpha = 19/2, \beta = 9/2, S = 5$: the six paired rows hold 16 lattice points while the offset-free lower bound guarantees 12 for any height split with $\alpha + \beta = 14$.

Put $B_k = \max_{S \ge 0} G_k(S)$, $\Gamma_k = 2\gamma_k/s_k$, and $\delta_k = d_k/s_k$. Level S is *full* ($L = \lfloor S/2 \rfloor + 1$, $G(S) = T(S+1)$) precisely when

$$S - \mathbf{1}_{\{S \text{ odd}\}}\, \delta < \Gamma. \tag{6}$$

When $S \ge \Gamma$ and $d > 0$, writing $z = (S - \Gamma)/\delta$, $e = \mathbf{1}_{\{S \text{ odd}\}}$, and $f = \lfloor (z-e)/2 \rfloor + 1$ shows that the truncation loss satisfies

$$T(S+1) - G(S) \le f(f+e) \le \frac{(z+2)^2}{4}. \tag{7}$$

Even-n height coupling (Lemma C). Introduce the two nonnegative slope differences $u = x_2 - x_1$ and $w = x_3 - x_2$, so that

$$\begin{aligned} 3(x_1,x_2,x_3) &= (1 - 2u - w,\ 1 + u - w,\ 1 + u + 2w), \\ (d_1,d_2,d_3) &= (w,\ u + w,\ u). \end{aligned} \tag{8}$$

Direct parity substitution in (3) gives, for every $n \ge 3$,

$$T_*(n) = T(h-2) + 2T(h-3), \qquad D(n) = T_*(n) + \mathbf{1}_{\{n \text{ even}\}}. \tag{9}$$

For even $n = 2c + 4$ ($c \ge 1$), define the coupling region $K(c)$ by

$$(3c+8)u - (3c+5)w < 1, \quad (3c+5)u + (6c+13)w < 1. \tag{10}$$

Lemma 10 (Even-n Height Coupling). *Let $c \ge 1$, $n = 2c + 4$, $u, w \ge 0$, $(u,w) \ne (0,0)$, and $(u,w) \in K(c)$. Then the right side of (5) is at least $T_*(n) + 1 = D(n)$ for every $t \in \mathbb{R}$.*

Proof. By (8), the inequalities (10) rewrite as $\gamma_2 > (2c-1)x_1 + c(u+w)$ and $\gamma_3 > (2c-1)x_1 + (c-1)u$, so level $2c-1$ is full in pairs 2 and 3; since $s_1 > 2/3$, pair 1 is full at level $2c$, giving baseline $(c+1)^2 + 2c(c+1) = T_*(n)$.

Write $\theta = t - n/2$, so pair k has thresholds $\gamma_k/2 \pm \theta$, and set

$$\begin{aligned} a &= \gamma_1/2 - cx_3 = (c+2)u + w > 0, \\ b &= cx_1 - \gamma_2/2 = u - (c+1)w. \end{aligned}$$

Clearly $a > b$. If $|\theta| < a$, both middle rows of length $c+1$ in pair 1 are full, adding $c + 1 \ge 1$ over the baseline pairing bound of $c + 1$. If $|\theta| \ge a > b$, exchange the two corners of pair 2 if needed so the row of length $c+1$ accepts at least one point, and pair it with the row of length c: if either is full they contribute at least $c+1$, and otherwise their untruncated thresholds sum to $(\gamma_2 - (2c-1)x_1)/d_2 > c$, again gaining $+1$ while all other rows keep their baseline guarantees. □

Universal upper bound and slope localization.

Proof of Proposition 7 and Slope Localization. For $n = 2$ and $n = 3$, $m(2) = U(2) = 7$ and $m(3) = U(3) = 19$; for $3 \le n \le 15$, Theorem 2 (Section 4) gives $m(n) = F(n) \le U(n)$. Assume now $n \ge 16$ (and the argument already proves the bound for all $n \ge 4$ in the central regime). The column bound $2n^2 \le U(n)$ handles $x_3 \ge 1/2$. Otherwise (5) applies. For $0 < p \le q$ and $b > 0$, summing the j-rows of $\Delta(p,q;b)$ with $m = \lceil b/q \rceil$ yields

$$\Delta(p,q;b) \ge \frac{b^2}{2pq} + \frac{b}{2p} \ge 2\left(\frac{b}{p+q}\right)^2 + \frac{b}{p+q},$$

since the excess of $(mb - qm(m-1)/2)/p$ over $b^2/(2pq) + b/(2p)$ is $q(m - b/q)(1 - m + b/q)/(2p) \ge 0$. Applying this to $\max(\alpha_k, 0)$ and $\max(\beta_k, 0)$ and using convexity gives

$$\Delta(p_k,q_k;\alpha_k) + \Delta(p_k,q_k;\beta_k) \ge \psi(y_k) := A_+^2 + A_+,$$

where $y_k = 1/s_k$ and $A = 2h - ny_k$. Set $a_0 = (h-1)(h-3)/4$ and define the tangent function $\ell(y) = a_0 - \frac{9}{4}n(n-3)(2/3 - 1/y)$. For $A \ge 0$,

$$\psi(y) - \ell(y) = \frac{n(2y-3)^2(ny - h + 2)}{4y} \ge 0. \tag{11}$$

For $A \le 0$, $-4ny\,\ell(y) = (5h^2 - 2h - 15)(ny - 2h) + (h-3)^2(h+2) > 0$, so $\psi(y) \ge \ell(y)$ holds universally. Because $\sum_k 1/y_k =$

$\sum_k s_k = 2$, the linear terms cancel, leaving $\sum_k \ell(y_k) = 3a_0$ and therefore cutCount $\le C(n) - 3a_0 = (9n^2 + 6n - 20)/4$, proving Proposition 7.

Moreover, any plane cutting at least $F(n)$ cubes leaves total excess $\sum_k(\psi(y_k) - \ell(y_k)) \le U(n) - F(n) \le n/2$. At $y = y_3 = 1/s_3 \in [3/2, 2)$, if $ny \ge 2h$ then $-\ell(y) \ge (h-3)^2(h+2)/(8n) > n/2$ for $n \ge 16$, impossible. Hence (11) applies to y_3 and gives $(2y_3 - 3)^2(ny_3 - h + 2) \le 2y_3$. Since $4y_3(ny_3 - h + 2) - 3(n + 6) = (2y_3 - 3)(2ny_3 + n + 6) \ge 0$ and $u + 2w = 3x_3 - 1 = (2y_3 - 3)/y_3$, every competitor with $n \ge 16$ lies in the localized cone

$$u + 2w \le 3r(n), \qquad r(n) := \sqrt{\frac{8}{27(n+6)}}. \tag{12}$$

□

On this shrinking cone, Appendix A proves that for every $n \ge 700$ and $(u, w) \in (0, \infty)^2$ with $u + 2w \le 3r(n)$, either $B_1 + B_2 + B_3 \ge D(n)$ or n is even and $(u, w) \in K((n-4)/2)$. Combined with Lemma 10 and Proposition 6, this yields Theorem 4 and Corollary 5 unconditionally over $\mathbb{R}$.

4. FINITE CERTIFICATES: FARKAS-BSP TREES AND VERTEX ARRANGEMENTS

For a fixed $n \ge 5$, the normalized slope cone $Z(R) = \{u, w \ge 0 : u + 2w \le R\}$ is a compact triangle, with $R = 1/2$ for $n < 16$ and any rational $R \in [3r(n), 1/2]$ for $n \ge 16$. For fixed integers $S \ge 0$ and $1 \le \ell \le \lfloor S/2 \rfloor + 1$, Lemma 9 certifies $G_k(S) \ge T(S+1) - T(S+1-2\ell)$ whenever the affine form

$$H_{k,S,\ell}(u, w) := \gamma_k - p_k S - (\ell - 1)d_k > 0 \tag{13}$$

holds (including $d_k = 0$ on a cone edge). Both coupling tests in (10) are affine in (u, w) as well.

Pure-kernel Farkas-BSP trees ($5 \le n \le 32$). To close $5 \le n \le 32$ unconditionally inside Lean 4 without floating point or external C code, we recursively split $Z(R)$ by affine lines $H > 0$ vs. $H \le 0$ (and optionally $u > w$ vs. $w > u$, valid since perturbation gives $u \ne w$) until every leaf certifies either three level bounds (13) summing to at least $D(n)$ or both strict inequalities (10). On each leaf, a strict inequality $H > 0$ is certified by a nonnegative integer Farkas identity

$$mH = c + \sum_i \lambda_i H_i, \qquad m > 0,\ c, \lambda_i \ge 0,$$

using the leaf's active ancestor forms H_i with either $c > 0$ or some $\lambda_i > 0$ on a strict ancestor. The soundness theorem `checkCertN_sound` proves by induction over the tree that a passing integer word stream implies cutCount $\le F(n)$ for every real plane. Across $5 \le n \le 32$, the 28 trees stored in `SmallNAndDim.lean` total 1,299 leaves (235 for $5 \le n \le 15$ and 1,064 for $16 \le n \le 32$) and check purely by kernel reduction (`decide`, zero `native_decide`), proving Theorem 2 and Corollary 3. Figure 3 plots the complete 21-leaf partition for $n = 10$ (20 pairing leaves plus 1 coupling leaf $K(3)$).

Vertex-arrangement principle and external sweeps ($33 \le n \le 699$).

Proof of Proposition 8. Fix $n \ge 16$ and a finite family $\mathcal{A}$ of nondegenerate affine forms containing the three side lines of $Z(R_1)$ ($R_1 \ge 3r(n)$), all nondegenerate boundary lines of (13) within the configured windows $[\mathrm{slo}_k, \mathrm{shi}_k]$ and truncation depth F_p, and (for even n) both lines of (10). The truncated valuation $\mathrm{arrVal}(p)$ is monotone with respect to positive half-plane dominance: if every $\ell \in \mathcal{A}$ with $\ell(q_1) > 0$ also satisfies $\ell(q_2) > 0$, then $\mathrm{arrVal}(q_1) \le \mathrm{arrVal}(q_2)$. Given any nonzero $p \in Z(R_1)$, intersect $Z(R_1)$ with the closed half-planes $\{\ell \le 0 : \ell \in \mathcal{A}, \ell(p) \le 0\}$. This yields a nonempty compact convex polygon containing p that cannot collapse to $\{0\}$; its extreme points are arrangement vertices of $\mathcal{A}$, so at least one nonzero vertex $v \in Z(R_1)$ lies in this polygon. By construction every $\ell \in \mathcal{A}$ positive at v is positive at p, so $D(n) \le \mathrm{arrVal}(v) \le \mathrm{arrVal}(p)$. Unfolding $\mathrm{arrVal}(p)$ gives either $B_1 + B_2 + B_3 \ge D(n)$ or n even with $p \in K((n-4)/2)$, completing the proof via Lemma 9, Lemma 10, and (12). □

Two independently written C programs (`arr_verify.c` and `myarr.c`, using distinct level windows and cone radii) execute exact 128-bit integer determinant arithmetic over all 685 sizes $16 \le n \le 700$, checking $3(n+6)R_p^2 \ge 8R_q^2$ for $R_1 = R_p/R_q \ge 3r(n)$ and verifying every nonzero in-cone intersection of two nonparallel lines (counting incidences with multiplicity):

Table 1. Independent exact-integer vertex-arrangement sweeps covering $16 \le n \le 700$ (discharging the 667 non-Lean sizes $33 \le n \le 699$ in Theorem 1).

Program	Sizes	In-cone incidences	Failures
`arr_verify.c`	685	335,094,616	0
`myarr.c`	685	402,548,904	0

Both implementations record minimum margin 0 at every size (for instance, at $n = 16$ and $(u, w) = (0, 1/49)$, full levels $12, 12, 10$ yield $49 + 49 + 36 = 134 = D(16)$). Because Theorem 2 already closes $3 \le n \le 32$ and Theorem 4 closes $n \ge 700$ inside Lean, Theorem 1 relies on Table 1 solely for $33 \le n \le 699$.

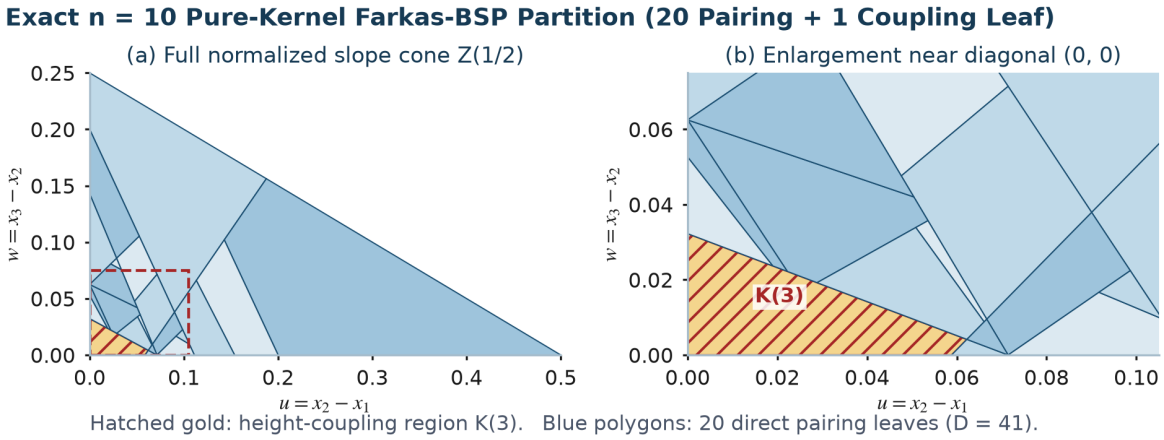


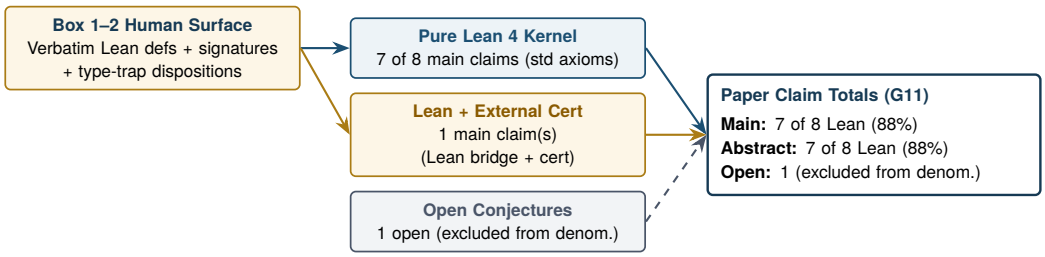
Figure 3. Complete 21-leaf pure-kernel Farkas-BSP partition of $Z(1/2)$ for $n = 10$ ($D(10) = 41$), decoded directly from `SmallNAndDim.lean` and verified for exact rational area sum $1/16$. Left: entire normalized cone $Z(1/2)$ with dashed zoom box. Right: enlargement near the diagonal showing the single hatched coupling leaf $K(3)$ (Lemma 10) surrounded by 20 direct pairing leaves.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 36f5eec0342e (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Open in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 164/164 public theorems (source-level approx) | WARN: Fidelity: 10/10 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

CheeseCutM.OpenCube (CheeseCutM/Defs.lean:57-60) — Section 2 (open unit cube (i,i+1) x (j,j+1) x (k,k+1))

```
def OpenCube (z : Cell) : Set (ℝ × ℝ × ℝ) :=
  {p | (z.1 : ℝ) < p.1 ∧ p.1 < (z.1 : ℝ) + 1 ∧
       (z.2.1 : ℝ) < p.2.1 ∧ p.2.1 < (z.2.1 : ℝ) + 1 ∧
       (z.2.2 : ℝ) < p.2.2 ∧ p.2.2 < (z.2.2 : ℝ) + 1}
```

CheeseCutM.Meets (CheeseCutM/Defs.lean:63-64) — Section 2 (plane meets open unit cube interior)

```
def Meets (a b c t : ℝ) (z : Cell) : Prop :=
  (plane a b c t ∩ OpenCube z).Nonempty
```

CheeseCutM.IsNormal (CheeseCutM/Defs.lean:67-67) — Section 2 (nonzero normal vector in R^3)

```
def IsNormal (a b c : ℝ) : Prop := (a, b, c) ≠ (0, 0, 0)
```

CheeseCutM.cutCount (CheeseCutM/Defs.lean:78-79) — Section 2 (number of open unit cubes cut by plane)

```
noncomputable def cutCount (n : ℕ) (a b c t : ℝ) : ℕ :=
  (cutSet n a b c t).card
```

CheeseCutM.maxCutCount (CheeseCutM/Defs.lean:82-83) — Section 1 (maximum m(n) over all nonzero real planes)

```
noncomputable def maxCutCount (n : ℕ) : ℕ :=
  sSup {k | ∃ a b c t : ℝ, IsNormal a b c ∧ cutCount n a b c t = k}
```

CheeseCutM.F (CheeseCutM/Defs.lean:86-86) — Equation (1) (Barany-Frankl formula floor((9n^2+4n-17)/4))

```
def F (n : ℕ) : ℕ := (9 * n ^ 2 + 4 * n - 17) / 4
```

CheeseCutM.Ubound (CheeseCutM/Defs.lean:100-100) — Proposition 7 (universal upper bound floor((9n^2+6n-20)/4))

```
def Ubound (n : ℕ) : ℕ := (9 * n ^ 2 + 6 * n - 20) / 4
```

CheeseCutM.ArrVerifyCfg (CheeseCutM/VertexPrinciple.lean:354-359) — Section 4.2 (vertex-arrangement certificate parameters)

```
structure ArrVerifyCfg where
  R1 : ℝ
  slo : Fin 3 → ℕ
  shi : Fin 3 → ℕ
  Fp : ℕ
  TUP : ℕ
```

CheeseCutM.ArrVerifyPass (CheeseCutM/VertexPrinciple.lean:472-475) — Proposition 8 (vertex-arrangement pass predicate)

```
def ArrVerifyPass (n : ℕ) (cfg : ArrVerifyCfg) : Prop :=
  3 * rad n ≤ cfg.R1 ∧
  (∀ k : Fin 3, cfg.slo k ≤ cfg.shi k) ∧
  (∀ v : ℝ × ℝ, IsVertex cfg.R1 (arrLines n cfg) v → v ≠ (0, 0) → D n ≤ arrVal n cfg
    ↪ v)
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): CheeseCutM.F_small_values (F 3 = 19 ∧ F 4 = 35 ∧ F 5 = 57 ∧ F 6 = 82); CheeseCutM.maxCutCount_one (maxCutCount 1 = 1); CheeseCutM.maxCutCount_two (maxCutCount 2 = 7); CheeseCutM.cutCount_two_unperturbed_central (cutCount 2 1 1 1 3 = 6).

Supporting Semantic Closure Index (non-Box-1 helpers): CheeseCutM.Cell (CheeseCutM/Defs.lean), CheeseCutM.plane (CheeseCutM/Defs.lean), CheeseCutM.cells (CheeseCutM/Defs.lean), CheeseCutM.cutSet (CheeseCutM/Defs.lean), CheeseCutM.D (CheeseCutM/Defs.lean), CheeseCutM.T2 (CheeseCutM/Defs.lean), CheeseCutM.Tstar (CheeseCutM/Defs.lean), CheeseCutM.rad (CheeseCutM/Phase4.lean), CheeseCutM.inK (CheeseCutM/Pairing.lean), CheeseCutM.AffForm (CheeseCutM/VertexPrinciple.lean), CheeseCutM.evalForm (CheeseCutM/VertexPrinciple.lean), CheeseCutM.NonDegen (CheeseCutM/VertexPrinciple.lean), CheeseCutM.detForm (CheeseCutM/VertexPrinciple.lean), CheeseCutM.inZone (CheeseCutM/VertexPrinciple.lean), CheeseCutM.IsVertex (CheeseCutM/VertexPrinciple.lean), CheeseCutM.sideU (CheeseCutM/VertexPrinciple.lean), CheeseCutM.sideW (CheeseCutM/VertexPrinciple.lean), CheeseCutM.sideTop (CheeseCutM/VertexPrinciple.lean), CheeseCutM.KForm1 (CheeseCutM/VertexPrinciple.lean), CheeseCutM.KForm2 (CheeseCutM/VertexPrinciple.lean), CheeseCutM.cap (CheeseCutM/Pairing.lean), CheeseCutM.hpForm (CheeseCutM/VertexPrinciple.lean), CheeseCutM.arrLines (CheeseCutM/VertexPrinciple.lean), CheeseCutM.isFullHp (CheeseCutM/VertexPrinciple.lean), CheeseCutM.Ltrunc (CheeseCutM/VertexPrinciple.lean), CheeseCutM.Gtrunc (CheeseCutM/VertexPrinciple.lean), CheeseCutM.lastFull (CheeseCutM/VertexPrinciple.lean), CheeseCutM.btPair (CheeseCutM/VertexPrinciple.lean), CheeseCutM.arrVal (CheeseCutM/VertexPrinciple.lean), CheeseCutM.Delta (CheeseCutM/Phase4.lean), CheeseCutM.triNat (CheeseCutM/Phase4.lean), CheeseCutM.Lval (CheeseCutM/Pairing.lean), CheeseCutM.Gval (CheeseCutM/Pairing.lean), CheeseCutM.x1 (CheeseCutM/Phase4.lean), CheeseCutM.x2 (CheeseCutM/Phase4.lean), CheeseCutM.x3 (CheeseCutM/Phase4.lean), CheeseCutM.s1 (CheeseCutM/Phase4.lean), CheeseCutM.s2 (CheeseCutM/Phase4.lean), CheeseCutM.s3 (CheeseCutM/Phase4.lean), CheeseCutM.alpha1 (CheeseCutM/Phase4.lean), CheeseCutM.alpha2 (CheeseCutM/Phase4.lean), CheeseCutM.alpha3 (CheeseCutM/Phase4.lean), CheeseCutM.beta1 (CheeseCutM/Phase4.lean), CheeseCutM.beta2 (CheeseCutM/Phase4.lean), CheeseCutM.beta3 (CheeseCutM/Phase4.lean), CheeseCutM.missedCount (CheeseCutM/Phase4.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

CheeseCutM.theoremM_all_n_from_arr_33_699 (CheeseCutM/SmallNAndDim.lean:2124-2137) src:02b2b418fa15f5

```
theorem theoremM_all_n_from_arr_33_699
    (h_arr_33_699 : ∀ n : ℕ, 33 ≤ n → n ≤ 699 → ∃ cfg : ArrVerifyCfg, ArrVerifyPass n
      ↪ cfg)
    (n : ℕ) (hn : 3 ≤ n) :
    maxCutCount n = F n ∧ maxCutCount n < F n + 2
```

Assumptions: n >= 3 plus external vertex-certificate predicate ArrVerifyPass on the finite window 33 <= n <= 699 (discharged by two independent exact-integer C runs). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.maxCutCount_eq_F_3_to_32 (CheeseCutM/SmallNAndDim.lean:2086-2088) src:c98612f47a1e2a

```
theorem maxCutCount_eq_F_3_to_32 (n : ℕ) (hn3 : 3 ≤ n) (hn32 : n ≤ 32) :
    maxCutCount n = F n
```

Assumptions: 3 <= n <= 32 (pure-kernel Farkas-BSP trees; zero external hypotheses). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.maxCutCount_ten (CheeseCutM/SmallNAndDim.lean:2099-2101) src:f950bb57988f40

```
theorem maxCutCount_ten : maxCutCount 10 = 230
```

Assumptions: None (closed unconditional theorem maxCutCount 10 = 230 over R). Applicability: Unconditional (closed statement; no outer premise witness needed).

CheeseCutM.maxCutCount_eq_F_ge_700 (CheeseCutM/Main.lean:53-56) src:9b3bfd868f2922

```
theorem maxCutCount_eq_F_ge_700 (n : ℕ) (hn : 700 ≤ n) :
    maxCutCount n = F n
```

Assumptions: n >= 700 (pure-kernel uniform analytic proof over R; zero external hypotheses). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.barany_frankl_plus_two_refuted_ge_700 (CheeseCutM/Main.lean:60-63) src:84840470c2f0cf

```
theorem barany_frankl_plus_two_refuted_ge_700 (n : ℕ) (hn : 700 ≤ n) :
    maxCutCount n < F n + 2
```

Assumptions: n >= 700 (unconditional refutation maxCutCount n < F n + 2 over R). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.barany_frankl_lower_bound (CheeseCutM/Defs.lean:889-901) src:17ef6ec1998f3a

```
theorem barany_frankl_lower_bound (n : ℕ) (hn : 3 ≤ n) :
    ∃ a b c t : ℝ, IsNormal a b c ∧ cutCount n a b c t = F n
```

Assumptions: n >= 3 (explicit tilted plane cutting exactly F(n) open unit cubes in both parities). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.theoremU_bound_all_ge_2 (CheeseCutM/SmallNAndDim.lean:2106-2119) src:0773610f50a433

```
theorem theoremU_bound_all_ge_2 (n : ℕ) (hn : 2 ≤ n) (a b c t : ℝ) (hv : IsNormal a b
  ↪ c) :
    cutCount n a b c t ≤ Ubound n
```

Assumptions: n >= 2 and nonzero normal IsNormal a b c in R^3. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.prop_V_arr_verify_sound (CheeseCutM/VertexPrinciple.lean:709-770) src:4bb208eaec0a11

```
theorem prop_V_arr_verify_sound (n : ℕ) (hn : 16 ≤ n) (cfg : ArrVerifyCfg)
    (hpass : ArrVerifyPass n cfg) :
    ∀ a b c t : ℝ, IsNormal a b c → cutCount n a b c t ≤ F n
```

Assumptions: n >= 16 and arrangement configuration cfg satisfying ArrVerifyPass n cfg. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.lemma_P_pairing_bound (CheeseCutM/Pairing.lean:212-232) src:f861c6acfd6345

```
theorem lemma_P_pairing_bound {p q : ℝ} (hp : 0 < p) (hpq : p ≤ q) (α β : ℝ) (S : ℕ) :
    Gval p q (α + β) S ≤ Delta p q α + Delta p q β
```

Assumptions: 0 < p <= q, real thresholds alpha, beta, and level S in N. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

CheeseCutM.lemma_C_coupling (CheeseCutM/Pairing.lean:654-906) src:97d86dcbfaef30

```
theorem lemma_C_coupling (c : ℕ) (hc : 1 ≤ c) (u w t : ℝ) (hu : 0 ≤ u) (hw : 0 ≤ w)
    (hne : (u, w) ≠ (0, 0)) (hK : inK c u w) :
    D (2 * c + 4) ≤ missedCount (2 * c + 4) u w t
```

Assumptions: c >= 1, u, w >= 0, (u, w) != (0, 0), (u, w) in K(c), any real height t. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
D	nat_sub	- 3	intended_convention	3*n <= 3*n^2 holds for every n in N (no Nat underflow).
D	nat_sub	- F	proved_guard (F_le_diag)	Guarded by proved lemma F_le_diag for all n >= 3.
F	nat_sub	- 17	intended_convention	For n >= 2, 9*n^2 + 4*n >= 44 > 17 (all theorems take n >= 3; n = 1 checked separately).
F	nat_div	/ 4	intended_convention	Nat division by 4 implements floor((9n^2 + 4n - 17)/4) exactly for nonnegative numerators.
Ltrunc	nat_sub	- Fp	intended_convention	Clamped below by max 1 (cap S - Fp) by design of truncated window.
Ltrunc	nat_sub	- 1	intended_convention	lo = max 1 (cap S - Fp) >= 1 guarantees lo - 1 >= 0.
T2	nat_div	/ 4	intended_convention	Implements floor((m+1)^2/4) for m >= 1.
Ubound	nat_sub	- 20	intended_convention	For n >= 2, 9*n^2 + 6*n >= 48 > 20 (theoremU_bound_all_ge_2 takes 2 <= n).
Ubound	nat_div	/ 4	intended_convention	Implements floor((9n^2 + 6n - 20)/4) for nonnegative numerators.
arrLines	nat_sub	- cfg.Fp	intended_convention	Window lower bound max 1 (cap S - cfg.Fp) intentionally clamps at 1.
arrLines	nat_sub	- 4	intended_convention	Evaluated only under 16 <= n, so n - 4 >= 12.
arrLines	nat_div	/ 2	intended_convention	Guarded by if Even n, so (n - 4) / 2 = c is exact.
arrVal	nat_sub	- 4	intended_convention	Evaluated only under 16 <= n, so n - 4 >= 12.
arrVal	nat_div	/ 2	intended_convention	Guarded by if Even n, so (n - 4) / 2 = c is exact.
cap	nat_div	/ 2	intended_convention	Nat division implements floor(S/2) + 1.
hpForm	nat_sub	- Sr	intended_convention	Real subtraction over R (signed field, no Nat underflow).
hpForm	nat_sub	- 3	intended_convention	Real subtraction over R (signed field, no Nat underflow).
hpForm	nat_sub	- 2	intended_convention	Real subtraction over R (signed field, no Nat underflow).
hpForm	nat_sub	- 4	intended_convention	Real subtraction over R (signed field, no Nat underflow).
inK	nat_sub	- (3 * (c : ℝ) + 5)	intended_convention	Real subtraction in R (signed field, no Nat underflow).
rad	real_log_rpow	Real.sqrt	intended_convention	Positivity of radicand 8/(27*(n+6)) > 0 holds for all n in N (proved by rad_pos and rad_sq).
rad	real_div	(27 * ((n : ℝ) + 6))	intended_convention	Denominator 27*(n+6) >= 162 > 0 unconditionally.

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
instDecidablePredStdArith PlaneZ	local_instance_ or_notation	instDecidablePredStd ArithPlaneZ	intended_convention	Standard DecidablePred instance (unused by 3D proofs).
Lval	nat_sub	- p	intended_convention	Real subtraction over R (signed field).
Lval	real_div	(q - p)	intended_convention	Evaluated only in the branch p < q, where q - p > 0.
alpha1	nat_sub	- 1	intended_convention	Evaluated for n >= 3 (n - 1 >= 2).
alpha2	nat_sub	- 1	intended_convention	Evaluated for n >= 3 (n - 1 >= 2).
alpha3	nat_sub	- 1	intended_convention	Evaluated for n >= 3 (n - 1 >= 2).
beta1	nat_sub	- 1	intended_convention	Evaluated for n >= 3 over R.
beta2	nat_sub	- 1	intended_convention	Evaluated for n >= 3 over R.
beta3	nat_sub	- 1	intended_convention	Evaluated for n >= 3 over R.
triNat	nat_div	/ p	intended_convention	Real division over R under 0 < p <= q.
triNat	nat_div	/ q	intended_convention	Real division over R under 0 < p <= q.
x1	nat_div	/ 3	intended_convention	Exact real division by constant 3 in R.
x2	nat_div	/ 3	intended_convention	Exact real division by constant 3 in R.
x3	nat_div	/ 3	intended_convention	Exact real division by constant 3 in R.

[a]Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Exact law m(n) = F(n) and m(n) < F(n) + 2 for all n >= 3 (Lean + 33..699 C certs)	Lean + Cert	CheeseCutM.theoremM_all_n_from_arr_33_699 + recompute.py (20835a7c4381) [archived; replay pending]	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel for n <= 32, n >= 700, & soundness + 2 independent C exact-integer vertex runs for 33 <= n <= 699.
Thm. 2: Unconditional small-box law m(n) = F(n) for all 3 <= n <= 32	Lean Kernel	CheeseCutM.maxCutCount_eq_F_3_to_32	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (28 exact N Farkas-BSP trees, 1,299 leaves).
Cor. 3: Exact resolution m(10) = 230 of Barany-Frankl's open case	Lean Kernel	CheeseCutM.maxCutCount_ten	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (unconditional over R).
Thm. 4: Unconditional asymptotic-range law m(n) = F(n) for all n >= 700	Lean Kernel	CheeseCutM.maxCutCount_eq_F_ge_700	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (uniform analytic proof over R).
Cor. 5: Unconditional refutation m(n) < F(n) + 2 for all n >= 700	Lean Kernel	CheeseCutM.barany_frankl_plus_two_refuted_ge_700	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (unconditional over R).
Prop. 6: Barany-Frankl tilted-plane lower bound cut Count = F(n) for all n >= 3	Lean Kernel	CheeseCutM.barany_frankl_lower_bound	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (explicit tilted plane in both parities).
Prop. 7: Universal upper bound cutCount <= floor((9n^2 + 6n - 20)/4) for all n >= 2	Lean Kernel	CheeseCutM.theoremU_bound_all_ge_2	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (unconditional over R; U(10) = 235).
Prop. 8: Vertex-principle certificate soundness Arr VerifyPass => cutCount <= F(n) (n >= 16)	Lean Kernel	CheeseCutM.prop_V_arr_verify_sound	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (n >= 16, ArrVerifyPass n cfg).
Lem. 9: Opposite-row pairing lower bound Delta(alpha)+Delta(beta) >= G(S)	Lean Kernel	CheeseCutM.lemma_P_pairing_bound	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (0 < p <= q, real alpha, beta, S in N).
Lem. 10: Even-n height coupling missedCount >= T*(n) + 1 on K(c)	Lean Kernel	CheeseCutM.lemma_C_coupling	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (c >= 1, u, w >= 0, (u,w) != (0,0), (u,w) in K(c)).
Conj. 11: Maximizer classification in 3D, Lean closure of 33..699, and optimal 4D normals	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open problems (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: propext, Classical.choice, Quot.sound. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 6c1cb081fed4 | Lean mode: committed_receipt | Closure SHA-256: 52e6c472da5d | Git anchor: 36f5eec0342e | Axioms SHA-256: e708f6d864ab | Fidelity SHA-256: ecfefb3cca48. External validation receipts: Pinned commit | Palomar record.

6. EXTENSIONS, PRIOR WORK, AND OPEN QUESTIONS

Companion Lean-verified extensions. The library `lean/CheeseCutM/` also formalizes three related exact facts unconditionally in the pure kernel: (i) in two dimensions, $m_2(n) = 2n - 1$ for all $n \geq 1$ (`maxCutCount2D_eq`), attained by $x + y = n - 1/2$; (ii) for standard arithmetic planes in digital geometry in the sense of Andres [4], an integral normal $(a, b, c) \in \mathbb{N}^3$ with $W = a + b + c > 0$ and $\mu \in \mathbb{Z}$ satisfies $\mu \leq ai + bj + ck < \mu + W$ on $\{0, \dots, h\}^3$ precisely for the open cubes cut at height $t = \mu + W - 1/2$, transferring the sharp bound $\leq F(n)$ to all $3 \leq n \leq 32$ and $n \geq 700$ (`stdArithPlane_card_le_F_uncond`); and (iii) in four dimensions at $n = 3$, the preferred tilt direction reverses: every translate of $(1, 1, 1, 1)$ cuts at most 61 cubes and every translate of $(2, 2, 2, 3)$ cuts at most 62, whereas $(1, 2, 2, 2) \cdot x = 21/2$ cuts 63 cubes (`cutCount4D_tilted_surprise_three`), which is optimal over all normals in $\mathbb{R}^4$ because $3^4 - 2^4 = 65$ diagonal chains lose at least 2 via the two disjoint one-cell pairs $\{(0, 2, 0, 0), (0, 2, 2, 2)\}$ and $\{(2, 0, 0, 0), (2, 0, 2, 2)\}$.

Prior work and searches logged. Bárány and Frankl [6] posed the three-dimensional cube-cutting problem (summarized in Oberwolfach Report 30/2020 [5]), established $m(2) = 7, m(3) = 19, m(4) = 35$ and $230 \leq m(10) \leq 246$, introduced the tilted plane (2) and the diagonal-chain framework, and proved the d-dimensional asymptotic law $N^d(n) = V_d n^{d-1}(1 + o(1))$ in [7]. Aliev [2] proved their normalized section-volume conjecture ($V_3 = 9/4$); Marichal and Mossinghoff [9] analyzed hypercube diagonal slabs via Eulerian numbers; Nayar and Tkocz [10] and Kalarickal, Rotunno, Singh, and Tkocz [8] surveyed extremal cube sections while still quoting the 2021 bounds on $m(n)$; and Ambrus, Bárány, Frankl, and Varga [3] studied the distinct two-dimensional chessboard-piercing question. The $n = 3$ diagonal bisector (19 cubes) appeared as Problem 30 of the 1995 AHSME [1]. Searches logged across OEIS, arXiv, zbMATH, CORE, Google Scholar, and citing literature found no prior proof of $m(n) = F(n)$ for $n \geq 5$ or test of the +2 remark; however, the paywalled printed *Monthly* pagination [6] [to check], the official 1995 AHSME print booklet [1] [to check], MathSciNet, Croft–Falconer–Guy, and Brass–Moser–Pach remain owed before journal submission.

Conjecture 11 (Extremizer Classification and 4D Tilt Family). *Characterizing all maximizing planes in $[0, n]^3$ (in particular whether every maximizer for $n \geq 5$ cuts three full consecutive diagonal layers), eliminating the finite $33 \leq n \leq 699$ external certificate window inside Lean 4, and determining the optimal normal family in $[0, n]^4$ for general n remain open.*

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, certificate decoders, figure scripts, and ledgers live in `papers/cheese-cut/draft4/`, with the Lean 4 library in `lean/CheeseCutM/` (registered in the Palomar Registry as `PALOMAR-2026-11-22-000015` v2 via mechanical Lean checking and automated editorial review) and both exact-integer C verifiers and archived logs in `docs/goal2/records/phase5/B-cheese/`. Running `python3 papers/cheese-cut/draft4/recompute.py` executes the Section 1 Python box verbatim, checks all 28 Farkas-BSP trees (1,299 leaves), verifies all 685 archived rows of both C runs, and recompiles both C programs to reproduce $n = 33$ and $n = 699$ (pass `-a` for a full 685-size C replay). AI research agents built and directed by Tyler B. Hughes discovered the proof architecture, wrote and audited the Lean 4 formalization and C verifiers, generated the figures, and drafted this manuscript. No human mathematician has yet refereed the proof.

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A. UNIFORM INEQUALITIES FOR $N \geq 700$

This appendix proves that for every $n \geq 700$ and $(u, w) \in (0, \infty)^2$ satisfying (12) ($u + 2w \leq 3r(n)$), either $B_1 + B_2 + B_3 \geq D(n)$ or n is even and $(u, w) \in K((n-4)/2)$, completing Theorem 4 and Corollary 5.

Accounting identity. Write $\eta_k = s_k - 2/3$, $\kappa_k = s_k/d_k$, and $\Gamma_k = h - 3 + g_k$, where

$$g_k = \frac{3n\eta_k}{s_k}, \quad j_k = \lceil g_k \rceil, \quad J_k = h - 3 + j_k, \quad \varepsilon_k = j_k - g_k, \quad z_k = \kappa_k \varepsilon_k.$$

For $n \geq 700$, (12) yields $r < 0.0204862$, $0.64618 \leq s_k \leq 0.70764$, and $d_2 = u + w \leq 3r$, ensuring $\Gamma_k = 4h - 2n/s_k > 0$ and $J_k > 0$. Define

$$E = -\sum_{k=1}^{3} g_k, \qquad N = \sum_{k=1}^{3} j_k, \qquad Q = \sum_{k=1}^{3} g_k^2 - 2(h-2)E.$$

Using $\sum_k \eta_k = 0$ and $\eta_k/s_k = \frac{3}{2}\eta_k - \frac{3}{2}\eta_k^2/s_k$ gives

$$E = \frac{9n}{2} \sum_{k=1}^{3} \frac{\eta_k^2}{s_k} > 0, \qquad \sum_{k=1}^{3} \varepsilon_k = N + E,$$
$$Q = 9n \sum_{k=1}^{3} \frac{\eta_k^2}{s_k} \left(\frac{n(1 - s_k)}{s_k} + 3 \right) \geq 0. \tag{14}$$

Level $J_k - 1 < \Gamma_k$ is full by (6). At level J_k, the integer loss $c_k = T(J_k + 1) - G_k(J_k)$ obeys $4c_k \le (z_k + 2)^2$ by (7). Our three integer lower bounds on $B_1 + B_2 + B_3$ are $V_0 = \sum_k T(J_k)$, V_a (raising component a by 1 and subtracting c_a), and V_{ab} (raising $a \ne b$ and subtracting $c_a + c_b$).

For $M_k = h - 3 + \mu_k$, set $\pi(M) = \#\{k : M_k \text{ even}\} - e_*$ with $e_* = 1$ (odd n) or 2 (even n). From $4T(m) = (m+1)^2 - \mathbf{1}_{\{m \text{ even}\}}$,

$$4\Big(\sum_k T(M_k) - T_*\Big) = 2(h-2)\Big(\sum_k \mu_k - 1\Big) + \sum_k \mu_k^2 - 1 - \pi(M). \tag{15}$$

Evaluating (15) at $J + e_a$ and $J + e_a + e_b$, substituting $j_i = g_i + \varepsilon_i$, and dropping nonnegative terms $2(h - 2 + g_i)\varepsilon_i + \varepsilon_i^2$ yields

$$\begin{aligned} 4(V_a - T_*) + \pi(J + e_a) &\ge Q + 2g_a - 4 \\ &\quad + \varepsilon_a(2\Gamma_a + 4 - 4\kappa_a - \kappa_a^2\varepsilon_a), \end{aligned} \tag{16}$$

$$\begin{aligned} 4(V_{ab} - T_*) + \pi(J + e_a + e_b) &\ge 2(h-2) + Q + 2g_a + 2g_b - 7 \\ &\quad - z_a^2 - 4z_a - z_b^2 - 4z_b. \end{aligned} \tag{17}$$

Because $\pi \le 2$ (odd n) and $\pi \le 1$ (even n), a right-hand side strictly exceeding 1 implies $V \ge D(n)$ by integrality and (9).

Case $N \ge 1$. Apply (15) with $\mu = j$. For $N \ge 2$ the right side is positive; for $N = 1$ and odd n, $1 + \pi(J) = \#\{k : j_k \text{ odd}\} \le \sum j_k^2$. For $N = 1$ and even $n = 2c + 4$, the bound succeeds unless $j = (1, 0, 0)$. Then $g_3 > -1$ is the second inequality of (10); if the first failed we would have $n(u - w) \ge 2x_1$, while $g_1 \le 1$ gives $s_1 \ge n(2u + w) \ge 2n(u - w) \ge 4x_1$, forcing $x_1 \le 1/5$ and $x_3 \ge 2/5$, contradicting $x_3 \le 1/3 + r$ with $r < 1/15$.

Case $N \le 0$: raising one or two levels. Set $d = u + w$, $X = nd$, $\phi = \eta_2/d = (w - u)/(3d) \in [-1/3, 1/3]$, and $H = 1 + 3\phi^2 \in [1, 4/3]$, so $\sum_k \eta_k^2 = Hd^2/2$. All terminating decimals below denote exact rationals verified in `recompute.py`. Substituting $0.64618 \le s_k \le 0.70764$ into (14) gives

$$\begin{aligned} Q &\ge 2.627HX^2, \quad & 3.1795HX^2/n \le E &\le 3.4821HX^2/n, \\ |g_2| &\le 1.5476X, & \kappa_2 &\le 0.70764n/X, \end{aligned} \tag{18}$$

together with the sharp mixed bound

$$\kappa_2^2 E \le n(2 + 4.295d), \tag{19}$$

which follows from $\kappa_2^2 E/n \le 9Hs_2^2/(4s_3)$, the algebraic identity

$$\begin{aligned} d(8+d)/3 + 2s_3 - 9Hs_2^2/4 &= \tfrac{1}{3} - 3\phi^2 + d^2\big(\tfrac{1}{3} - \tfrac{9}{4}H\phi^2\big) \\ &\quad + d\big(\tfrac{1}{3} - \phi\big)(5 + 3\phi + 9\phi^2) \ge 0, \end{aligned}$$

and $(8 + d)/(3s_3) < 4.295$.

Subcase $E \ge 1$. Take $a = 2$ in (16). The ε_2-term is concave on $[0, 1]$, so its minimum occurs at $\varepsilon_2 \in \{0, 1\}$. Writing $2\Gamma_2 + 4 = 2n - 4 + 2g_2$ and $(\kappa_2^2 - 2n)_+ \le 4.295X$ from (19) gives a lower bound

$$2.627HX^2 - 10.486X - 8 - 2.8306n/X,$$

increasing for $X \ge 2$. Since (18) forces $X \ge X_0 = \sqrt{0.28718\, n/H}$ ($0.4640\sqrt{n} \le X_0 \le 0.5359\sqrt{n}$), substituting X_0 yields at least $0.7544n - 11.72\sqrt{n} - 8 > 209$ for all $n \ge 700$.

Subcase $E < 1$, $X \ge 6$. Since $\sum \varepsilon_k = N + E \ge 0$ and $N \le 0$ is an integer, $N = 0$ and $\varepsilon_k \le E$. From (18), $X < 0.56082\sqrt{n}$ and $d < 0.0212$, sharpening the ranges to $s_1 \le 0.6809$, $s_3 \ge 0.6525$, $0.6595 \le s_2 \le 0.6738$, whence

$$Q \ge 3.097HX^2, \quad E \le 3.4483HX^2/n, \quad \kappa_2^2 E \le 2n + 4.2X,$$

and $\kappa_2 \le 0.6738n/X$. Evaluating (16) at endpoints $\varepsilon_2 \in \{0, E\}$ uses $E(4.2X + 4) \le 0.3268HX^2$ and $4E\kappa_2 \le 9.294HX$. For $\phi \ge 0$ ($g_2 \ge 0$), the right side is at least $H(2.7702X^2 - 9.294X) - 4 > 39$. For $\phi = -\tau < 0$, $g_2 \ge -4.549\tau X$ and $2E|g_2| \le 0.2218HX^2$, giving

$$H(2.5484X^2 - 9.294X) - 9.098\tau X - 4 \ge 2.5484X^2 - 12.327X - 4 \ge 13.7.$$

Subcase $E < 1$, $X < 6$. Apply (17) with $a = 2$ and $b \in \{1, 3\}$ minimizing z_b/s_b. Since $d_1 + d_3 = d_2$, averaging gives $z_2/s_2 + z_b/s_b \le (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/d_2 = E/d_2 \le 3X/s_3$, so $z_2 + z_b \le 18(0.6809/0.6525) < 18.79$ and $z_2^2 + 4z_2 + z_b^2 + 4z_b < 428.3$. With $g_1 > 0$, $g_2 \ge -9.098$, and $g_3 \ge -18.391$, the right side of (17) exceeds $2(h-2) - 54.98 - 7 - 428.3 \ge 903.72 > 1$, finishing the proof.

REFERENCES

(1) American High School Mathematics Examination (AHSME). *46th Annual AHSME (1995)*, Problem 30, Mathematical Association of America, **1995** [to check].

(2) Aliev, I. On the volume of hyperplane sections of a d-cube. *Acta Math. Hungar.* **2021**, *163* (2), 547–551. DOI: 10.1007/s10474-020-01057-y.

(3) Ambrus, G.; Bárány, I.; Frankl, P.; Varga, D. Piercing the chessboard. *SIAM J. Discrete Math.* **2023**, *37* (3), 1457–1471. DOI: 10.1137/21M146048X.

(4) Andres, É. Discrete linear objects in dimension n: the standard model. *Graphical Models* **2003**, *65* (1–3), 92–111. DOI: 10.1016/S1524-0703(03)00004-3.

(5) Bárány, I. (joint with P. Frankl). Cells in the box and a hyperplane. In *Discrete Geometry*, *Oberwolfach Rep.* **2020**, *17* (2), Report No. 30/2020, 1477–1480. DOI: 10.4171/OWR/2020/30.

(6) Bárány, I.; Frankl, P. How (not) to cut your cheese. *Amer. Math. Monthly* **2021**, *128* (6), 543–552 (author manuscript read in full; printed pages [to check]). DOI: 10.1080/00029890.2021.1901463.

(7) Bárány, I.; Frankl, P. Cells in the box and a hyperplane. *J. Eur. Math. Soc.* (*JEMS*) **2023**, *25* (7), 2863–2877. DOI: 10.4171/JEMS/1252.

(8) Kalarickal, J.; Rotunno, D.; Singh, S.; Tkocz, T. Convexity properties of sections of 1-symmetric bodies and Rademacher sums. *Proc. Amer. Math. Soc. Ser. B* **2025**, *12*, 187–196. DOI: 10.1090/bproc/237.

(9) Marichal, J.-L.; Mossinghoff, M. J. Slices, slabs, and sections of the unit hypercube. *Online J. Anal. Comb.* **2008**, *3*, Art. 1, 11 pp. arXiv:math/0607715.

(10) Nayar, P.; Tkocz, T. Extremal sections and projections of certain convex bodies: a survey. In *Harmonic Analysis and Convexity*, Adv. Anal. Geom. **9**, De Gruyter, Berlin, **2023**, pp. 343–390. DOI: 10.1515/9783110775389-008.