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Working Draft

The Climb to a Square: Structural Dynamics, No Periodic Branch Patterns, and Verification Below $100{,}001^2$

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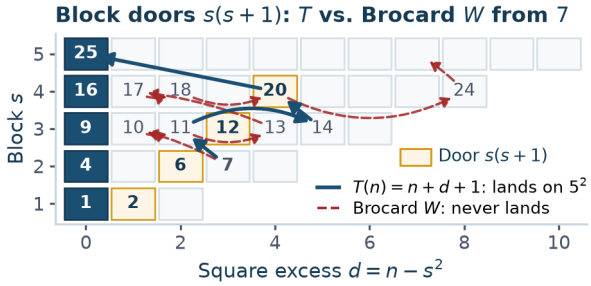
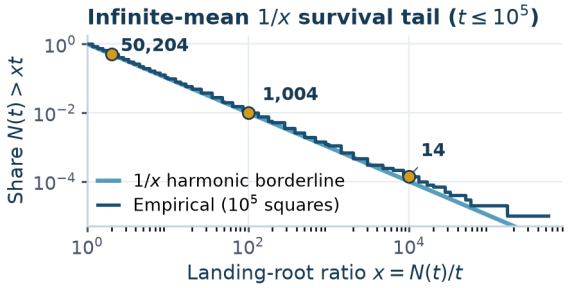
Supporting Information: `recompute.py` (standalone exact Python verifier for walks, invariants, families, and branch patterns)

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.





ABSTRACT: Write $n = s^2 + d$ ($s = \lfloor\sqrt{n}\rfloor$, $0 \le d \le 2s$) and iterate $T(n) = n + d + 1 = 2n + 1 - \lfloor\sqrt{n}\rfloor^2$. We conjecture every $n \ge 0$ reaches a square. We prove: (1) T is strictly increasing, lands only from doors $s(s+1) \mapsto (s+1)^2$, has a growing-clock form, and satisfies $T(x) + 1 = W(x+1)$ off shifted squares for Brocard's 1904 walk $W(n) = n + d$; (2) T is injective with non-images $r^2 - 1$ (Wenders 2012 / Mathar, OEIS `A094761`), giving $s + 1$ chains per block with one landing via its door; (3) $N(2^k - 1) = 2^k$, $N(2^k - 2) = 2^k + 1$, and $(2^k - k)^2 - 1 \to (2^k - 1)^2$; and (4) no orbit has an eventually periodic block-doubling pattern. Standalone replay certifies $n < 1{,}001^2$ and 1,048,575 patterns ($E \le 20$); multi-program censuses verify $n < 100{,}001^2 = 10{,}000{,}200{,}001$ (slowest $80{,}867^2 \to 43{,}471{,}242{,}310^2$, 86,942,332,668 steps) and a $1/x$ survival tail. The conjecture is unsettled; no Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 6 in Lean Kernel (0%) · 2 Cert Only · 4 written · 1 open (not counted)
Abstract claims: 0 of 6 in Lean Kernel (0%) · 2 Cert Only · 4 written · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 11B83, 11B37, 11Y55 **Keywords:** square excess, integer iteration, Brocard recursion, growing-modulus doubling, infinite-mean survival tail, experimental number theory

1. INTRODUCTION AND MAIN RESULTS

Every nonnegative integer n lies at a unique nonnegative distance above a perfect square: writing $s = \lfloor\sqrt{n}\rfloor$ and $d = n - s^2 \in \{0, 1, \dots, 2s\}$ (the *square excess* or *leftover*, OEIS `A053186`), we have $n = s^2 + d$. Consider the one-line deterministic map

$$T(n) := n + d + 1 = 2n + 1 - \lfloor\sqrt{n}\rfloor^2. \tag{1}$$

Because $d + 1 \ge 1$, the iterates $n, T(n), T^2(n), \dots$ grow strictly at every step, so no orbit can ever cycle. Starting from $n = 7 = 2^2 + 3$, for instance, produces

$$7 \to 11 \to 14 \to 20 \to 25 = 5^2,$$

hitting a perfect square in 4 steps. Without the additive $+1$, the map $W(n) = n + d$ goes back to H. Brocard's 1904 Question 2837 (OEIS `A094761`, `A094764` [4, 5] `[to check]`) and was later placed in a broader multiplicative family $n \mapsto n + rd$ ($r \ge 2$) by S. H. Weintraub [9] (`[to check]`). Under W, a simple parity obstruction guarantees that no non-square ever reaches a square. Adding 1 reverses that fate completely: T is simply W shifted by 1 away from shifted squares ($T(x) + 1 = W(x+1)$ whenever $x + 1$ is not a square), and that unit shift opens a single landing door in every square block.

What makes the landing question for T mathematically sharp is its exact block-population law. As we prove below, $\mathbb{Z}_{\ge 0}$ decomposes into disjoint infinite forward chains rooted at the integers $r^2 - 1$ ($r \ge 1$), exactly $s + 1$ chains traverse block $B_s = [s^2, (s+1)^2 - 1]$, and exactly one of those $s + 1$ chains exits onto $(s+1)^2$ through the block's unique door $s(s+1)$. Under an independent random-selection heuristic across blocks, a climb alive at block t survives past block S with probability

$$\prod_{s=t}^{S-1} \left(1 - \frac{1}{s+1}\right) = \frac{t}{S}.$$

Because $t/S \to 0$ as $S \to \infty$, the heuristic predicts that every climb eventually lands; yet because $\sum_{S \ge t} t/S = \infty$, the expected landing root is infinite. The map T therefore sits directly on the harmonic *infinite-mean borderline* between cer-

tain landing and positive-probability escape.

In this paper, we establish the exact structural theory of T, prove that no orbit can evade squares by locking onto a periodic block-doubling pattern (in contrast to the neighbouring map $n \mapsto n + 2d$ in OEIS A094766 [6], which admits eternal quadratic families), and record exact standalone and multi-program computational evidence through $n < 100{,}001^2 = 10{,}000{,}200{,}001$. We state the main results and the governing conjecture upfront.

Proposition 1 (Monotonicity, Unique Landing Door, Clock Form, and Brocard Shift). *For every integer $n \geq 0$, write $n = s^2 + d$ with $s = \lfloor \sqrt{n} \rfloor$ and $0 \leq d \leq 2s$.*

1. (Strict growth.) *$T(n) > n$ for all $n \geq 0$; in particular, T has no periodic orbits.*
2. (Unique landing door.) *$T(n)$ is a perfect square if and only if $d = s$, equivalently $n = s^2 + s = s(s+1)$, in which case $T(n) = (s+1)^2$. In particular, $0 \notin T(\mathbb{Z}_{\geq 0})$.*
3. (Clock representation.) *Under the bijection $n = s^2 + d \leftrightarrow (h, m) = (d+1, 2s+1)$ onto pairs of integers with $m \geq 1$ odd and $1 \leq h \leq m$, the step T acts as*

$$(h, m) \longmapsto \begin{cases} (2h,\ m) & \text{if } 2h \leq m, \\ (2h - m,\ m + 2) & \text{if } 2h > m. \end{cases}$$

   Perfect squares correspond to $h = 1$, and the door of block s corresponds to the midpoint $h = s + 1$.
4. (Brocard shift and parity contrast.) *Whenever $x + 1$ is not a perfect square, $T(x) + 1 = W(x + 1)$, where $W(n) = n + d$. Under W and under the $+2$ map $n \mapsto n + d + 2$, no non-square maps to a perfect square. Consequently, every T-orbit from $n \geq 0$ reaches a perfect square if and only if every W-orbit from a non-square $m \geq 2$ passes through an integer of the form $k^2 + 1$.*

Proposition 2 (Injectivity, Chain Decomposition, and Block Census). *Define block $B_s := \{s^2, s^2 + 1, \ldots, (s+1)^2 - 1\}$ for each $s \geq 0$.*

1. (Injectivity and missing images.) *The map $T : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is injective, and its set of non-images is precisely $\{r^2 - 1 : r \geq 1\} = \{0, 3, 8, 15, 24, \ldots\}$. (Under the shift $x \mapsto x + 1$, this is equivalent to Wenders's 2012 proof of Mathar's conjecture in OEIS A094761 [4], predated by Mathieu's 1905 statement without proof.) Hence $\mathbb{Z}_{\geq 0}$ partitions into pairwise disjoint forward chains rooted at the sources $r^2 - 1$ ($r \geq 1$).*
2. (Block census.) *Every chain rooted at $r^2 - 1$ with $r \leq s + 1$ traverses B_s exactly once, entering at a unique even square excess $d \in \{0, 2, \ldots, 2s\}$ (odd clock hand $h \in \{1, 3, \ldots, 2s+1\}$) and exiting into B_{s+1} without returning. Thus exactly $s + 1$ chains traverse B_s, and exactly one of them—the unique chain entering at $h = \mathrm{odd}(s+1)$, the odd part of $s+1$—exits through the door $s(s+1)$ onto $(s+1)^2$.*
3. (Induced square map.) *Every $n \geq 0$ eventually reaches a perfect square under T if and only if every chain contains infinitely many perfect squares. Defining $N(t)$ as the root of the least square $(N(t))^2 > t^2$ reached from t^2 (whenever such a square exists), the partial map $t \mapsto N(t)$ is injective on its domain.*

Proposition 3 (One-Block Landing Criterion and Explicit Infinite Families). *Let $s \geq 1$ and $n = s^2 + d \in B_s$.*

1. (One-block law.) *The orbit of n reaches $(s+1)^2$ before entering B_{s+2} if and only if $(d+1)2^j = s + 1$ for some integer $j \geq 0$. In particular, for $t \geq 1$, $N(t) = t + 1$ holds if and only if $t + 1 = 2^k$ for some integer $k \geq 1$.*
2. (Near-power-of-2 families.) *For every integer $k \geq 2$, the square start $(2^k - 2)^2$ satisfies $N(2^k - 2) = 2^k + 1$, and the source $(2^k - k)^2 - 1$ reaches $(2^k - 1)^2$ in exactly k steps of T.*

In Collatz-type problems, cycles are ruled out separately; here $T(n) > n$ rules out cycles in one line, so the natural algebraic obstruction to landing is an *eternal periodic branch family*—an orbit whose clock hand follows a fixed affine formula $h_n = \alpha_r m_n + \beta_r$ along a periodic doubling pattern, analogous to R. Israel's eternal quadratic trajectories for $n \mapsto n + 2d$ (OEIS A094766 [6]). Our main structural theorem rules out every such pattern unconditionally for all period lengths.

Theorem 4 (Non-Existence of Eventually Periodic Branch Patterns). *Let $(h_n, m_n)_{n \geq 0}$ be the sequence of block-entry states along any forward chain of T, so $m_{n+1} = m_n + 2$ (m_n odd), $1 \leq h_n \leq m_n$ is odd, and $h_{n+1} = 2^{e_n} h_n - m_n$, where $e_n \geq 1$ is the unique integer with $2^{e_n - 1} h_n \leq m_n < 2^{e_n} h_n$. Then the branch-exponent sequence $(e_n)_{n \geq 0}$ is never eventually periodic. In particular, no orbit of T can eventually satisfy an affine relation $h_n = \alpha_r m_n + \beta_r$ ($r \equiv n \pmod P$) along any periodic branch pattern of length $P \geq 1$.*

On the computational side, we separate what is directly replayed by our standalone Python script (`recompute.py`) from what rests on committed multi-program C censuses.

Proposition 5 (Direct Walks, Small-Range Invariants, and Branch-Pattern Enumeration). *Standalone exact Python arithmetic via `recompute.py` certifies three finite checks:*

1. *Every integer n with $1 \leq n \leq 3{,}000$ reaches a square in 2,386,142 combined steps, with slowest start $n = 2{,}602 = 51^2 + 1$ taking 78,220 steps to reach $39{,}252^2 = 1{,}540{,}719{,}504$ (and 78,221 steps from 51^2).*
2. *Across $0 \leq n \leq 100{,}000$, strict growth, the unique door $d = s$, clock equivalence, the Brocard shift $T(x) + 1 = W(x + 1)$ (off shifted squares), zero non-square landings for W and $n + d + 2$, and injectivity (100,001 distinct images; all 31 missing images below 1,000 equal to $r^2 - 1$) hold with 0 exceptions; the block census holds across blocks $0 \leq s \leq 999$; and exact block-jump replay proves that all 2,001 chains born at blocks $0 \leq s \leq 1{,}000$ land (every $n < 1{,}001^2 = 1{,}002{,}001$), matching direct walks for $1 \leq t \leq 300$ with 0 disagreements.*
3. *Among all 1,048,575 periodic branch patterns with total doubling exponent $1 \leq E \leq 20$, exactly 20 (the all-ones patterns $E = P$) admit integer affine solutions, all at $h = m + 2 > m$ outside the clock domain $[1, m]$.*

Proposition 6 (Exhaustive Census Below $100{,}001^2$ and Empirical $1/x$ Survival Tail). *Independent multi-program exact censuses (two independent C codes sharing no code, reproduced record-for-record by a third block-jumping program in blind re-review, plus independent verification below $20{,}001^2 = 400{,}040{,}001$; committed receipts in `survival.out` and repository audit logs) establish:*

1. *Every integer n with $0 \leq n < 100{,}001^2 = 10{,}000{,}200{,}001$ reaches a perfect square under T (all 200,001 chains born at blocks $0 \leq s \leq 10^5$—100,000 from squares t^2 and 100,001*

from sources $r^2 - 1$—followed to landing with no step cap).

2. *The slowest square climb with $1 \le t \le 10^5$ starts at $80{,}867^2$ and lands on $43{,}471{,}242{,}310^2$ in $86{,}942{,}332{,}668$ applications of T (so $80{,}867^2 + 1$ takes $86{,}942{,}332{,}667$ steps; ratio $N(80{,}867)/80{,}867 \approx 537{,}564.7$).*
3. *Across the $100{,}000$ square starts t^2 $(1 \le t \le 10^5)$, the counts satisfying $N(t) > xt$ are $50{,}204$ $(x = 2)$, $9{,}940$ $(x = 10)$, $1{,}004$ $(x = 100)$, 110 $(x = 1{,}000)$, and 14 $(x = 10{,}000)$, with mean $\ln(N(t)/t) = 1.00020$ against the $1/x$ model's predicted mean 1.*

Conjecture 7 (The Climb-to-a-Square Conjecture and Statistical Companions). *The following statements remain unsettled:*

1. (Main conjecture.) *For every integer $n \ge 0$, there exists an integer $k \ge 0$ such that $T^k(n)$ is a perfect square. (Equivalently, every orbit of Brocard's map W from a non-square $m \ge 2$ passes through $k^2 + 1$ for some $k \ge 1$.) Even the weaker assertion that a positive asymptotic density of integers $n \ge 0$ reach a square remains unsettled.*
2. (Statistical companions.) *For every real $x \ge 1$, the proportion of $t \in \{1, \dots, X\}$ with $N(t) > xt$ tends to $1/x$ as $X \to \infty$; and the proportion of squares s^2 $(1 \le s \le X)$ not lying in the image of N tends to $1/2$ as $X \to \infty$.*

Figure 1 summarizes the relationship between the structural theorems, standalone and multi-program computational checks, and the unsettled conjecture boundary.

Concept in Action: Climb from 7 and Slow Start 2,602

```
from math import isqrt
def step(n): return 2 * n + 1 - isqrt(n) ** 2
path7 = [7]
while isqrt(path7[-1]) ** 2 != path7[-1]:
    path7.append(step(path7[-1]))
x, k = 2602, 0
while isqrt(x) ** 2 != x:
    x, k = step(x), k + 1
assert path7 == [7, 11, 14, 20, 25]
assert k == 78220 and isqrt(x) == 39252
```

Roadmap. Section 2 fixes block and clock notation and illustrates worked climbs alongside Brocard's walk (Figure 2). Section 3 gives complete proofs of Propositions 1–3 and Theorem 4. Section 4 presents the standalone checks in `recompute.py`, the multi-program exhaustive census below $100{,}001^2$, the harmonic-borderline heuristic, and the empirical $1/x$ survival tail and record climb lengths (Figure 3). Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 documents logged prior-work searches, unread caveats (`[to check]`), and open questions, and Section 7 provides data availability and AI disclosure.

2. DEFINITIONS, BLOCK GEOMETRY, AND WORKED EXAMPLES

For every integer $s \ge 0$, define the *s-th square block*

$$B_s := \{s^2 + d : 0 \le d \le 2s\} = \{s^2, s^2 + 1, \dots, (s+1)^2 - 1\},$$

which contains $|B_s| = 2s + 1$ consecutive integers. Each $n \in \mathbb{Z}_{\ge 0}$ belongs to a unique block B_s with $s = \lfloor \sqrt{n} \rfloor$ and has *square excess* (or *leftover*) $d = n - s^2 \in \{0, 1, \dots, 2s\}$. The map T of (1) adds $d + 1$ to n:

$$T(s^2 + d) = s^2 + 2d + 1.$$

We also compare T with Brocard's unshifted map $W(s^2 + d) = s^2 + 2d$ (OEIS `A094761` [4]) and the $+2$ variant $T_{+2}(s^2 + d) = s^2 + 2d + 2$.

A starting integer n that is already a perfect square $(d = 0)$ counts as landed in 0 steps under Conjecture 7(1). To study the jump from one square to the next, however, we apply T at least once: since $T(t^2) = t^2 + 1$, whenever the climb from $t^2 + 1$ $(t \ge 1)$ reaches a square, we write $N(t)$ for the root of that landing square, so $T^k(t^2) = N(t)^2$ with $k \ge 1$ minimal. Note carefully the one-step convention difference: the climb from the non-square $t^2 + 1$ takes $k - 1$ steps, whereas starting from the square t^2 takes k steps. For example, $t = 51$ has $T(51^2) = 2{,}602$, and $2{,}602$ lands on $39{,}252^2$ in $78{,}220$ steps, so the square start 51^2 lands on $39{,}252^2$ in $78{,}221$ steps.

Worked examples. Placing block B_s on row s and excess d on column d turns T into a staircase walk (Figure 2, Panel A):

- *Start* 2 (1 *step*): $2 = 1^2 + 1$ sits at the door $1 \cdot 2$ of B_1 and steps directly to $2 + 1 + 1 = 4 = 2^2$.
- *Start* 3 (2 *steps*): $3 = 1^2 + 2 \to 6 = 2^2 + 2$ (door $2 \cdot 3$ of B_2) $\to 9 = 3^2$.
- *Start* 5 (5 *steps*) *and start* 7 (4 *steps*): $5 = 2^2 + 1 \to 7 = 2^2 + 3 \to 11 = 3^2 + 2 \to 14 = 3^2 + 5 \to 20 = 4^2 + 4$ (door $4 \cdot 5$ of B_4) $\to 25 = 5^2$.
- *Start* 15 (27 *steps*): $15 = 3^2 + 6 \to 22 \to 29 \to 34 \to \dots \to 315 \to 342 = 18^2 + 18$ (door $18 \cdot 19$ of B_{18}) $\to 361 = 19^2$.

In clock coordinates $(h, m) = (d + 1, 2s + 1)$, the climb from 7 reads

$$(4,5) \xrightarrow{\text{wrap}} (3,7) \xrightarrow{\times 2} (6,7) \xrightarrow{\text{wrap}} (5,9) \xrightarrow{\text{wrap}} (1,11),$$

where $(1, 11)$ corresponds to $5^2 + 0 = 25$. By contrast, Brocard's walk W from 7 yields $7, 10, 11, 13, 17, 18, 20, 24, 32, 39, \dots$ (Figure 2, Panel B), remaining permanently off $d = 0$.

3. PROOFS OF THE STRUCTURAL RESULTS

This section gives complete, self-contained proofs of the four written results of the paper: Propositions 1–3 and Theorem 4.

Proof of Proposition 1. Write $n = s^2 + d$ with $s = \lfloor \sqrt{n} \rfloor \ge 0$ and $0 \le d \le 2s$.

Part (1). Since $d \ge 0$, $T(n) = n + d + 1 \ge n + 1 > n$. Strictly increasing orbits cannot revisit any state, so T has no cycles.

Part (2). Split into two cases according to whether $d < s$ or $d \ge s$:

- If $0 \le d < s$, then $1 \le 2d + 1 \le 2s - 1 < 2s + 1$, so $T(n) = s^2 + (2d+1) \in B_s$ has odd square excess $2d + 1 \ne 0$, hence is not a perfect square.
- If $s \le d \le 2s$, then writing $(s+1)^2 = s^2 + 2s + 1$, we obtain

$$T(n) = s^2 + 2d + 1 = (s + 1)^2 + 2(d - s),$$

where $0 \le 2(d - s) \le 2s < 2(s + 1)$. Hence $T(n) \in B_{s+1}$ with even excess $2(d - s)$, which vanishes if and only if $d = s$, i.e., $n = s^2 + s = s(s + 1)$ and $T(n) = (s + 1)^2$. Since $T(n) \ge 1$ for all $n \ge 0$, $0 \notin T(\mathbb{Z}_{\ge 0})$.

Part (3). Under $h = d + 1$ and $m = 2s + 1$, the inequalities $d < s$ and $2h \le m$ are equivalent (since $2(d + 1) \le 2s +$

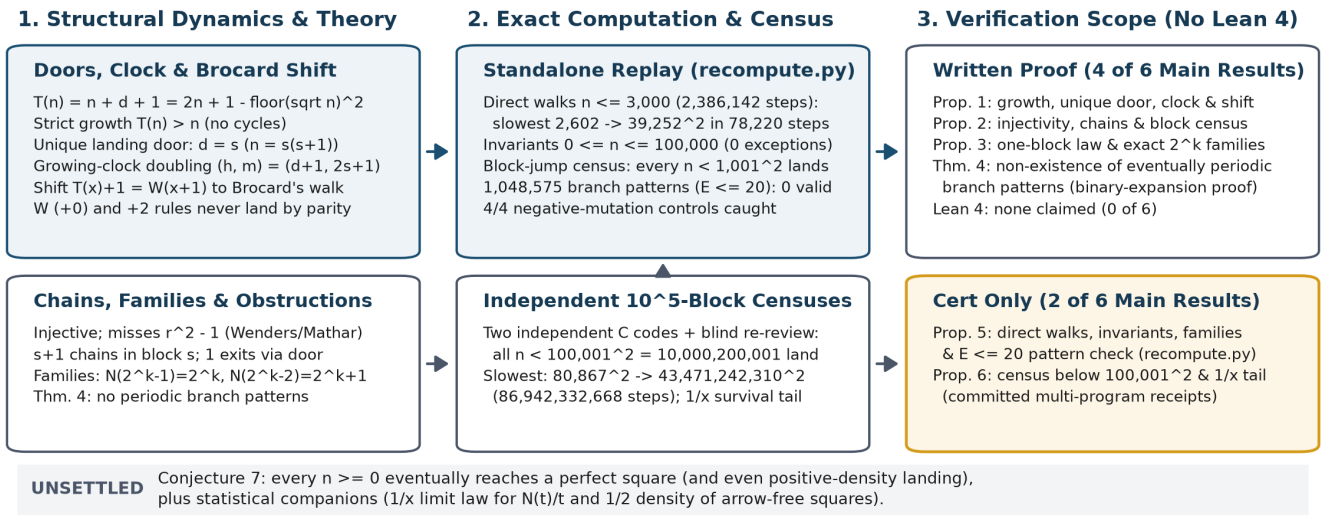


Figure 1. Discovery and verification architecture for Propositions 1–6, Theorem 4, and Conjecture 7. No Lean 4 formalization is claimed (0 of 6 proved main results in Lean 4; 4 of 6 by written proof; 2 of 6 by exact computational certificate and committed multi-program census receipts). Conjecture 7 (universal landing, positive-density landing, and the $1/x$ and $1/2$ statistical companions) remains unsettled.

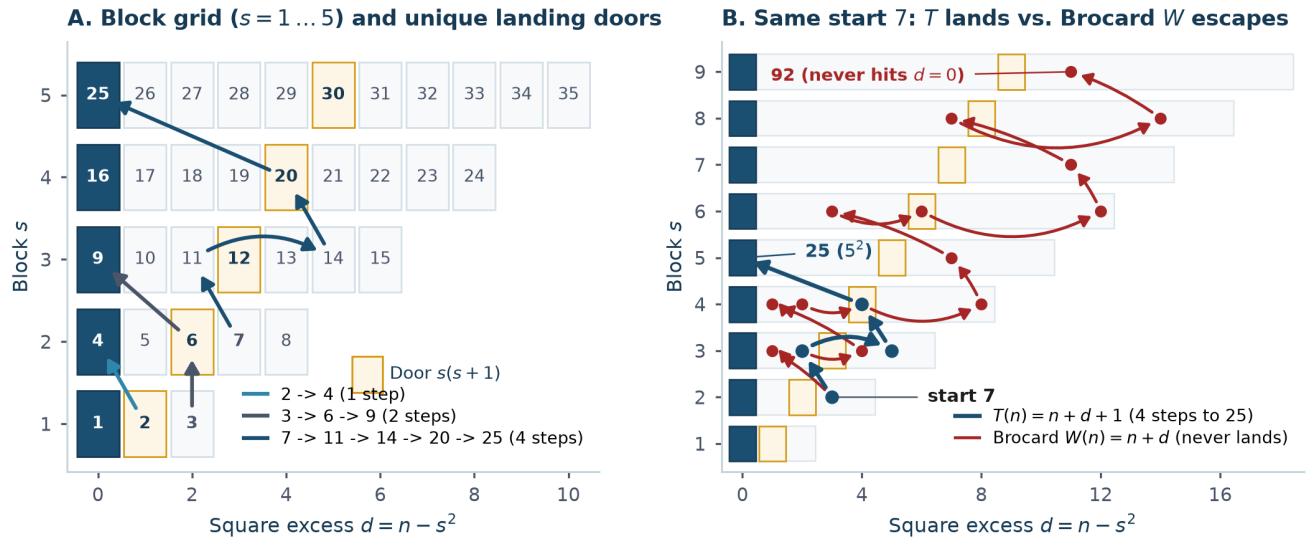


Figure 2. Block geometry and contrast with Brocard's walk. Panel A: square blocks $B_1, \ldots, B_5$ arranged by excess $d = n - s^2$, highlighting perfect squares at $d = 0$ (navy), unique landing doors $s(s+1) \in \{2, 6, 12, 20, 30\}$ (gold), and the climbs $2 \to 4$ (1 step), $3 \to 6 \to 9$ (2 steps), and $7 \to 11 \to 14 \to 20 \to 25$ (4 steps). Panel B: direct comparison from $n = 7$ across blocks $1 \le s \le 9$: T lands on $25 = 5^2$ in 4 steps, whereas Brocard's map $W(n) = n + d$ produces $7, 10, 11, 13, 17, 18, 20, 24, 32, 39, 42, 48, 60, 71, 78, 92, \ldots$ (OEIS A094764 [5]) and never reaches the left edge $d = 0$.

$1 \Leftrightarrow 2d + 1 \le 2s \Leftrightarrow d < s$ for integers). When $2h \le m$, $T(n) = s^2 + (2d+1) \in B_s$ has resulting excess $2d + 1 = 2h - 1$, giving hand $2h$ and unchanged modulus m. When $2h > m$, $T(n) = (s+1)^2 + 2(d-s) \in B_{s+1}$ has resulting excess $2d - 2s = 2h - 1 - m$, giving hand $2h - m$ and modulus $2(s+1) + 1 = m + 2$. Squares have $d = 0 \Leftrightarrow h = 1$, and the door $d = s$ has $h = s + 1 = (m+1)/2$.

Part (4). Let $x = s^2 + d \ge 0$ be such that $x+1$ is not a perfect square. Then $d < 2s$ (since $d = 2s$ gives $x + 1 = (s+1)^2$), so $x + 1 = s^2 + (d+1)$ lies in the same block B_s with excess $d + 1$. Thus

$$W(x+1) = (x+1) + (d+1) = (x + d + 1) + 1 = T(x) + 1.$$

Under W, if $n = s^2 + d$ is non-square ($1 \le d \le 2s$), then for $d \le s$ we have $W(n) = s^2 + 2d \in B_s$ with even excess $2d \ge 2$, while for $d > s$ we have $W(n) = (s+1)^2 + (2d - 2s - 1) \in B_{s+1}$ with odd excess $2d - 2s - 1 \ge 1$. Under $T_{+2}(n) = n + d + 2$, if $d < s$ then $2d + 2 \le 2s$, so $T_{+2}(n) = s^2 + (2d+2) \in B_s$ with positive even excess $2d + 2 \ge 2$; if $d \ge s$ then $T_{+2}(n) = (s+1)^2 + (2d - 2s + 1) \in B_{s+1}$ with odd excess $2d - 2s + 1 \ge 1$. In both maps, no non-square ever lands on

a square. Finally, note by Part (2) that $T(y)$ can never equal r^2-1 for any $r \geq 1$ (inside B_s the image has odd excess at most $2s-1 < 2s$, and entering B_{s+1} it has even excess at most $2s < 2(s+1)$). Hence, starting from any non-square $m \geq 2$ and setting $x_0 = m-1 \geq 1$, the integer $x_j + 1$ is never a perfect square for any $j \geq 0$ (at $j = 0$, $x_0 + 1 = m$ is non-square by hypothesis; for $j \geq 1$, $x_j \in T(\mathbb{Z}_{\geq 0})$ cannot equal r^2-1). Induction using $T(x)+1 = W(x+1)$ therefore yields $W^j(m) = T^j(m-1)+1$ for all $j \geq 0$, so $W^j(m) = k^2+1$ if and only if $T^j(m-1) = k^2$. For the remaining starts $x_0 = r^2-1$ ($r \geq 1$), $T(0) = 1^2$ already lands, while for $r \geq 2$ we have $r^2 < T(r^2-1)+1 = r^2+2r-1 < (r+1)^2$, reducing landing from r^2-1 under T to hitting k^2+1 from the non-square $T(r^2-1)+1$ under W. □

Proof of Proposition 2. Part (1). By the two-case formula in the proof of Proposition 1(2), for every $s \geq 0$:

- As d runs over $\{0, 1, \dots, s-1\}$, $T(s^2+d) = s^2 + (2d+1)$ bijects onto the odd-excess elements $\{s^2+1, s^2+3, \dots, s^2+2s-1\}$ of B_s.
- As d runs over $\{s, s+1, \dots, 2s\}$, $T(s^2+d) = (s+1)^2 + 2(d-s)$ bijects onto the even-excess elements $\{(s+1)^2+0, (s+1)^2+2, \dots, (s+1)^2+2s\}$ of B_{s+1}.

Therefore, inside each block B_{s+1} ($s \geq 0$), every odd excess $1, 3, \dots, 2s+1$ has a unique preimage in B_{s+1}, every even excess $0, 2, \dots, 2s$ has a unique preimage in B_s, and the single top element $(s+1)^2+2(s+1) = (s+2)^2-1$ has no preimage. In block $B_0 = \{0\}$, 0 has no preimage (since $T(n) \geq 1$). Thus T is injective on $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{\geq 0} \setminus T(\mathbb{Z}_{\geq 0}) = \{r^2-1 : r \geq 1\}$. Because T is injective and strictly increasing, tracing preimages backwards from any $n \geq 0$ terminates at a unique non-image r^2-1, partitioning $\mathbb{Z}_{\geq 0}$ into disjoint forward chains.

Part (2). From the case split above, every transition either stays inside B_s (mapping to an odd excess) or advances from B_s to an even excess of B_{s+1}. Hence a chain never skips a block and never revisits an earlier block. Every chain born at $r^2-1 \in B_{r-1}$ starts at excess $2(r-1)$ (which is even), and every transition into B_s from B_{s-1} lands at an even excess in $\{0, 2, \dots, 2s-2\}$, while no inside-block step produces an even excess. Consequently, every chain born at or before B_s enters B_s at a unique even excess $d \in \{0, 2, \dots, 2s\}$ (corresponding to an odd clock hand $h = d+1 \in \{1, 3, \dots, 2s+1\}$). Because there are $s+1$ even excesses in B_s, exactly $s+1$ chains traverse B_s. Inside B_s, the clock hand strictly doubles ($h \mapsto 2h$) until it exceeds $(2s+1)/2$; it hits the door $h = s+1$ if and only if $h \cdot 2^j = s+1$ for some $j \geq 0$, which for odd $h \in \{1, 3, \dots, 2s+1\}$ holds if and only if $h = \mathrm{odd}(s+1)$, the unique odd part of $s+1$.

Part (3). If every $n \geq 0$ reaches a square, then for any chain C and any $x \in C$, the iterate $T(x)$ reaches a square strictly larger than x, so C contains infinitely many squares; the converse is immediate since every n lies on some chain. Finally, if $N(t_1) = N(t_2) = u$, then both t_1^2 and t_2^2 reach u^2 without hitting an intermediate square; by injectivity of T, walking backwards from u^2 to the immediately preceding square yields a unique value, so $t_1 = t_2$. □

Proof of Proposition 3. Part (1). In clock coordinates, $n = s^2 + d \in B_s$ has hand $h = d+1$ on modulus $m = 2s+1$. While staying inside B_s, T doubles the hand ($h \mapsto 2h \mapsto 4h \mapsto \dots$). It exits B_s onto $(s+1)^2$ if and only if one of these doublings lands on the door $s+1$, i.e., $(d+1)2^j = s+1$ for some $j \geq 0$. For a square start t^2 ($d = 0$, $h = 1$), this condition becomes $2^j = t+1$, i.e., $t+1$ is a power of 2.

Part (2). Fix $k \geq 2$. First consider the square start t^2 with $t = 2^k-2$, i.e., block $s_0 = 2^k-2$, modulus $m_0 = 2^{k+1}-3$, and entry hand $h_0 = 1$. Since $k \geq 2$ implies $2^k \leq 2^{k+1}-3 < 2^{k+1}$, the hand doubles k times inside B_{s_0} and wraps to

$$h_1 = 2^{k+1} - (2^{k+1}-3) = 3$$

in block $s_1 = 2^k-1$ ($m_1 = 2^{k+1}-1$). Because $3 \cdot 2^{k-1} \leq 2^{k+1}-1 < 3 \cdot 2^k$ for all $k \geq 1$, the hand $h_1 = 3$ doubles $k-1$ times inside B_{s_1} and wraps to

$$h_2 = 3 \cdot 2^k - (2^{k+1}-1) = 2^k + 1$$

in block $s_2 = 2^k$ ($m_2 = 2^{k+1}+1$). Now $h_2 = 2^k+1 = s_2+1$ is the door of B_{s_2}, so one wrap carries it onto $(2^k+1)^2$. Since $h_1 = 3 \neq 1$ and $h_2 = 2^k+1 \geq 5 \neq 1$, no earlier square is visited, establishing $N(2^k-2) = 2^k+1$.

Second, consider the source $(2^k-k)^2-1$, which is the top element ($h_0 = m_0$) of block $s_0 = 2^k-k-1 \geq 1$. For $0 \leq j \leq k-1$, set $s_j = s_0 + j = 2^k-k-1+j$, $m_j = 2s_j+1 = 2^{k+1}-2k-1+2j$, and define the top-hand deficit $e_j := m_j - h_j$. We verify by induction on $j \in \{0, \dots, k-1\}$ that $e_j = 2^{j+1}-2$ and $2h_j > m_j$ (so each block exits in a single wrap step). At $j = 0$, $e_0 = 0 = 2^1-2$ and $2h_0 = 2m_0 > m_0$. Assuming the formula at $j \leq k-2$, note that $2^{j+2} \leq 2^k$, so

$$m_j - 2e_j = 2^{k+1}-2k-1+2j-(2^{j+2}-4) \geq 2^k-2k+3+2j > 0$$

for all $k \geq 2$ and $j \geq 0$, which gives $2h_j = 2m_j - 2e_j > m_j$. Hence one application of T wraps immediately to block s_{j+1} ($m_{j+1} = m_j+2$) with

$$h_{j+1} = 2h_j - m_j = m_j - 2e_j = m_{j+1} - (2e_j+2),$$

giving $e_{j+1} = 2e_j+2 = 2^{j+2}-2$. At $j = k-1$, we have $s_{k-1} = 2^k-2$, $m_{k-1} = 2^{k+1}-3$, and $e_{k-1} = 2^k-2$, so $h_{k-1} = m_{k-1} - e_{k-1} = 2^k-1 = s_{k-1}+1$, the door of block s_{k-1}. One further wrap lands on $(2^k-1)^2$ at step k. For $0 \leq j \leq k-1$, $h_j > m_j/2 > 1$ (since $m_j \geq 3$), so $(2^k-1)^2$ is the earliest square reached. □

Proof of Theorem 4. Suppose for contradiction that the block-entry sequence $(h_n, m_n)_{n \geq 0}$ of some orbit of T has an eventually periodic branch-exponent sequence $(e_n)_{n \geq 0}$. Shifting the index n to the start of the periodic tail, we may assume that $(e_n)_{n \geq 0}$ is purely periodic with period $P \geq 1$, so $e_{n+P} = e_n \geq 1$ for all $n \geq 0$, $m_n = m_0 + 2n$, $1 \leq h_n \leq m_n$, and $h_{n+1} = 2^{e_n} h_n - m_n$.

Let $E := \sum_{i=0}^{P-1} e_i \geq P \geq 1$, and for each $0 \leq i \leq P-1$ define the tail sum $F_i := \sum_{j=i+1}^{P-1} e_j$ (with $F_{P-1} = 0$). Iterating $h_{kP+i+1} = 2^{e_i} h_{kP+i} - (m_{kP}+2i)$ across $i = 0, 1, \dots, P-1$ gives the exact one-period affine transition

$$h_{(k+1)P} = 2^E h_{kP} - A m_{kP} - B, \tag{2}$$

where

$$A := \sum_{i=0}^{P-1} 2^{F_i}, \qquad B := \sum_{i=0}^{P-1} 2i \cdot 2^{F_i}.$$

Set $D := 2^E - 1 \geq 1$ (which is odd) and define the unique rational numbers

$$\alpha_0 := \frac{A}{D}, \qquad \beta_0 := \frac{B + 2P\alpha_0}{D}.$$

By construction, (α_0, β_0) satisfies $\alpha_0 = 2^E\alpha_0 - A$ and $2P\alpha_0 + \beta_0 = 2^E\beta_0 - B$. Writing $u_k := h_{kP} - (\alpha_0 m_{kP} + \beta_0)$, substituting into (2) gives

$$u_{k+1} = 2^E u_k \qquad \text{for all } k \geq 0,$$

so $u_k = 2^{kE}u_0$. However, since $1 \leq h_{kP} \leq m_{kP} = m_0 + 2kP$, we have $|u_k| = O(k)$ as $k \to \infty$. Because $2^E \geq 2$, exponential growth $2^{kE}|u_0| = O(k)$ forces $u_0 = 0$. Therefore

$$h_{kP} = \alpha_0 m_{kP} + \beta_0 \qquad \text{for all } k \geq 0. \tag{3}$$

Applying the same argument to the shifted periodic sequence starting at index $r \in \{0, 1, \ldots, P-1\}$ yields rational constants (α_r, β_r) such that $h_{kP+r} = \alpha_r m_{kP+r} + \beta_r$ for all $k \geq 0$. Comparing coefficients of m_{kP+r} in $h_{kP+r+1} = 2^{e_r}h_{kP+r} - m_{kP+r}$ yields $\alpha_{r+1} = 2^{e_r}\alpha_r - 1$ for every $r \geq 0$.

Now inspect the binary expansion of $\alpha_0 = A/(2^E - 1)$. Since $e_i \geq 1$, the exponents $F_0 > F_1 > \cdots > F_{P-1} = 0$ are P strictly decreasing integers in $\{0, 1, \ldots, E-1\}$. Hence $1 \leq A \leq \sum_{j=0}^{E-1} 2^j = 2^E - 1$, so $0 < \alpha_0 \leq 1$.

Case 1: $\alpha_0 = 1$. Then $A = 2^E - 1$, which forces $P = E$ and $e_i = 1$ for all i. Then $F_i = E - 1 - i$ and

$$B = \sum_{i=0}^{E-1} 2i \cdot 2^{E-1-i} = 2(2^E - 1 - E),$$

so $\beta_0 = (B + 2E)/(2^E - 1) = 2$. Equation (3) then gives $h_0 = m_0 + 2 > m_0$, contradicting $h_0 \leq m_0$.

Case 2: $0 < \alpha_0 < 1$. Write $\alpha_0 = c/q$ in lowest terms with $1 \leq c < q$. Because $q \mid D = 2^E - 1$, q is odd and $q \geq 3$. The binary expansion of $\alpha_0 = A/(2^E - 1)$ is purely periodic of E-bit period having 1-bits precisely at positions $e_0, e_0 + e_1, \ldots, E$ (with $e_0 = E - F_0$), and at least one 0-bit because $\alpha_0 < 1$. Hence every α_r lies strictly in $(0, 1)$, and e_r is uniquely determined by α_r as the unique positive integer satisfying $2^{e_r - 1}\alpha_r < 1 < 2^{e_r}\alpha_r$ (the gap to the next 1-bit of α_r).

Let $d := \mathrm{ord}_q(2) \geq 2$ be the multiplicative order of 2 modulo q. Then the binary expansion of c/q has minimal bit-period d, containing some number P_0 of 1-bits per period. Because $0 < c/q < 1$, each minimal period contains at least one 1-bit and at least one 0-bit, giving

$$1 \leq P_0 \leq d - 1 \leq \varphi(q) - 1 \leq q - 2. \tag{4}$$

After P_0 applications of the map $\alpha \mapsto 2^e\alpha - 1$, the binary expansion of α_0 has shifted left by exactly d bits, returning to $\alpha_{P_0} = \alpha_0$. Because α_r uniquely determines e_r, the branch-exponent sequence $(e_n)_{n \geq 0}$ is already periodic with period P_0. Repeating the derivation of (3) with period P_0 (noting that the resulting rational slope is again α_0) yields $h_{P_0} - h_0 = \alpha_0(m_{P_0} - m_0) = 2P_0\alpha_0 = 2P_0 c/q$. Since $h_{P_0}, h_0 \in \mathbb{Z}$, we must have $q \mid 2P_0 c$. As $\gcd(c, q) = 1$ and q is odd ($\gcd(2, q) = 1$), this forces $q \mid P_0$, contradicting (4).

In both cases we reach a contradiction, so $(e_n)_{n \geq 0}$ cannot be eventually periodic. The final sentence follows immediately because any eventually affine orbit along a periodic branch pattern produces an eventually periodic branch-exponent sequence. □

■ 4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Standalone Verification (`recompute.py`). All finite claims in Proposition 5 are verified from scratch using standard-library Python integer arithmetic in `recompute.py` (output saved verbatim in `recompute.out`; runtime roughly 10 seconds):

- *Direct walks* $1 \leq n \leq 3{,}000$. Every start lands, totaling 2,386,142 steps. The 11 record starts in $2 \leq n \leq 3{,}000$ are $(2,1)$, $(3,2)$, $(5,5)$, $(8,6)$, $(15,27)$, $(26,30)$, $(63,78)$, $(99,492)$, $(325,2{,}659)$, $(1{,}445,22{,}739)$, and $(2{,}602,78{,}220)$ (landing on $39{,}252^2 = 1{,}540{,}719{,}504$).
- *Structural invariants* $(0 \leq n \leq 100{,}000)$ *and block census* $(0 \leq s \leq 999)$. Zero exceptions are found for $T(n) > n$, the door condition $T(n)$ square $\Leftrightarrow d = s$, clock-step equivalence, the Brocard shift $T(x) + 1 = W(x + 1)$ off shifted squares, zero non-square landings for W and T_{+2}, and injectivity (100,001 distinct images; all 31 non-images below 1,000 are $r^2 - 1$, $1 \leq r \leq 31$). On $0 \leq n \leq 10^6$, all 1,000,001 integers partition into disjoint chains from $r^2 - 1$ with $s + 1$ chains per block $(0 \leq s \leq 999)$. The square roots visited by the chain from 0 up to 174,225 are $0, 1, 2, 5, 20, 24, 70, 355, 563, 612, 673, 4{,}317, 4{,}745, 49{,}196, 174{,}225$.
- *Exact families and block-jump census below* $1{,}001^2$. For $1 \leq t \leq 2{,}000$, $N(t) = t + 1$ holds precisely at $t \in \{1, 3, 7, 15, 31, 63, 127, 255, 511, 1{,}023\}$; Proposition 3(2) is verified for $2 \leq k \leq 16$. Using exact bit-shift block-jumping (e chosen so $2^{e-1}h \leq m < 2^e h$, $h' = 2^e h - m$), `recompute.py` matches direct step-by-step walks across all $1 \leq t \leq 300$ with 0 mismatches and follows all 2,001 chains born at blocks $0 \leq s \leq 1{,}000$ to landing (certifying every $n < 1{,}001^2 = 1{,}002{,}001$, with largest square landing root 148,574 at $t = 934$ and largest source landing root 30,915,712).
- *Exhaustive branch-pattern check* $(1 \leq E \leq 20)$ *and negative controls.* Enumerating all $\sum_{E=1}^{20} 2^{E-1} = 1{,}048{,}575$ compositions $(e_0, \ldots, e_{P-1})$ confirms that only the 20 all-ones patterns $(E = P)$ satisfy the integrality congruence $(gD) \mid (BD + 2PA)$ (all at $h = m + 2 > m$), while 11 repeated non-trivial patterns with $q \mid P$ (such as $E = 6, P = 3, q = 3$) fail integrality as guaranteed by Theorem 4. Four negative-mutation controls (replacing $+1$ by $+0$ or $+2$, shifting the door to $d = s - 1$, and perturbing $N(2) = 5$ to 4) are all caught (4/4).

Multi-Program Exhaustive Census Below $100{,}001^2$. Because every chain traverses every block from its birth onward (Proposition 2(2)), verifying that every integer $n < (X + 1)^2$ reaches a perfect square is equivalent to following all $2X + 1$ chains born at blocks $0 \leq s \leq X$ (X square starts t^2 for $1 \leq t \leq X$ and $X + 1$ source starts $r^2 - 1$ for $1 \leq r \leq X + 1$) until each hits a square. Two independent C implementations sharing no code carried out this exact population-and-follower census at $X = 10^5$ (every $n < 100{,}001^2 = 10{,}000{,}200{,}001$, with zero step caps and every long survivor resumed to exact landing), and a third independent block-jumping C implementation in a shielded blind re-review reproduced all 200,001 landings record for

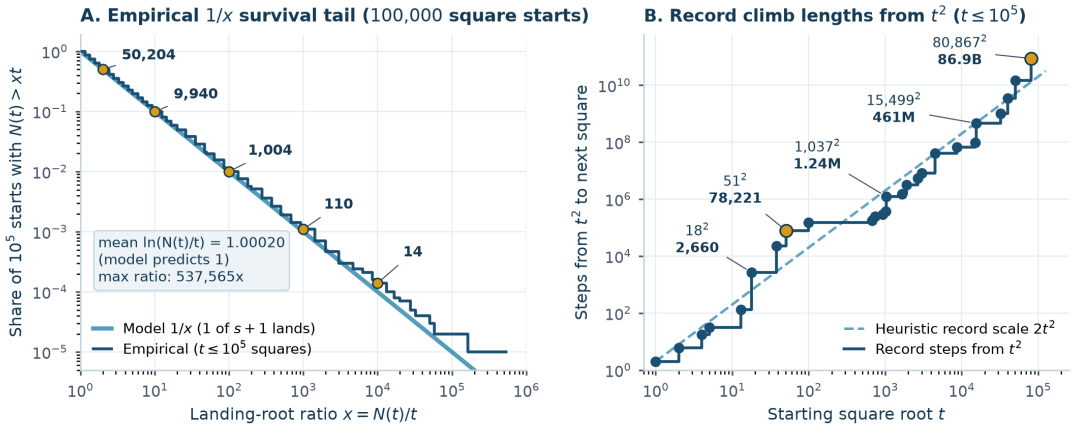


Figure 3. Heavy-tailed survival statistics and record climb lengths across 100,000 square starts ($1 \le t \le 10^5$). Panel A: empirical share of square starts with $N(t) > xt$ (navy step curve) against the infinite-mean harmonic borderline $1/x$ (teal), with checkpoints at $x = 2, 10, 100, 1{,}000, 10{,}000$ (counts 50,204, 9,940, 1,004, 110, 14) and mean $\ln(N(t)/t) = 1.00020$. Panel B: all 26 record step counts from squares t^2 through $t = 80{,}867$ (86,942,332,668 steps to $43{,}471{,}242{,}310^2$) alongside the heuristic $2t^2$ record-growth envelope.

record; a fourth independent deck-simulation program verified every $n < 20{,}001^2 = 400{,}040{,}001$ (slowest source $7{,}106^2 - 1 \to 467{,}727{,}972^2$). Table 1 lists selected milestone record square climbs across $1 \le t \le 10^5$.

Table 1. Selected record climbs from squares t^2 ($1 \le t \le 10^5$). Rows with $t \le 1{,}000$ are replayed directly by `recompute.py`; rows with $t > 1{,}000$ come from the committed multi-program 10^5-block census (`land_steps.out`).

Start t^2	Landing $N(t)^2$	Steps from t^2	Ratio $N(t)/t$
1^2	2^2	2	2.0
18^2	$1{,}362^2$	2,660	75.7
51^2	$39{,}252^2$	78,221	769.6
100^2	$75{,}065^2$	149,223	750.7
934^2	$148{,}574^2$	295,641	159.1
$1{,}037^2$	$621{,}253^2$	1,241,213	599.1
$15{,}499^2$	$230{,}394{,}489^2$	460,726,820	14,865.1
$50{,}560^2$	$7{,}222{,}040{,}837^2$	14,443,872,347	142,841.0
$80{,}867^2$	$43{,}471{,}242{,}310^2$	86,942,332,668	537,564.7

Additional targeted checks past 10^5 followed 988 structured starts up to $t = 2^{26} + 5$ (powers of 2 and neighbours, factorials, primorials, Fibonacci numbers, repunits, and near-miss hands), all of which landed (the longest landing on $2{,}911{,}045{,}757^2$, ratio 694). In the blind re-review, the block-jumping program extended independent verification to every start below $150{,}001^2 = 22{,}500{,}300{,}001$ (and to $200{,}001^2 = 40{,}000{,}400{,}001$ when combined with an earlier re-review unit's records); we keep our headline Proposition 6 anchored to the fully triply reproduced bound $n < 100{,}001^2$.

The Harmonic-Borderline Heuristic and $1/x$ Survival Tail. By Proposition 2(2), each block B_s hosts $s + 1$ active chains and lands exactly one of them. Modeling the choice of landing chain at each block as independent and uniform across the $s + 1$ active chains yields survival probability

$$\mathbb{P}(N(t) > S) \approx \prod_{s=t}^{S-1} \frac{s}{s+1} = \frac{t}{S},$$

so for any $x \ge 1$, $\mathbb{P}(N(t) > xt) \approx 1/x$. Under this Pareto-1 law, $\ln(N(t)/t)$ is unit-exponential with mean 1, while $\mathbb{E}[N(t)/t] = \int_1^\infty x^{-1}\,dx = \infty$. Moreover, because crossing block s takes on average ≈ 2 steps of T (since the expected doubling exponent of a uniform odd hand in $[1, 2s + 1]$ approaches 2), a climb landing on $N(t)^2$ takes roughly $2(N(t) - t)$ steps—matching Table 1 closely (for $t = 80{,}867$, $86{,}942{,}332{,}668/43{,}471{,}242{,}310 = 1.999996$). Under the $1/x$ tail, the largest ratio among $t \le X$ grows like X, so record step counts grow roughly like $2t^2$ (Figure 3, Panel B).

Table 2. Empirical survival tail across all 100,000 square starts t^2 ($1 \le t \le 10^5$, committed receipt `survival.out`) compared against the harmonic-borderline $1/x$ model.

Ratio x	Count $N(t) > xt$	Empirical share	Model $1/x$	$x \times$ share
2	50,204	0.502040	0.500000	1.0041
10	9,940	0.099400	0.100000	0.9940
100	1,004	0.010040	0.010000	1.0040
1,000	110	0.001100	0.001000	1.1000
10,000	14	0.000140	0.000100	1.4000

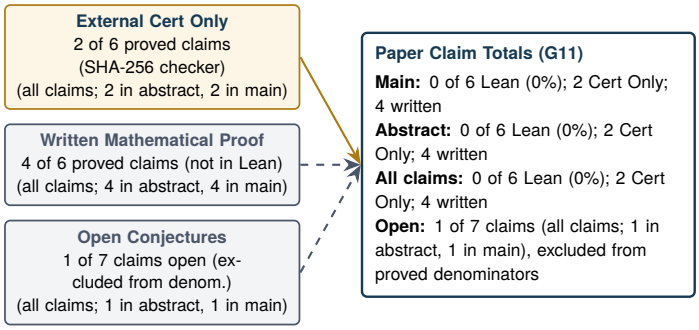
Table 2 and Figure 3 (Panel A) show close agreement across four decades of x: the empirical mean of $\ln(N(t)/t)$ is 1.00020 (against 1). Random-window tests at $t \in [10^6, 2 \cdot 10^6)$ (13,000 samples), $t \in [10^7, 2 \cdot 10^7)$ (1,900 samples), and 310 random sources $r^2 - 1$ in $[10^8, 2 \cdot 10^8)$ show no departure beyond two standard errors, and binning by $\mathrm{frac}(\log_2 t)$ yields $\chi^2 \in \{6.25, 8.55, 3.47\}$ on 7 degrees of freedom (no log-periodic signal). Likewise, 3,332,394,771 of the 9,999,900,000 non-squares below $(10^5)^2$ land on a square $\le 10^{10}$ (share 0.333243, matching the model's $\frac{1}{X^2}\sum_{t=1}^{X} 2t(1 - t/X) \to 1/3$), and 50,095 of the 10^5 squares s^2 ($1 \le s \le 10^5$) receive no arrow under N (with exact identity $2F(X) - X = 1 + A_{\mathrm{sq}}(X) - A_{\mathrm{src}}(X)$ linking arrow-free squares $F(X)$ to the balance of live square-born versus source-born chains). We emphasize that these heuristic derivations and empirical checks are *not* proofs: finite searches cannot exclude an escaping orbit, and the heavy $1/x$ tail guarantees arbitrarily slow climbs under Conjecture 7(1).

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 2 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 1: Strict growth T(n) > n, unique landing door n = s(s+1) -> (s+1)^2, clock form (h, m) = (d+1, 2s+1), and Brocard shift T(x)+1 = W(x+1) with parity obstruction for W and n+d+2	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof (Section 3); finite checks through n = 100,000 in recompute.py. No Lean 4 formalization.
Prop. 2: Injectivity of T with missing images {r^2 - 1 : r >= 1} (equivalent to Wenders 2012 / Mathar in OEIS A094761), s+1 chains per block with unique landing hand odd(s+1), and injectivity of N	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof (Section 3); finite checks through n = 100,000 and blocks s <= 999 in recompute.py. No Lean 4 formalization.
Prop. 3: One-block landing criterion (d+1)2^j = s+1 (so N(t) = t+1 if and only if t+1 = 2^k) and infinite families N(2^k - 2) = 2^k + 1 and (2^k - k)^2 - 1 -> (2^k - 1)^2 in k steps (k >= 2)	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof by induction on clock deficits (Section 3); verified for k <= 16 and t <= 2,000 in recompute.py. No Lean 4 formalization.
Thm. 4: No orbit of T has an eventually periodic block-doubling sequence (e_n); in particular, no orbit is eventually an affine function of m = 2s + 1 along a periodic branch pattern	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof via periodic binary expansions and minimal period P_0 <= q - 2 (Section 3); corroborated up to E <= 20 (1,048,575 patterns) in recompute.py. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 5: Exact standalone replay of direct walks n <= 3,000 (slowest 2,602 -> 39,252^2 in 78,220 steps), invariants n <= 100,000, block-jump census n < 1,001^2, and 1,048,575 branch patterns (E <= 20)	Cert Only	recompute.py (ec6a490611ee) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Certified by standalone exact Python script recompute.py (standard library only, ~10 s runtime, 4/4 negative-mutation controls).
Prop. 6: Exhaustive multi-program census showing every n < 100,001^2 = 10,000,200,001 lands (slowest 80,867^2 -> 43,471,242,310^2 in 86,942,332,668 steps) and 1/x survival tail on 100,000 square starts	Cert Only	recompute.py (ec6a490611ee) [archived; replay pending]; survival.py (e381b778b1e3) [regenerable; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Backed by committed multi-program audit receipts (two independent C codes + blind block-jumping C re-review + independent deck census below 20,001^2); raw 5.4 MB segment table is retired outside git (regenerable deterministically); recompute.py replays n < 1,001^2 directly.
Conj. 7: Every n >= 0 eventually reaches a perfect square under T (equivalently every Brocard W-orbit from non-square m >= 2 hits k^2 + 1), plus 1/x survival limit and 1/2 arrow-free square density	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled main conjecture and statistical companions (excluded from G11 denominators).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 1f489b13bde2 | Fidelity SHA-256: d7dc63db863f.

6. PRIOR WORK AND OPEN QUESTIONS

Nearest prior literature. The unshifted +0 map $W(n) = n + d$ was posed by H. Brocard in 1904 (Question 2837, *L'Intermédiaire des Mathématiciens* **11** (1904), p. 259, with a reply by E. Mathieu in **12** (1905), pp. 181–182; documented in OEIS `A094761`, `A094763`, `A094764` [4, 5] `[to check]`). In OEIS `A094761`, J. Wenders (2012) proved R. J. Mathar's (2009) conjecture characterizing injectivity and the missing images of W, which corresponds under the unit shift $x \mapsto x+1$ to Proposition 2(1). S. H. Weintraub [9] (`[to check]`) studied the multiplicative recursion $a_{k+1} = a_k + rS(a_k)$ for fixed integers $r \geq 2$ (where $S(n) = d$ is the square excess), asking which starts converge (reach a square) and conjecturing for $2 \leq r \leq 8$ that starts that are squares modulo r either converge or predictably diverge (per zbMATH review `Zbl 1062.11009`). For $r = 2$, OEIS `A094765` and `A094766` [6] record eternal quadratic trajectories (R. Israel), contrasting directly with Theorem 4.

Three further neighbours share either the landing shape or the doubling mechanism for distinct maps:

- B. Cloitre (2009, OEIS `A156344` [7]) conjectured that every integer reaches a perfect square under the rational-ceiling map $x \mapsto x\lceil\sqrt{x}\rceil/\lfloor\sqrt{x}\rfloor$—the nearest printed conjecture of the same shape, for a different rule (from 7, Cloitre's map takes 8 steps to 49, whereas T takes 4 steps to 25).
- OEIS `A159016` and `A159258` [8] study $a \mapsto a + \lfloor\sqrt{a}\rfloor$, which is known to contain infinitely many squares for every start.
- J.-P. Allouche, M. Stipulanti, and J.-Y. Yao [1] (`[to check]`) analyze doubling modulo a *fixed* odd integer $x \mapsto 2x \bmod (2n+1)$; the clock form of Proposition 1(3) adds the feature that the odd modulus increments by 2 at every wrap.

Logged prior-work searches. Three independent literature logs passed the automated search-gate check: 41 returning queries (33 OEIS, 7 arXiv, 1 zbMATH) plus 15 web searches and manual searches of Lagarias's two annotated $3x+1$ bibliographies [2, 3]; 27 returning queries (22 OEIS, 4 arXiv, 1 zbMATH), 5 web searches, and manual inspection of roughly 45 OEIS entries citing `A053186`; and 6 targeted queries on Weintraub [9] (3 OEIS, 2 arXiv, 1 zbMATH). Subsequent keyed searches on 2026-11-19 (7 CORE requests returning 20 records) and 2026-11-22 (6 Google Scholar queries) likewise returned zero records stating Conjecture 7 or the +1 map T. Thus Conjecture 7 was not found in the searches logged.

Unread caveats and still-owed sources (`[to check]`). The main bibliographic caveat concerns S. H. Weintraub [9] (`[to check]`): the printed journal pages 528–530 in *The American Mathematical Monthly* were paywalled and not obtained. Its zbMATH review (`Zbl 1062.11009`) was read verbatim and mentions only $n + rS(n)$ with $r \geq 2$; in addition, a blind re-review inspected an archived 4-page 2003 author preprint (which states the $r = 1$ parity fact as the 1991 Putnam problem and contains no +1 or shifted rule), but any remark added in the printed 3-page version remains unverified. Also flagged

`[to check]` in `references-checked.md` are: printed volumes of Brocard (1904) and Mathieu (1905) (consulted via OEIS and an archived scan in blind re-review), full text of Allouche, Stipulanti, and Yao [1], the subscription MathSciNet review `MR2076587` and citation lists for Weintraub [9], Guy's *Unsolved Problems in Number Theory*, and the `seqfan` mailing-list archive.

Open directions. Beyond Conjecture 7(1) itself, three concrete intermediate problems stand out: (i) prove that a positive lower density of integers $n \geq 0$ (or of squares t^2) land on a square under T; (ii) prove that infinitely many chains pass through infinitely many squares (or even that the single chain rooted at 0 contains infinitely many squares); and (iii) determine whether any deterministic equidistribution theorem for the growing-modulus doubling map $(h, m) \mapsto (2h \bmod m,\ m + 2)$ can yield nontrivial upper bounds on the landing-ratio tail $N(t)/t$.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, and the standalone exact verifier live in `papers/climb-to-a-square/draft1/`. Running `python3 papers/climb-to-a-square/draft1/recompute.py` completes in roughly 10 seconds using only the Python standard library and deterministically reproduces `recompute.out` (all hand climbs, direct walks $1 \leq n \leq 3{,}000$, structural invariants $0 \leq n \leq 100{,}000$, block census $0 \leq s \leq 999$, exact infinite families, independent block-jump census below $1{,}001^2$, exhaustive enumeration of all 1,048,575 periodic branch patterns for $E \leq 20$, 4/4 negative-mutation controls, and the Section 1 Python snippet). Committed summary receipts for the 10^5-block multi-program census live in the repository's `climb-to-a-square` documentation archive (`survival.out`, `land_steps.out`, and companion verification logs, SHA-256 pinned in Section 5); the raw 5.4 MB segment table from the 10^5-block C census is retired outside the git repository and is regenerable from the exact deterministic block-jump algorithm. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes proposed the +1 map and conjecture, proved the structural propositions and Theorem 4, wrote and independently rechecked the census and verification programs, conducted the logged prior-work searches, and drafted this manuscript. No human mathematician has yet refereed the proof.

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