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Working Draft

The Cube-Root-of-6 Map: Hidden $3n$ Structure, Cycle Bounds, and Verification through 3.621×10^9

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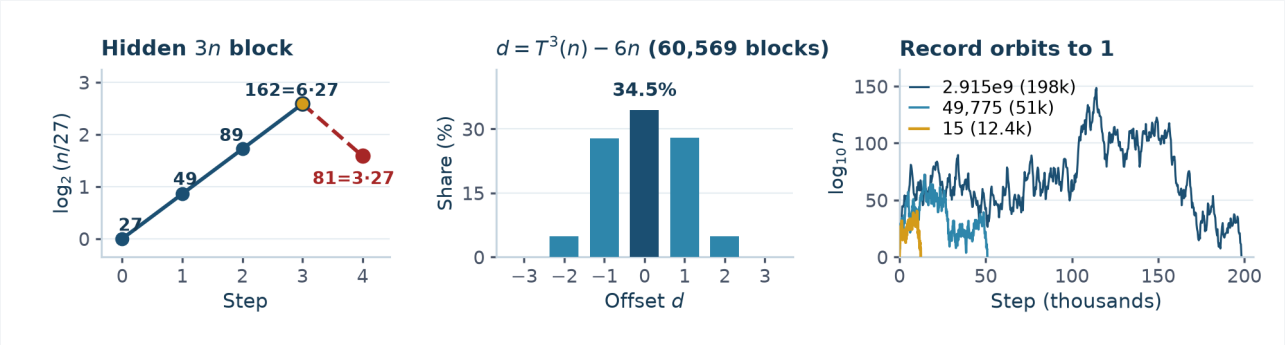
Ⓢ *Supporting Information:* `recompute.py` *(standalone exact-integer Python verifier with zero floating-point step decisions)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Define $T(n) = n/2$ for even $n \geq 1$ and $T(n) = \mathrm{round}(n \cdot 6^{1/3})$ for odd $n \geq 1$, member $(6^{1/3}, 1/2)$ of the Mignosi–Brocco–Althöfer–Zipproth Collatz family. We conjecture every $n \geq 1$ reaches $1 \to 2 \to 1$ (unsettled). We prove five results: (1) $(2m-1)^3 < 48n^3 < (2m+1)^3$; (2) three odd steps satisfy $T^3(n) = 6n + d$ ($d \in \{-3, \ldots, 3\}$), with $T^4(n) = 3n$ at $d = 0$, while j consecutive $3n$ passes need $|e(n)| < 1/(2 \cdot 3^{j-1})$; (3) two- and four-step descents; (4) a cycle lower bound in $|a \ln(6^{1/3}) - b \ln 2|$; and (5) polygon-area density $p_{r-3}/4$ for odd starts with $\geq r$ odd steps (fixed $r \geq 3$). Exact integer replay (`recompute.py`) verifies starts 15 (12,356 steps, 41-digit peak), 49,775 (50,984), 2,915,156,363 (198,333), 60,569 blocks (34.5% at $d = 0$), and $n \leq 100{,}000$. Sweeps to 3,621,000,001 (dual-code $\leq 10^9 + 1$; single-program $> 10^9$, 1.4% audit $> 2 \times 10^9$) force nontrivial cycles to have $\geq 266{,}345$ steps. No Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 7 in Lean Kernel (0%) · 2 Cert Only · 5 written · 1 open (not counted)
Abstract claims: 0 of 7 in Lean Kernel (0%) · 2 Cert Only · 5 written · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 11B83, 11J70, 11K06, 37P99 **Keywords:** Collatz-type maps, irrational multipliers, nearest-integer rounding, Diophantine approximation, Weyl equidistribution, cycle lower bounds

1. INTRODUCTION AND MAIN RESULTS

Take a positive integer n: if n is even, halve it; if n is odd, multiply it by the cube root of 6 ($\beta := 6^{1/3} = 1.8171205928\ldots$) and round to the nearest integer. Because β is irrational, $n\beta$ never lands halfway between two consecutive integers, so the map

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ \mathrm{round}(n \cdot 6^{1/3}) & \text{if } n \text{ is odd} \end{cases}$$

is uniquely defined on $\mathbb{Z}_{\geq 1}$, with $T(1) = 2$ and $T(2) = 1$. On the odd integers, passing to the odd part makes T the exact member $(x, y) = (6^{1/3}, 1/2)$ of the irrational-multiplier Collatz family $n \mapsto$ odd part of $\lfloor xn + y \rfloor$ studied by Althöfer and Zipproth (2025, open research note accompanying [1]) and the shifted ceiling family $\lceil \beta n + \alpha \rceil$ of Mignosi [6] (`[to check]`) and Brocco [2] (`[to check]`).

Although $\beta < 2$ places the map in a regime where independent parities would already drift steadily downward ($(\ln\beta - \ln 2)/2 \approx -0.04795$ per step), the identity $\beta^3 = 6 = 2 \cdot 3$ embeds a hidden copy of the classical $3n$ multiplier inside T. Whenever three consecutive odd steps occur, their combined multiplication by $\beta^3 = 6$ lands on $6n + d$ with a small integer offset $d \in \{-3, -2, -1, 0, 1, 2, 3\}$. In roughly one third of all three-odd-step blocks (34.5% across odd starts $3 \leq n \leq 20{,}001$), the offset vanishes ($d = 0$), so $T^3(n) = 6n$ is even and the ensuing halving produces $T^4(n) = 3n$—which is odd again and immediately re-enters the rising branch. This parity feedback raises the empirical odd-step fraction from 0.500 to approximately 0.5313, within 0.006 of the exact balance threshold $\ln 2/(\ln 2 + \ln\beta) \approx 0.53716$, yielding a measured drift of only about -0.0077 per step (roughly twelve times nearer balance than the classical $3n + 1$ map at -0.09589). As a result, small starts wander for thousands of steps—start 15 requires 12,356 steps and reaches a 41-digit peak before falling to 1—yet every tested trajectory up to 3,621,000,001 ultimately reaches the single cycle $1 \to 2 \to 1$.

The central global convergence statement (Conjecture 8 in Section 6) remains completely unsettled. In this paper we

assemble five elementary written structural results (Propositions 1 and 3 and Theorems 2, 4, and 5) alongside two computational propositions (Propositions 6 and 7), keeping proved statements, exact live replays, external multi-billion-step sweeps, and open questions strictly separated.

Proposition 1 (Exact Integer Cube Criterion). *For every odd integer $n \geq 1$, the product $n \cdot 6^{1/3}$ is never a half-integer, so $T(n) = \mathrm{round}(n \cdot 6^{1/3})$ is uniquely defined. Moreover, for every integer $m \geq 1$,*

$$T(n) = m \iff (2m-1)^3 < 48n^3 < (2m+1)^3,$$

equivalently $T(n) = \lfloor (\lfloor (48n^3)^{1/3} \rfloor + 1)/2 \rfloor$.

Theorem 2 (Hidden $3n$ Structure and Closure of the $3n$ Channel). *Set $\beta = 6^{1/3}$, and for every odd integer $n \geq 1$ define the rounding error $e(n) := T(n) - \beta n \in (-1/2, 1/2)$.*

1. (Three-odd-step identity.) *Whenever n, $T(n)$, and $T^2(n)$ are all odd,*

$$T^3(n) = 6n + d, \quad d = \beta^2 e(n) + \beta e(T(n)) + e(T^2(n)),$$

where $d \in \{-3, -2, -1, 0, 1, 2, 3\}$. In particular, when $d = 0$, $T^3(n) = 6n$ is even and $T^4(n) = 3n$ is odd.

2. (Finite $3n$-channel exit.) *For every odd $n \geq 1$ and every integer $j \geq 1$, if the orbit of n executes j consecutive $d = 0$ tripling blocks $n \to \cdots \to 3n \to \cdots \to 3^j n$ (each consisting of three odd steps followed by one halving), then $|e(n)| < 1/(2 \cdot 3^{j-1})$. Since $e(n) \neq 0$, no orbit remains in the $3n$ channel indefinitely.*

Proposition 3 (Elementary Descent Classes). *For every odd integer $n \geq 3$:*

1. *If $T(n)$ is even, then $T^2(n) < n$.*
2. *If $T(n)$ is odd and $4 \mid T^2(n)$, then $T^4(n) < n$.*

Theorem 4 (Cycle-Length Lower-Bound Inequality). *Let $C \subset \mathbb{Z}_{\geq 1}$ be a nontrivial cycle of T (meaning $C \neq \{1, 2\}$) with least element $m = \min C$, consisting of $a \geq 1$ odd steps and $b \geq 1$ halvings (total length $L = a + b$). Then $m \geq 3$ is odd, $a \ln \beta \neq b \ln 2$, and*

$$m < \frac{1}{2\beta(1 - \exp(-|a \ln \beta - b \ln 2|/a))}. \tag{1}$$

Consequently, if every integer $n \leq S$ eventually reaches 1, then any nontrivial cycle has $m > S$ and its step counts (a, b) satisfy

$$|a \ln \beta - b \ln 2| < -a \ln\left(1 - \frac{1}{2\beta S}\right). \tag{2}$$

Because three odd steps give $6n + d$, the parity of $T^3(n)$ equals the parity of d. Iterating the error recursion shows that runs of consecutive odd steps correspond to shrinking convex polygons in the error plane $(e_0, e_1) \in (-1/2, 1/2)^2$.

Theorem 5 (Fixed-Length Odd-Run Density Law). *For an odd integer $n \geq 1$, let $\mathrm{run}(n) \geq 1$ denote the number of consecutive odd steps starting at n (so $\mathrm{run}(n) \geq r$ means that $n, T(n), \ldots, T^{r-1}(n)$ are all odd). Then for every fixed integer $r \geq 3$, the set of odd positive integers n with $\mathrm{run}(n) \geq r$ has exact natural density $p_{r-3}/4$ inside the odd positive integers, where $p_0 = 1$ and p_t is the total Euclidean area of a finite union of convex polygons (each having at most 6 sides) in $(-1/2, 1/2)^2$.*

Theorem 5 holds for each fixed $r \geq 3$ with no uniformity claim in r; the numerically observed area decay ratio $\rho \approx 0.53308$ and the resulting record-run law $L(N) \approx 1.10184 \log_2 N - 0.44$ are heuristic rather than proved.

Proposition 6 (Exact Replay of Worked Orbits, Block Census, and Cycle-Floor Arithmetic). *Pure exact integer evaluation via Proposition 1 (zero floating-point decisions) in `recompute.py` establishes:*

1. *Start 3 reaches 1 in 7 steps ($3 \to 5 \to 9 \to 16 \to 8 \to 4 \to 2 \to 1$); start 27 gives the smallest $d = 0$ block ($27 \to 49 \to 89 \to 162 \to 81$); start 93 is the smallest odd integer taking two consecutive $d = 0$ blocks ($93 \to \cdots \to 279 \to \cdots \to 837$).*
2. *Start 15 reaches 1 in 12,356 steps (6,635 odd; stopping time 12,353; 41-digit peak 2.5944×10^{40}, 135 bits, at step 10,159). Start 49,775 attains the longest stopping time (38,546) for $2 \leq n \leq 100{,}000$, reaching 1 in 50,984 steps (27,378 odd; 241-bit peak at step 14,527). Start 2,915,156,363 reaches 1 in 198,333 steps (106,519 odd; stopping time 185,127; 494-bit peak at step 114,199). The 64-bit start 13,885,776,609,203,266,287 takes 70 consecutive odd steps (climbing to 124 bits) before reaching 1 in 33,713 steps (257-bit peak).*
3. *Across the 404,820 distinct integers visited before 1 by orbits of odd starts 3 through 20,001, exactly 60,569 are odd integers followed by two further odd steps; their offsets $d = T^3(n) - 6n$ have exact counts 8, 2,935, 16,876, 20,891, 16,918, 2,909, and 32 for $d = -3, \ldots, +3$, with $d = 0$ occurring in $20{,}891/60{,}569 = 34.49\%$ ($\approx 34.5\%$) of blocks and 0 outside $\{-3, \ldots, 3\}$.*
4. *Every integer $2 \leq n \leq 100{,}000$ drops below itself (6,193,555 total stopping steps; 0 certified fixed-point fallbacks).*
5. *Exhaustive 80-digit evaluation of (1) across all $1 \leq a \leq 266{,}344$ shows that any nontrivial cycle of length $L \leq 20$ satisfies $m < 88.14$; $L \leq 400$ satisfies $m < 20{,}952.14$; $L \leq 3{,}000$ satisfies $m < 12{,}574{,}884.79$; no nontrivial cycle of length $< 88{,}369$ can have $m > 2{,}000{,}000{,}001$; and no nontrivial cycle of length $< 266{,}345$ can have $m > 3{,}620{,}904{,}121.47$ (in particular, $m > 3{,}621{,}000{,}001$).*

Proposition 7 (Extended Sweep through 3.621×10^9 and Cycle Floor). *Combining the committed stopping-time sweeps stratified in Table 3 (live exact replay in `recompute.py` for $2 \leq n \leq 100{,}000$; two independent C/GMP programs on odd starts up to $10^9 + 1$; a single falsifier C/GMP program from $10^9 + 3$ through $2 \times 10^9 + 1$; and a single escape-attack C/GMP program from $2 \times 10^9 + 3$ through 3,621,000,001, audited by an independent reviewer via a 1.4% sample re-run of 11,000,009 starts and contiguous chunk-log checks), every positive integer $n \leq 3{,}621{,}000{,}001$ drops below itself and hence reaches 1 by induction. Combined with Theorem 4 and Proposition 6(5), any nontrivial cycle of T has at least 266,345 steps (143,069 odd steps and 123,276 halvings).*

Figure 1 summarizes the algebraic mechanism, the multi-layer computational verification, and the explicit verification boundary.

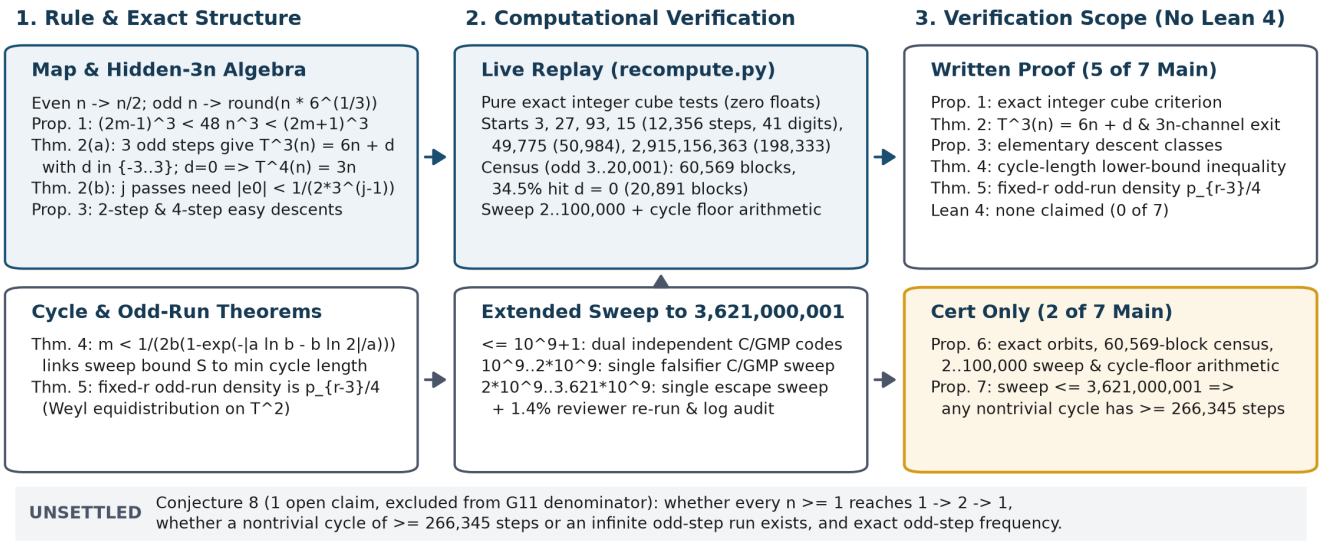


Figure 1. Discovery and verification architecture for Propositions 1, 3, 6, and 7, Theorems 2, 4, and 5, and Conjecture 8. No Lean 4 formalization is claimed (0 of 7 proved main results in Lean 4; 5 of 7 by written proof; 2 of 7 by external certificate, with provenance stratified honestly between archived live replay in `recompute.py` and non-archived multi-billion-step C/GMP sweeps). Conjecture 8 (1 open claim) is separated from the proved denominator.

Concept in Action: Exact Cube-Test Orbit of 3 and Hidden $3n$ at 27

```
def icbrt(x):
    r = 1 << ((x.bit_length() + 2) // 3)
    while (s := (2 * r + x // (r * r)) // 3) < r: r = s
    return r
T = lambda n: n // 2 if n % 2 == 0 else (icbrt(48 * n**3) + 1) // 2
orbit = lambda n, k: [n] if k == 0 else [n] + orbit(T(n), k - 1)
assert orbit(3, 7) == [3, 5, 9, 16, 8, 4, 2, 1]
assert orbit(27, 4) == [27, 49, 89, 162, 81]  # 162 = 6*27, 81 = 3*27
```

Roadmap. Section 2 fixes notation, works through start 3 by hand with exact cube brackets, illustrates the hidden-$3n$ blocks and double channel at 93, and contrasts the near-balance arithmetic with the $\sqrt{2}$ relative. Section 3 gives complete self-contained proofs of Propositions 1 and 3 and Theorems 2, 4, and 5. Section 4 presents the exact replay tables, record height trajectories, three-odd-step census, stratified provenance of the sweep through 3,621,000,001, cycle-floor convergent table, relative drift comparison, and negative-mutation controls. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 details prior work, unread sources (`[to check]`), and Conjecture 8, and Section 7 documents reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND WORKED EXAMPLES

Throughout, $\beta := 6^{1/3} = 1.8171205928\ldots$, and $T\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ is the map defined in Section 1. For any $n \geq 1$, its *orbit* is the sequence $n, T(n), T^2(n), \ldots$, where $T^0(n) = n$ and $T^{k+1}(n) = T(T^k(n))$. We call $k \geq 1$ the *stopping time* of $n \geq 2$ if k is the least positive integer with $T^k(n) < n$ (finite whenever such a k exists), and the *total stopping time* if k is the least nonnegative integer with $T^k(n) = 1$. If every integer $2 \leq n \leq S$ has a finite stopping time, then strong induction on n implies that every $1 \leq n \leq S$ reaches 1. For odd $n \geq 1$, we write

$$e(n) := T(n) - \beta n \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

for the signed rounding error, and $\mathrm{run}(n) := \min\{k \geq 1 : T^k(n) \text{ is even}\}$ for the length of the initial block of consecutive odd steps starting at n (with $\mathrm{run}(n) = \infty$ if $T^k(n)$ is odd for all $k \geq 0$).

Worked example 1: Start 3 by hand and $T(1..20)$. Table 1 traces the orbit of $n = 3$ using only integer cubes via Proposition 1. Every odd step is decided strictly by $(2m-1)^3 < 48n^3 < (2m+1)^3$, with zero decimal approximation: $3 \to 5 \to 9 \to 16 \to 8 \to 4 \to 2 \to 1$, reaching 1 in 7 steps (and entering $1 \to 2 \to 1$ because $3^3 = 27 < 48 \cdot 1^3 = 48 < 5^3 = 125$). The first ten odd values are $T(1) = 2$, $T(3) = 5$, $T(5) = 9$, $T(7) = 13$, $T(9) = 16$, $T(11) = 20$, $T(13) = 24$, $T(15) = 27$, $T(17) = 31$, and $T(19) = 35$, while every even n halves to $n/2$.

Table 1. Exact integer-cube trace of start 3 (7 steps to 1) and the cycle step $T(1) = 2$, verified by `recompute.py`.

Step	n	$T(n)$	Exact rule / cube bracket
0	3	5	$9^3 = 729 < 48 \cdot 3^3 = 1{,}296 < 11^3 = 1{,}331$
1	5	9	$17^3 = 4{,}913 < 48 \cdot 5^3 = 6{,}000 < 19^3 = 6{,}859$
2	9	16	$31^3 = 29{,}791 < 48 \cdot 9^3 = 34{,}992 < 33^3 = 35{,}937$
3	16	8	even: $16/2 = 8$
4	8	4	even: $8/2 = 4$
5	4	2	even: $4/2 = 2$
6	2	1	even: $2/2 = 1$
7	1	2	$3^3 = 27 < 48 \cdot 1^3 = 48 < 5^3 = 125$

Worked example 2: Hidden-$3n$ blocks and the smallest double pass. Start 15 opens with $15 \to 27 \to 49 \to 89 \to 162 \to 81 \to 147 \to 267 \to 485 \to 881 \to 1{,}601 \to 2{,}909 \to \cdots$.

The initial three odd steps from 15 yield $T^3(15) = 89 = 6 \cdot 15 - 1$ (smallest odd start with $d = -1$; since $d = -1$ is odd, 89 remains odd). Stepping forward one index, the three odd steps from 27 yield $T^3(27) = 162 = 6 \cdot 27$ (smallest $d = 0$ block), and one halving yields $T^4(27) = 81 = 3 \cdot 27$, which is odd again. The smallest $d = +1$ block is $51 \to 93 \to 169 \to 307 = 6 \cdot 51 + 1$. The smallest odd start executing two consecutive $d = 0$ blocks ($n \to \cdots \to 3n \to \cdots \to 9n$ in 8 steps) is $n = 93$:

$$93 \to 169 \to 307 \to 558 \to 279 \to 507 \to 921 \to 1{,}674 \to 837,$$

where $279 = 3 \cdot 93$ and $837 = 9 \cdot 93$.

Worked example 3: Near-balance arithmetic and why the $\sqrt{2}$ trick fails. Since $\ln \beta = \frac{1}{3} \ln 6 \approx 0.59725$ and $\ln 2 \approx 0.69315$, exact zero-drift balance requires an odd-step fraction

$$p_{\text{bal}} = \frac{\ln 2}{\ln 2 + \ln \beta} \approx 0.53716.$$

Under independent fair-coin parities ($p = 1/2$), the expected change in $\ln n$ per step would be $(\ln \beta - \ln 2)/2 \approx -0.04795$ (compared with $(\ln 3 - 2 \ln 2)/3 \approx -0.09589$ for the classical $3n + 1$ map at odd share $1/3$). In T, however, every $d = 0$ block terminates in an exact halving to the odd integer $3n$, creating a positive bias toward odd states that pushes the empirical odd share up to ≈ 0.5313 and reduces the drift magnitude by a factor of roughly six to -0.0077 per step.

This same algebraic factorization $\beta^3 = 6 = 2 \cdot 3$ explains why elementary closed-form descent arguments that succeed for the $\sqrt{2}$ relative fail here. For $n \mapsto \lfloor n\sqrt{2} \rfloor$ on odd integers, two odd steps multiply by $(\sqrt{2})^2 = 2$, so a single halving cancels the multiplier completely and leaves a tight closed-form descent on bounded windows. For $6^{1/3}$, three odd steps multiply by $6 = 2 \cdot 3$; one halving cancels only the factor 2, leaving $3n + d/2$ with an uncanceled factor of 3—the same arithmetic obstruction that underlies the classical Collatz problem. Moreover, because consecutive odd runs are empirically unbounded (reaching 70 steps below 2^{64} and 1,435 steps among constructed 2,473-bit algebraic-unit starts), no bounded-window descent lemma can cover all odd starts.

3. PROOFS OF THE STRUCTURAL RESULTS

We now prove Propositions 1 and 3 and Theorems 2, 4, and 5 in full.

Proof of Proposition 1. Let $n \geq 1$ be an odd integer. Suppose for contradiction that $n \cdot 6^{1/3} = k + 1/2 = (2k+1)/2$ for some integer k. Multiplying by 2 and cubing both sides gives

$$48n^3 = 8 \cdot 6n^3 = (2n \cdot 6^{1/3})^3 = (2k+1)^3,$$

which is impossible because $48n^3$ is even while $(2k+1)^3$ is odd. Thus $n \cdot 6^{1/3}$ is never a half-integer, so its nearest integer $T(n)$ is unique.

For any integer $m \geq 1$, the condition $T(n) = m$ holds if and only if $m - 1/2 < n \cdot 6^{1/3} < m + 1/2$, which upon multiplying by 2 is equivalent to $2m - 1 < 2n \cdot 6^{1/3} < 2m + 1$. Because $x \mapsto x^3$ is strictly increasing on $\mathbb{R}$, this holds if and only if

$$(2m-1)^3 < (2n \cdot 6^{1/3})^3 = 48n^3 < (2m+1)^3.$$

Finally, let $q = \lfloor (48n^3)^{1/3} \rfloor$, so $q^3 \leq 48n^3 < (q+1)^3$ (with strict inequality on the left because $48n^3$ is not a cube: $v_3(48n^3) \equiv 1 \pmod 3$). Since $(2m-1)^3 < 48n^3 < (2m+1)^3$ holds if and only if $q \in \{2m-1, 2m\}$, we have $m = \lfloor (q+1)/2 \rfloor$. □

Proof of Theorem 2. For part (1), suppose $n_0 := n$, $n_1 := T(n)$, and $n_2 := T^2(n)$ are all odd, and write $e_i := e(n_i) = n_{i+1} - \beta n_i$ for $i \in \{0, 1, 2\}$, where $n_3 := T^3(n)$. Substituting successively yields

$$\begin{aligned} n_3 &= \beta n_2 + e_2 = \beta(\beta n_1 + e_1) + e_2 \\ &= \beta^2(\beta n_0 + e_0) + \beta e_1 + e_2 = \beta^3 n_0 + (\beta^2 e_0 + \beta e_1 + e_2). \end{aligned}$$

Since $\beta^3 = 6$, setting $d := n_3 - 6n_0$ gives $d = \beta^2 e_0 + \beta e_1 + e_2 \in \mathbb{Z}$. Because $|e_i| < 1/2$ for $i = 0, 1, 2$, the triangle inequality gives

$$|d| < \frac{\beta^2 + \beta + 1}{2}.$$

To bound the right-hand side strictly below 4 using exact integer arithmetic, observe from $6^4 = 1{,}296 < 1{,}331 = 11^3$ that $\beta < 11/6$. Hence

$$\begin{aligned} \frac{\beta^2 + \beta + 1}{2} &< \frac{(11/6)^2 + 11/6 + 1}{2} \\ &= \frac{121 + 66 + 36}{72} = \frac{223}{72} < 4 \end{aligned}$$

(numerically $(\beta^2 + \beta + 1)/2 \approx 3.0595$). Because d is an integer with $|d| < 4$, we conclude $d \in \{-3, -2, -1, 0, 1, 2, 3\}$. When $d = 0$, $T^3(n) = 6n$ is even, so $T^4(n) = 6n/2 = 3n$, which is odd because n is odd.

For part (2), suppose an odd $n \geq 1$ executes $j \geq 1$ consecutive $d = 0$ tripling blocks, so that for each $r \in \{0, 1, \ldots, j-1\}$ the orbit starting at $3^r n$ takes three odd steps to $6 \cdot 3^r n$ and halves to $3^{r+1} n$. Let $n_1 = T(n)$, $n_2 = T^2(n)$ (both odd), and $e_0 = e(n)$, $e_1 = e(n_1)$, $e_2 = e(n_2)$. We prove by induction on $r \in \{0, 1, \ldots, j-1\}$ that

$$\begin{aligned} T(3^r n) &= 3^r n_1, \qquad e(3^r n) = 3^r e_0, \\ T^2(3^r n) &= 3^r n_2, \qquad e(3^r n_1) = 3^r e_1, \qquad e(3^r n_2) = 3^r e_2, \end{aligned}$$

and hence $|3^r e_i| < 1/2$ for $i \in \{0, 1, 2\}$. The base case $r = 0$ holds by definition. Suppose the claim holds for $r < j - 1$. Then $|3^r e_0| < 1/2$, so $|3^{r+1} e_0| < 3/2$. Multiplying $\beta(3^r n) = 3^r n_1 - 3^r e_0$ by 3 gives

$$\beta(3^{r+1} n) = 3^{r+1} n_1 - 3^{r+1} e_0.$$

Since $3^{r+1} n_1$ is odd and $|3^{r+1} e_0| < 3/2$ (with $|3^{r+1} e_0| \neq 1/2$ by Proposition 1), the nearest integer to $3^{r+1} n_1 - 3^{r+1} e_0$ is $3^{r+1} n_1$ if $|3^{r+1} e_0| < 1/2$, and is $3^{r+1} n_1 \pm 1$ (which is even) if $1/2 < |3^{r+1} e_0| < 3/2$. Because pass $r+1$ requires $T(3^{r+1} n)$ to be odd, we must have $|3^{r+1} e_0| < 1/2$, forcing $T(3^{r+1} n) = 3^{r+1} n_1$ and $e(3^{r+1} n) = 3^{r+1} e_0$. Repeating the exact same parity argument on $\beta(3^{r+1} n_1) = 3^{r+1} n_2 - 3^{r+1} e_1$ forces $|3^{r+1} e_1| < 1/2$, $T^2(3^{r+1} n) = 3^{r+1} n_2$, and $e(3^{r+1} n_1) = 3^{r+1} e_1$. Finally, $\beta(3^{r+1} n_2) = 6 \cdot 3^{r+1} n - 3^{r+1} e_2$ with $|3^{r+1} e_2| < 3/2$; because pass $r+1$ has $d = 0$, we have $T^3(3^{r+1} n) = 6 \cdot 3^{r+1} n$, which forces $e(3^{r+1} n_2) = 3^{r+1} e_2$ and $|3^{r+1} e_2| < 1/2$.

Setting $r = j-1$ gives $|e(n)| = |e_0| < 1/(2\cdot 3^{j-1})$. Since $e(n) = n_1 - \beta n \neq 0$ (as β is irrational), $3^{j-1} < 1/(2|e(n)|)$ bounds j above, so no orbit can follow the $3n$ channel forever. □

Proof of Proposition 3. Let $n \geq 3$ be odd. As shown in the proof of Theorem 2, $48 \cdot 27 = 1{,}296 < 1{,}331 = 11^3$ implies $\beta < 11/6$.

For part (1), suppose $T(n)$ is even. Then $T(n) = \beta n + e(n) < \beta n + 1/2$, so

$$T^2(n) = \frac{T(n)}{2} < \frac{\beta}{2}n + \frac{1}{4} < \frac{11}{12}n + \frac{1}{4} = n - \frac{n-3}{12} \leq n,$$

and since the inequality $\beta < 11/6$ is strict, $T^2(n) < n$ for every odd $n \geq 3$.

For part (2), suppose $T(n)$ is odd and $4 \mid T^2(n)$. Applying $T(x) < \beta x + 1/2$ twice gives

$$T^2(n) < \beta\left(\beta n + \frac{1}{2}\right) + \frac{1}{2} = \beta^2 n + \frac{\beta+1}{2}.$$

Because $4 \mid T^2(n)$, the next two steps are halvings, yielding

$$T^4(n) = \frac{T^2(n)}{4} < \frac{\beta^2}{4}n + \frac{\beta+1}{8}.$$

Using $\beta < 11/6$ gives $\beta^2/4 < 121/144$ and $(\beta+1)/8 < 17/48 = 51/144$. Therefore, for every odd $n \geq 3$,

$$T^4(n) < \frac{121}{144}n + \frac{51}{144} = n - \frac{23n-51}{144} \leq n - \frac{18}{144} < n. \quad \square$$

Proof of Theorem 4. Let C be a nontrivial cycle of T with least element $m = \min C$, $a \geq 1$ odd steps, and $b \geq 1$ halvings. If m were even, then $T(m) = m/2 < m$ would lie in C, contradicting minimality of m. Thus m is odd, and since $C \neq \{1,2\}$ (the unique cycle containing 1), we have $m \geq 3$. Furthermore, $a\ln\beta \neq b\ln 2$: equality would imply $6^a = \beta^{3a} = 2^{3b}$, impossible for $a \geq 1$ because $3 \mid 6^a$ whereas $3 \nmid 2^{3b}$.

Taking the product of $T(x)/x$ around the cycle C gives

$$1 = \prod_{x\in C}\frac{T(x)}{x} = 2^{-b}\beta^a \prod_{\substack{x\in C\\ x \text{ odd}}}\left(1 + \frac{e(x)}{\beta x}\right).$$

Taking natural logarithms yields

$$b\ln 2 - a\ln\beta = \sum_{\substack{x\in C\\ x \text{ odd}}} \ln\left(1 + \frac{e(x)}{\beta x}\right).$$

For every odd $x \in C$, we have $x \geq m$ and $|e(x)| < 1/2$, so $|t_x| < \delta$, where $t_x := e(x)/(\beta x)$ and $\delta := 1/(2\beta m) \in (0,1)$. For any real t with $|t| < \delta < 1$, the elementary inequality $\ln(1+|t|) + \ln(1-|t|) = \ln(1-t^2) \leq 0$ implies $|\ln(1+t)| \leq -\ln(1-|t|) < -\ln(1-\delta)$. Applying the triangle inequality across all a odd elements of C gives

$$|a\ln\beta - b\ln 2| \leq \sum_{\substack{x\in C\\ x \text{ odd}}} |\ln(1+t_x)| < -a\ln\left(1 - \frac{1}{2\beta m}\right).$$

Dividing by $-a$ and exponentiating gives

$$1 - \frac{1}{2\beta m} < \exp\left(-\frac{|a\ln\beta - b\ln 2|}{a}\right),$$

which rearranges immediately to (1).

Finally, if every integer $n \leq S$ reaches 1, then any nontrivial cycle C must satisfy $m > S$, so $1/(2\beta m) < 1/(2\beta S)$ and $-\ln(1-1/(2\beta m)) < -\ln(1-1/(2\beta S))$, establishing (2). Moreover, for any pair (a,b) with $|a\log_2\beta - b| \geq 1/2$, we have $u := |a\ln\beta - b\ln 2|/a \geq (\ln 2)/(2a)$; using $1-e^{-u} > u/(1+u)$ in (1) gives

$$m < \frac{1}{2\beta}\left(\frac{1}{u}+1\right) \leq \frac{a}{\beta\ln 2} + \frac{1}{2\beta} < 0.80a + 0.28,$$

so for $a \leq 266{,}344$ every non-nearest b forces $m < 214{,}000$, far below the swept thresholds. □

Proof of Theorem 5. Fix an integer $r \geq 3$, and set $t := r-3 \geq 0$. Consider an odd integer $n = n_0 \geq 1$, and write $n_k := T^k(n)$ and $e_k := n_{k+1} - \beta n_k \in (-1/2, 1/2)$ as long as $n_0, \dots, n_k$ remain odd.

Step 1 (*Error recursion and digit words*). Whenever n_j, n_{j+1}, and n_{j+2} are odd, Theorem 2(1) gives $n_{j+3} = 6n_j + d_j$, with $d_j = \beta^2 e_j + \beta e_{j+1} + e_{j+2}$ in $\{-3,\dots,3\}$. Since $|e_{j+2}| < 1/2$, d_j is the nearest integer to $\beta^2 e_j + \beta e_{j+1}$, and n_{j+3} is odd if and only if d_j is odd (that is, $d_j \in \{-3,-1,1,3\}$). Thus, given that n_0, n_1, n_2 are odd, $\mathrm{run}(n_0) \geq r = t+3$ holds if and only if the first t offsets $d_0, \dots, d_{t-1}$ are all odd. Furthermore, along an odd block,

$$\begin{aligned} e_{j+3} &= n_{j+4} - \beta n_{j+3} = (6n_{j+1} + d_{j+1}) - \beta(6n_j + d_j) \\ &= 6(n_{j+1} - \beta n_j) + d_{j+1} - \beta d_j = 6e_j + d_{j+1} - \beta d_j. \end{aligned}$$

Iterating this linear recurrence shows that for every $0 \leq m \leq t+1$,

$$e_m = 6^{\lfloor m/3\rfloor} e_{m \bmod 3} + c_m(d_0, \dots, d_{m-2}),$$

where the constant c_m depends solely on the finite odd word $w = (d_0, \dots, d_{t-1})$, while $e_2 = d_0 - \beta^2 e_0 - \beta e_1$.

Step 2 (*Convex polygon cells*). For each fixed odd word $w \in \{-3,-1,1,3\}^t$, the condition that (e_0, e_1) generates word w is equivalent to the simultaneous inequalities $e_m \in (-1/2, 1/2)$ for $0 \leq m \leq t+1$. Grouping the inequalities by $m \bmod 3 \in \{0,1,2\}$ restricts each of the three linear forms e_0, e_1, and $e_2 = d_0 - \beta^2 e_0 - \beta e_1$ to a single open interval $I_0(w)$, $I_1(w)$, $I_2(w)$ (each an intersection of finitely many intervals). Thus the cell $C_w \subset (-1/2,1/2)^2$ realizing word w is the intersection of three parallel strips in three distinct directions—hence either empty or a convex polygon with at most 6 sides. Because distinct words $w \neq w'$ produce distinct sequences d_j, the cells $\{C_w : w \in \{-3,-1,1,3\}^t\}$ are pairwise disjoint, and their union $U_t := \bigcup_w C_w \subset (-1/2,1/2)^2$ is a finite disjoint union of convex polygons with total Euclidean area $p_t := \sum_w \mathrm{Area}(C_w)$ (with $U_0 = (-1/2,1/2)^2$ and $p_0 = 1$).

Step 3 (*Area-preserving torus map and Weyl equidistribution*). Define $\Phi(n) := (n\beta \bmod 2,\ n\beta^2 \bmod 2) \in \mathbb{T}_2^2 := (\mathbb{R}/2\mathbb{Z})^2$, where $\mathbb{T}_2^2$ has total area 4. For an odd integer $n = n_0$, write $x = n\beta \bmod 2 \in [0,2)$ and $y = n\beta^2 \bmod 2 \in [0,2)$. Then $n_1 = \mathrm{round}(n\beta)$ is odd if and only if $x \in (1/2, 3/2)$, in which case $e_0 = 1 - x \in (-1/2, 1/2)$. Next, $n_1\beta = n\beta^2 + \beta e_0$, so $n_2 = \mathrm{round}(n_1\beta)$ is odd if and only if $(y + \beta e_0) \bmod 2 \in (1/2, 3/2)$, in which case $e_1 = 1 - ((y+\beta e_0) \bmod 2) \in (-1/2, 1/2)$. For each fixed $x \in (1/2, 3/2)$, the map $y \mapsto 1 - ((y + \beta(1-x)) \bmod 2)$ is a rigid shift (with sign flip) of the interval onto

$(-1/2, 1/2)$, so the map

$$\Psi\colon \{(x,y) \in \mathbb{T}_2^2 : n_1, n_2 \text{ odd}\} \longrightarrow (-1/2,1/2)^2, \quad (x,y) \longmapsto (e_0, e_1)$$

is a bijection that preserves 2-dimensional Lebesgue measure. In particular, the pullback $V_t := \Psi^{-1}(U_t) \subset \mathbb{T}_2^2$ is a finite union of polygons (hence Jordan measurable) of area p_t out of the torus's total area 4.

Finally, as $n = 2k+1$ ranges over odd positive integers ($k = 0, 1, 2, \dots$), we have

$$\frac{1}{2}\Phi(2k+1) = \left(\frac{\beta}{2}, \frac{\beta^2}{2}\right) + k(\beta, \beta^2) \pmod{\mathbb{Z}^2}.$$

Because $x^3 - 6$ is irreducible over $\mathbb{Q}$ (by Eisenstein's criterion at the prime 2 or 3), the numbers $1, \beta, \beta^2$ are linearly independent over $\mathbb{Q}$. By Weyl's equidistribution theorem [9] ([to check]), the Kronecker sequence $k(\beta, \beta^2) \bmod \mathbb{Z}^2$ is equidistributed on $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. Since V_t is Jordan measurable with normalized measure $p_t/4$ on $\mathbb{T}_2^2$, the natural density of odd integers n satisfying $\Phi(n) \in V_t$ (equivalently $\mathrm{run}(n) \geq t+3 = r$) equals $p_t/4 = p_{r-3}/4$. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Exact Live Replay: Record Orbits and Three-Odd-Step Census. Every calculation in Proposition 6 is reproduced deterministically from scratch in under 7 seconds by `recompute.py` using only Python's standard library and exact integer arithmetic (Proposition 1 asserted at every odd step of every worked orbit and block census). Table 2 summarizes the five highlighted trajectories verified in exact arithmetic, and Figure 2 displays their height profiles ($\log_{10} n$ versus step). Notice how start 15 climbs to the 41-digit (135-bit) peak

25,944,260,013,186,661,896,399,793,144,320,345,058,518

at step 10,159 before dropping below 15 at step 12,353 and reaching 1 at step 12,356—over one hundred times longer than the classical Collatz start 27 (111 steps).

To quantify the hidden-3n mechanism empirically, `recompute.py` follows every odd start $3, 5, \dots, 20{,}001$ until it merges with a previously visited value or reaches 1, visiting exactly 404,820 distinct integers. Among these, 60,569 are odd integers n for which $T(n)$ and $T^2(n)$ are also odd. Every single offset $d = T^3(n) - 6n$ lies in $\{-3, \dots, 3\}$ with exact counts

$$d = -3\colon 8\ (0.01\%), \quad d = -2\colon 2{,}935\ (4.85\%),$$
$$d = -1\colon 16{,}876\ (27.86\%), \quad d = 0\colon 20{,}891\ (34.49\%),$$
$$d = +1\colon 16{,}918\ (27.93\%), \quad d = +2\colon 2{,}909\ (4.80\%),$$
$$d = +3\colon 32\ (0.05\%).$$

Thus $d = 0$ (exact $6n \to 3n$) accounts for 34.5% of all three-odd-step blocks, while even offsets overall ($d \in \{-2, 0, 2\}$) account for 44.14% and odd offsets ($d \in \{-3, -1, 1, 3\}$, continuing the odd run to at least 4 steps) account for 55.86% (Figure 3, Panel A).

Stratified Sweep Provenance and High-Height Stress Tests. Following every integer $n \in [2, S]$ until its first drop below n proves by strong induction that every $n \leq S$ reaches 1. Because the full sweep to $S = 3{,}621{,}000{,}001$ spans more than 1.8×10^9 odd starts and 2×10^{11} iterations, its verification is stratified honestly across four provenance tiers (Table 3):

1. *Archived live replay* ($2 \leq n \leq 100{,}000$). `recompute.py` cross-checks its certified fixed-point evaluator against pure exact cube tests on every odd $n \leq 20{,}001$ (0 disagreements), then verifies every $2 \leq n \leq 100{,}000$ (6,193,555 total stopping steps, 0 fallbacks, longest stopping time 38,546 at 49,775). In an earlier validation check, the exact cube test also agreed identically with 1,700-digit decimal arithmetic on 120,001 odd starts up to 4,900 bits.
2. *Dual-code coverage* ($3 \leq n \leq 10^9 + 1$). Two independently written C/GMP programs covered the entire range up to 10^9+1: an initial sweep through 10^9, independently rechecked on odd starts 3 to 1,000,001 (57,657,104 steps, max stop 90,399 at 299,571) and 900,000,003 to 1,000,000,001 (5,654,110,787 steps, max stop 176,318 at 902,717,049) by the falsifier unit, and on the full middle interval 1,000,003 to 900,000,001 (449,500,000 starts, 50,784,004,995 steps, max stop 177,960 at 329,809,307) by an independent reviewer's program.
3. *Single-program middle extension* ($10^9 + 3$ *to* 2,000,000,001). This range rests on the falsifier's C/GMP program alone (500,000,000 odd starts, 56,529,921,188 steps, 0 cycles, 0 unresolved, 0 exact fallbacks), with its two headline records—stopping time 176,913 at 1,089,006,337 and 243-digit peak at 1,449,917,797—confirmed in exact Python.
4. *Audited upper extension* (2,000,000,003 *to* 3,621,000,001). This range rests on the targeted escape unit's C/GMP program (810,500,000 odd starts across two contiguous chunks, 91,665,310,959 steps, 0 cycles, 0 unresolved, 0 exact fallbacks), recording longest stopping time 185,127 at start 2,915,156,363 and peak 950 bits at start 2,543,465,379. An independent reviewer checked both chunk logs for contiguity and re-ran 11 windows totaling 11,000,009 odd starts (1.4% sample; 1,249,095,552 steps) in a separate C/GMP code, reproducing all 6 reported chunk records identically.

Beyond contiguous sweeps, targeted stress tests followed large structured and random starts to 1: 18,078 special forms up to 1,380 bits (including $2^k \pm 1$, $3^k \pm 2$, Fibonacci numbers, and continued-fraction convergents of β and β^2), 148 constructed climbers up to 2,473 bits built from powers of the norm-1 unit $\eta = 109 + 60\beta + 33\beta^2 \in \mathbb{Z}[\beta]$ (producing initial all-odd bursts up to 1,435 steps, about 0.58 steps per bit, growing from 2,404 to 4,144 bits before their first halving), 8,300 random starts up to 4,096 bits, 881,000 dense-window starts at heights up to 2^{1000}, and 1,545 starts consisting of the 309 longest-odd-run records below 2^{64} (enumerated via LLL-reduced cell-lattice search, independently verified exhaustively through 2^{44}) together with their odd neighbors $\pm2, \pm4$. Every tested start reached 1.

Cycle-Floor Arithmetic, Drift Across Relatives, and Negative Controls. Table 4 records the exact 80-digit evaluation of Theorem 4 reproduced in `recompute.py` across all $1 \leq a \leq 266{,}344$. Because $a \ln\beta - b\ln 2 = \frac{1}{3}(a\ln 3 - (3b-a)\ln 2)$, the tightest cycle bounds correspond directly

Table 2. Worked and record trajectories verified step-by-step via the exact integer cube test $(2m-1)^3 < 48n^3 < (2m+1)^3$ in `recompute.py` (zero floating-point decisions).

Start n	Steps to 1	Odd steps	Stop time	Peak (bits / digits)	Peak step	Role
3	7	3	6	5 / 2	3	Hand example (Table 1)
15	12,356	6,635	12,353	135 / 41	10,159	Small long-wandering start
49,775	50,984	27,378	38,546	241 / 73	14,527	Stopping record on $[2, 10^5]$
2,915,156,363	198,333	106,519	185,127	494 / 149	114,199	Stopping record in $[2 \cdot 10^9, 3.621 \cdot 10^9]$
13,885,776,609,203,266,287	33,713	18,075	17,057	257 / 78	28,547	64-bit record (70 odd steps)

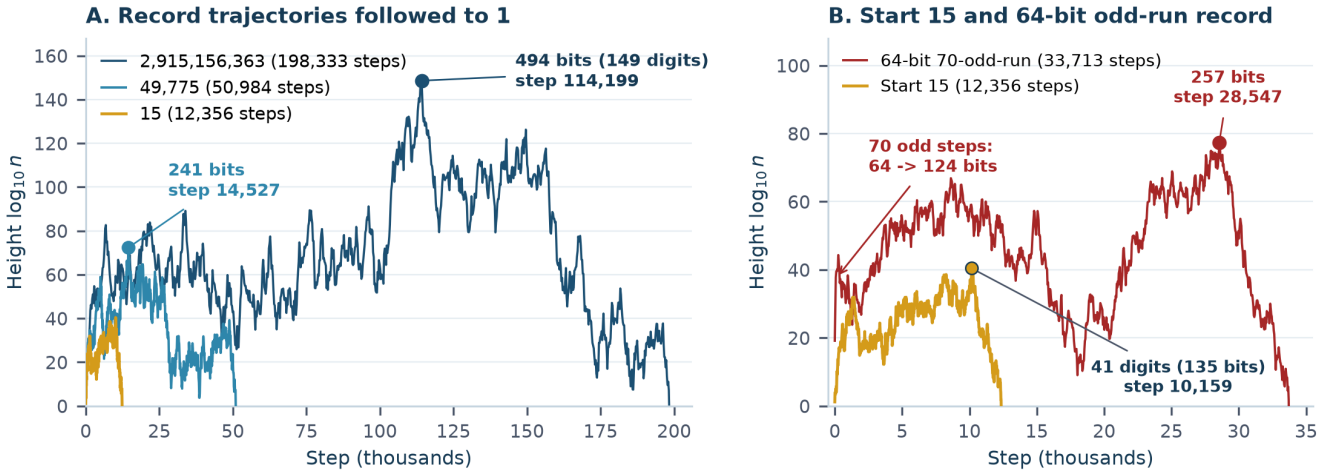


Figure 2. Exact height trajectories ($\log_{10} n$ versus step) verified in pure integer arithmetic by `recompute.py`. Panel A: record starts 15 (12,356 steps), 49,775 (50,984 steps, peak 241 bits at step 14,527), and 2,915,156,363 (198,333 steps, peak 494 bits / 149 digits at step 114,199). Panel B: detailed view of start 15 (41-digit peak at step 10,159) alongside the 64-bit 70-odd-run record 13,885,776,609,203,266,287 (33,713 steps, jumping from 64 to 124 bits during its initial 70 odd steps before peaking at 257 bits and descending to 1).

to continued-fraction convergents and semiconvergents of $\log_2 \beta = \frac{1}{3}\log_2 6$—the exact same near-misses between powers of 3 and powers of 2 that govern cycles in the classical $3n+1$ problem. Clearing the convergent (47,468, 40,901) ($L = 88{,}369$, bound 3.585×10^9) together with its multiples and the critical semiconvergent (95,601, 82,375) ($L = 177{,}976$, bound 3,620,904,121.47) raises the cycle floor directly to $L = 266{,}345$ ($a = 143{,}069$, $b = 123{,}276$, bound 1.087×10^{10}, corresponding to $3^{143{,}069} \approx 2^{226{,}759}$).

Figure 3 (Panel B) compares the measured drift per step (average $\Delta \ln n$ over 40 random 3,000-bit starts per map, up to 25,000 steps each) against the independent-parity coin-flip prediction $(\ln\beta - \ln 2)/2$ across nine related maps. On this sample, round$(n \cdot 6^{1/3})$ measures -0.00657 ± 0.00076 (and -0.00766 across a larger 64- to 4,096-bit regression sample with odd fraction 0.5313), sitting far above its coin-flip baseline of -0.04795. By contrast, its floor and ceiling siblings $\lfloor n \cdot 6^{1/3} \rfloor$ and $\lceil n \cdot 6^{1/3} \rceil$ sit directly on their coin-flip prediction (-0.04429 and -0.04600), because $\lfloor n \cdot 6^{1/3} \rfloor$ cannot produce the symmetric $d = 0$ channel, and both siblings exhibit additional small cycles. Notice also that $\lfloor n \cdot 15^{1/4} \rfloor$ ($\beta = 1.96799$) achieves a comparable drift (-0.00701) simply because 1.968 lies close to 2 (coin-flip -0.00807); what distinguishes round$(n \cdot 6^{1/3})$ is not proximity of β to 2, but the hidden-$3n$ parity correlation induced by $\beta^3 = 6$.

Negative-mutation controls. Five negative-mutation controls in `recompute.py` (5/5 caught) verify verifier sensitivity:

1. Switching to $\lfloor n \cdot 6^{1/3} \rfloor$ recovers the fixed point $\{1\}$ and the length-24 cycle at 11.
2. Switching to $\lceil n \cdot 6^{1/3} \rceil$ recovers the cycles $\{3, 6\}$ and $\{5, 10\}$.
3. Perturbing $T(3) = 5$ to 6 breaks the cube bracket ($11^3 = 1{,}331 > 1{,}296$).
4. Tightening $|d| \le 3$ to $|d| \le 2$ fails at odd start 4,831 ($T^3(4{,}831) = 28{,}989 = 6 \cdot 4{,}831 + 3$).
5. Lowering the sweep bound S to $2 \times 10^9 + 1$ re-admits the 88,369-step pair ($m < 3.585 \times 10^9$).

Table 3. Honest provenance stratification of the stopping-time verification through 3,621,000,001 (Proposition 7). Only the $2 \le n \le 100{,}000$ sweep is archived and replayed live inside `recompute.py`; the multi-billion-step C/GMP sweeps live in local non-archived run records (`regenerable_deterministic`).

Range (odd starts unless noted)	Verification coverage	Volume / records	Archive status
$2 \le n \le 100{,}000$ (all n)	Live replay (`recompute.py`)	6,193,555 steps; max stop 38,546 at 49,775	`archived_replayable`
$3 \le n \le 10^9 + 1$	Dual C/GMP sweeps	500,000,000 odd starts; 0 cycles, 0 unresolved	`regenerable_deterministic`
$10^9 + 3$ to 2,000,000,001	Single falsifier C/GMP	500,000,000 odd starts; 56,529,921,188 steps	`regenerable_deterministic`
2,000,000,003 to 3,621,000,001	Single escape + 1.4% audit	810,500,000 odd starts; 91,665,310,959 steps	`regenerable_deterministic`

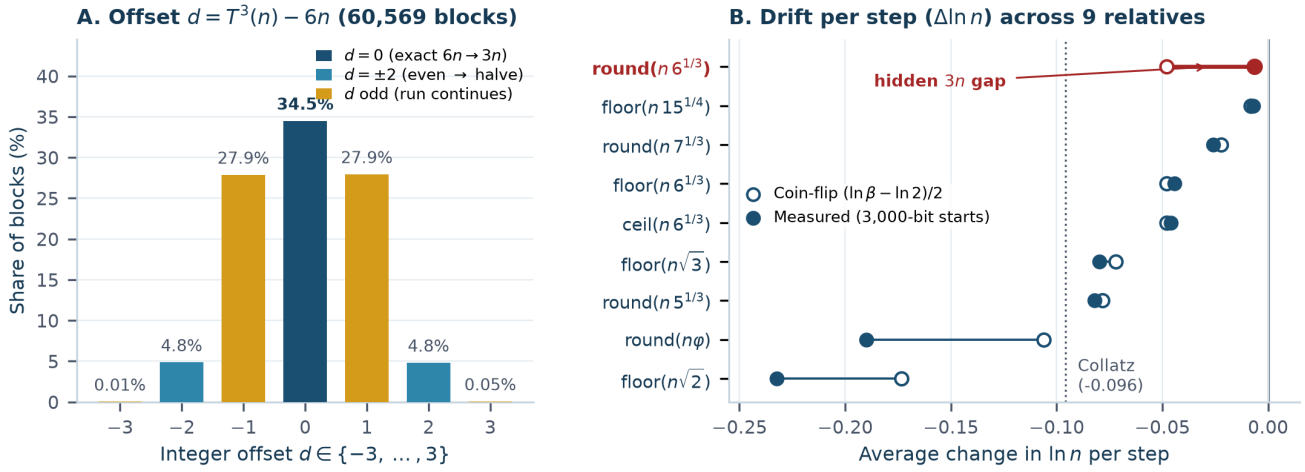


Figure 3. Hidden-$3n$ offset distribution and relative drift comparison. Panel A: exact distribution of $d = T^3(n) - 6n$ across all 60,569 three-odd-step blocks encountered by odd starts $3 \le n \le 20{,}001$ (`recompute.py`); $d = 0$ (20,891 blocks, 34.5%) halves to the odd integer $3n$, whereas $d = \pm 2$ (9.6%) halving leaves $3n \pm 1$ even and odd d (55.9%) extends the odd run. Panel B: measured drift per step on 40 random 3,000-bit starts (filled dots) versus independent-parity coin-flip drift $(\ln\beta - \ln 2)/2$ (open circles) for round$(n \cdot 6^{1/3})$ and eight relatives, illustrating the hidden-$3n$ gap separating round$(n \cdot 6^{1/3})$ (-0.0066) from its coin-flip baseline (-0.0480) and from its floor/ceiling siblings.

Table 4. Critical step-count pairs (a, b) from Theorem 4, verified exhaustively for $1 \le a \le 266{,}344$ at 80-digit precision in `recompute.py`.

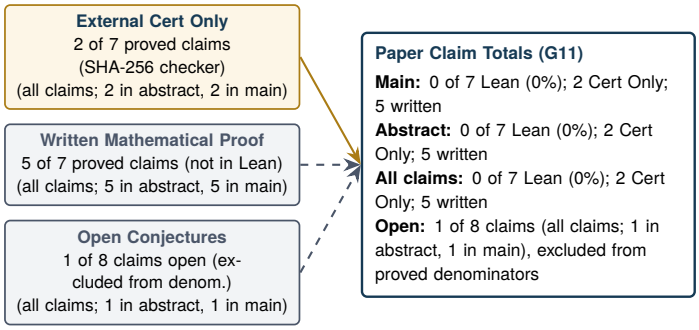
L	a	b	Bound on m	Status
13	7	6	88.13	Cleared ($L \le 20$)
296	159	137	20,952.13	Cleared ($L \le 400$)
1,238	665	573	12,574,884.79	Cleared ($L \le 3{,}000$)
88,369	47,468	40,901	3,585,106,060.05	Cleared (S)
177,976	95,601	82,375	3,620,904,121.47	Cleared (S)
266,345	143,069	123,276	10,869,697,131.22	**Unsettled floor**
354,714	190,537	164,177	2.438×10^{12}	Unsettled
443,083	238,005	205,078	1.787×10^{10}	Unsettled

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 5 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 5. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 1: Exact integer cube rounding criterion T(n) = m if and only if (2m - 1)^3 < 48 n^3 < (2m + 1)^3 for every odd n >= 1	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via parity of 48 n^3 vs (2k + 1)^3 and strict monotonicity of x^3 (Section 3). No Lean 4 formalization.
Thm. 2: Three consecutive odd steps satisfy T^3(n) = 6n + d with d in {-3..3} (d = 0 gives T^4(n) = 3n), and j consecutive d = 0 passes require |e(n)| < 1/(2 * 3^(j-1)), so no orbit stays in the 3n channel forever	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof using beta^3 = 6, beta < 11/6 (from 1296 < 1331), and induction on j (Section 3). No Lean 4 formalization.
Prop. 3: For every odd n >= 3, T(n) even implies T^2(n) < n, and T(n) odd with 4 | T^2(n) implies T^4(n) < n	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via beta < 11/6 (Section 3). No Lean 4 formalization.
Thm. 4: Any nontrivial cycle with a odd steps, b halvings, and minimum m satisfies m < 1 / (2 beta (1 - exp(-|a ln beta - b ln 2| / a)))	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof taking logarithms around the cycle product and using 6^a != 2^(3b) (Section 3). No Lean 4 formalization.
Thm. 5: For each fixed r >= 3, odd integers with run(n) >= r have exact natural density p_{r-3}/4 via convex polygon cells in (-1/2, 1/2)^2	Written	Written proof	Lean: none; fidelity: n/a	Written proof via area-preserving torus map and Weyl equidistribution for (beta, beta^2) using irreducibility of x^3 - 6 (Section 3); holds for fixed r only (decay ratio rho ~ 0.53308 and record law are heuristic). No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 6: Exact verification of worked and record orbits (3, 15, 27, 93, 49,775, 2,915,156,363, 13,885,776,609,203,266,287), 60,569-block census (34.5% at d = 0), sweep 2..100,000, and 80-digit cycle-floor arithmetic	Cert Only	recompute.py (480dd6cca02f) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Directly replayed in pure exact integer arithmetic (zero floats in step decisions) plus 80-digit Decimal cycle-floor scan (1 <= a <= 266,344) by recompute.py.
Prop. 7: Stopping-time sweep through 3,621,000,001 proves every n <= 3,621,000,001 reaches 1 and forces any nontrivial cycle to have at least 266,345 steps (143,069 odd, 123,276 halvings)	Cert Only	recompute.py (480dd6cca02f) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Stratified provenance (Table 3): 2..100,000 live in recompute.py; 3..10^9+1 dual independent C/GMP sweeps; 10^9+3.. 2,000,000,001 single falsifier C/GMP sweep; 2,000,000,003.. 3,621,000,001 single escape C/GMP sweep audited via 1.4% independent re-run (11,000,009 starts) and chunk-log check. Multi-billion-step logs are non-archived (regenerable_ deterministic); cycle-floor arithmetic (266,345) is archived and replayed in recompute.py.
Conj. 8: Every positive integer n >= 1 reaches 1 -> 2 -> 1 under T (equivalently no nontrivial cycles >= 266,345 steps and no unbounded orbits); plus infinite odd-run and invariant-measure questions	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open conjecture (excluded from G11 proved denominators).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 3b6e62ebe43e | Fidelity SHA-256: 9485224ab696.

6. PRIOR WORK AND OPEN QUESTIONS

The cube-root-of-6 map belongs to an established family of irrational-multiplier Collatz generalizations, credited to its authors as follows:

- *Althöfer and Zipproth (2025)*, in their open research note *Divergence in a Collatz-Type Recursion* (`althofer.de`, read in full, accompanying [1] `[to check]`), introduce the two-parameter family sending odd n to the odd part of $\lfloor xn+y \rfloor$ for $1 < x < 2$, $y > 0$. On odd integers, T is precisely their member $(x, y) = (6^{1/3}, 1/2)$. They prove divergence for every odd start at $(\sqrt{2}, 4)$ (refuting their initial family-wide finite-cycles hypothesis) and note numerical divergence for several pairs including the cube-root example $(4^{1/3}, 5.5)$, without naming $(6^{1/3}, 1/2)$ or reporting a cycle list for it.
- *Mignosi* [6] (*`[to check]`*) studies $T_\beta(n) = \lceil \beta n \rceil$ for odd n and $n/2$ for even n, conjecturing that for suitable β every orbit enters one of finitely many cycles. Extending Terras's stopping-time framework [8] (`[to check]`), he proves density-one descent when β is transcendental or a rational with even denominator, while noting that algebraic β can exhibit different behavior. Because $6^{1/3}$ is algebraic, it sits outside his density theorem; moreover, the hidden-3n parity correlation breaks the independence of parities required by Terras-type arguments.
- *Brocco* [2] (*`[to check]`*) proves counterexamples to Mignosi's conjecture for Pisot and Salem multipliers β, including shifted rules $\lceil \beta n + \alpha \rceil$ (note that round$(\beta n) = \lceil \beta n - 1/2 \rceil$). Because the non-real Galois conjugates of $6^{1/3}$ have modulus $6^{1/3} > 1$, $6^{1/3}$ is neither Pisot nor Salem, so Brocco's counterexamples do not apply.
- *Related surveys and online records.* Classical $3x + 1$ generalizations are surveyed in Lagarias [3] (`[to check]`) and his two annotated bibliographies [4, 5] (searched in full). Square-root relatives appear in OEIS `A264789` ($\sqrt{3}$) and `A265287` ($\sqrt{2}$), while a distinct borderline family $\lfloor n\alpha \rfloor / \lfloor n/\alpha \rfloor$ appears in MathOverflow threads `353434`/`353524` and Rafik [7] (`[to check]`). We also note transparently that round$(n \cdot 6^{1/3})$ was selected from an automated scan of 81 maps of the form $\lfloor nK^{1/M} \rfloor$, $\lceil nK^{1/M} \rceil$, or round$(nK^{1/M})$ tested up to 200,000, of which 15 (including round$(n \cdot 5^{1/3})$ and $\lfloor n \cdot 15^{1/4} \rfloor$) exhibited only the trivial cycle.

Prior-work searches logged. Logged searches comprised 24 OEIS queries by computed terms (map values, rounded multiples, step counts, trajectories, odd-part map, and record positions) and 20 OEIS word queries, 18 arXiv API queries, 6 web searches, full-text searches of both Lagarias bibliographies [4, 5], Matthews' generalized $3x+1$ pages, and Althöfer and Zipproth's August 2025 note; an independent reviewer added 8 OEIS queries, 16 MathOverflow/Math Stack Exchange API queries, 8 arXiv queries, zbMATH Open searches (including papers citing Mignosi and Brocco), and 2 web searches. On 2026-11-18, 11 CORE searches followed (CORE holds none of the known papers on these maps, so its empty return adds little), and on 2026-11-22 a 4-query Google Scholar sweep inspected 35 records. This member $(6^{1/3}, 1/2)$ and the conjecture that $1 \to 2 \to 1$ is its sole cycle were *not found in the searches logged*.

Unread and unsearched items (`[to check]`). The following sources remain unread in full or unsearched and are marked `[to check]` in `references-checked.md`: (1) Guy's *Unsolved Problems in Number Theory*; (2) the paywalled full texts of Mignosi [6] and Brocco [2] (inspected via Lagarias [4] and zbMATH reviews `Zbl 0837.11011` and `Zbl 0820.11007`); (3) the paywalled LNCS chapter Althöfer, Hartisch, and Zipproth [1]; and (4) the full texts of Terras [8], Lagarias [3], Weyl [9], and Rafik [7].

Conjecture 8 (Global Convergence and Open Questions for T). *The following questions remain unsettled:*

1. (Main convergence conjecture.) *Every positive integer $n \geq 1$ eventually reaches 1 under iteration of T; equivalently, $\{1, 2\}$ is the unique cycle of T and every orbit is bounded.*
2. (Long cycles.) *Decide whether T admits any nontrivial cycle of length at least 266,345 steps.*
3. (Infinite odd runs.) *Decide whether there exists an odd integer $n \geq 1$ with $\mathrm{run}(n) = \infty$ (whether any integer orbit lands on the measure-zero Cantor set $\bigcap_{t \geq 0} U_t$ of Theorem 5, a Mahler-type Diophantine question).*
4. (Invariant measure and drift.) *Establish rigorously the limiting odd-step frequency (empirically ≈ 0.5313), negative average drift (≈ -0.0077), and whether the polygon-area ratio p_t/p_{t-1} converges to a limit $\rho \approx 0.53308$.*

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure generation scripts, and the standalone exact-arithmetic verifier live in `papers/cube-root-of-6-map/draft1/`. Running `python3 papers/cube-root-of-6-map/draft1/recompute.py` completes in under 7 seconds with zero third-party dependencies and deterministically verifies Proposition 6, Tables 1, 2, and 4, the Section 1 Python snippet, and 5/5 negative-mutation controls. The multi-billion-step C/GMP sweeps underlying Proposition 7 ($n > 100{,}000$ up to 3,621,000,001) reside in local non-archived run records (`regenerable_deterministic`) rather than inside this draft directory; Table 3 states which ranges had dual-code versus single-program coverage. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes selected the map, proved the structural results, ran and audited the computational sweeps and exact verification scripts, and drafted this manuscript. No human mathematician has yet refereed the proof.

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