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Working Draft

Domino Flips on a Rectangle: Width-3 Rank Unimodality and a Computational Census

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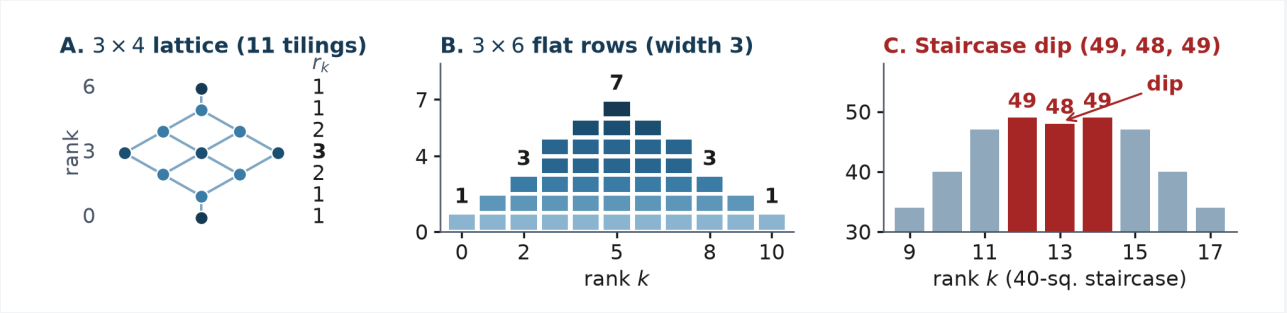
Ⓢ *Supporting Information:* `recompute.py` *(standalone exact Python verifier and negative-mutation controls)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Orienting 2×2 domino flips on an $m \times n$ rectangle (mn even) by a chessboard rule gives a graded distributive lattice $L(m,n)$ whose rank counts $r_0, \ldots, r_R$ mirror Sylvester's box-partition counts. Width $m = 2$ gives fence lattices, proved rank-unimodal by Kantarcı Oğuz and Ravichandran (2023). We prove by hand that every $3 \times 2k$ board ($k \geq 1$) is rank-symmetric and rank-unimodal: identifying $L(3,2k) \cong L_k \cong J(P_k)$ (a chain of $k-1$ hexagons), we derive $\sum_{k\geq 0} S_k(q)x^k = (1-xq^2)(1-xq^{-2})/((1-x)(1-cx+x^2))$ with $D_k = uV_{k-1}(c)$, realize D_k by admissible UM-words via signed-matching cancellation, and build a four-type reflection involution giving $D_k = \sum_{j=0}^{2k-1} f_{k,j}[2j+1]$ ($f_{k,j} \geq 0, k \geq 2$). We also prove forced palindromy (odd side or $m = n$) and log-concavity failure on $3 \times 4, 3 \times 6, 3 \times 8$. Local exact replay verifies 40,544 tilings (3×2–3×16) and the 40-square staircase $(8,7,6,6,6,6,1)$ (580 tilings, dip $49,48,49$); archived multi-program sweeps up to 17×18 (16×16 matching OEIS `A004003`) and 5,097 re-review boards ($m,n \geq 4$ to 18×20) find zero rectangular dips. Widths $m \geq 4$ remain unsettled; no Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 5 in Lean Kernel (0%) · 1 Cert Only · 4 written · 1 open (not counted)
Abstract claims: 0 of 5 in Lean Kernel (0%) · 1 Cert Only · 4 written · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 05B45, 05A15, 06D05 **Keywords:** domino tilings, flip lattice, rank unimodality, distributive lattices, generating functions, sign-reversing involution

1. INTRODUCTION AND MAIN RESULTS

A *domino tiling* covers a finite grid region of unit squares with non-overlapping 1×2 and 2×1 dominoes. Whenever two parallel dominoes fill a 2×2 block, rotating them by $90°$ performs a *flip*. Coloring the squares like a chessboard with the lower-left cell blue, orient a flip *upward* if two horizontal dominoes turn vertical on a block whose lower-left square is blue, or if two vertical dominoes turn horizontal on a block whose lower-left square is white. For any simply connected tileable region, this orientation agrees with Thurston's height order [11] (`[to check]`) and endows the tilings with a graded distributive lattice structure [3] (`[to check]`): a unique lowest tiling sits at rank 0, and every upward flip increases rank by 1.

Let r_k denote the number of domino tilings of an $m \times n$ rectangle (mn even) at rank k above the lowest tiling. A sequence $(r_0, r_1, \ldots, r_R)$ is *unimodal* if there exists an index k^* such that $r_0 \leq r_1 \leq \ldots \leq r_{k^*} \geq r_{k^*+1} \geq \ldots \geq r_R$ (flat peaks are permitted; no dip is allowed). This question is the domino-tiling counterpart of Sylvester's theorem that integer partitions fitting inside an $a \times b$ box are unimodal by size [9, 10] (`[to check]` for [9]). As Stanton [10] showed for Young's lattice of partitions inside non-rectangular shapes (such as $(8,8,4,4)$, whose counts dip $31,30,31$), unimodality fails on general regions even while holding on rectangles: for domino tilings, a 40-square staircase Young diagram already dips $49,48,49$.

For width $m = 2$, the flip lattice of the $2 \times n$ board is the order-ideal lattice of a zigzag fence poset, whose rank-unimodality was proved by Kantarcı Oğuz and Ravichandran [5] (following McConville, Sagan, and Smyth [6]), settling a conjecture of Morier-Genoud and Ovsienko. Width 2 is therefore in print and credited entirely to them. For widths $m \geq 3$, no product formula exists for the rank generating function, and the underlying poset of join-irreducibles is no longer a fence.

Our main written result settles every tileable rectangle of width 3. Since a $3 \times n$ rectangle has an even number of

squares if and only if $n = 2k$ is even, and transposing $3 \times 2k$ to $2k \times 3$ merely reverses the rank sequence, width 3 reduces to the family $3 \times 2k$ ($k \geq 1$). Write

$$[2j+1] := q^{-j} + q^{-(j-1)} + \cdots + q^{j} \qquad (j \geq 0)$$

for the symmetric flat row of $2j+1$ ones centered at exponent 0.

Theorem 1 (Width-3 Rank Symmetry and Unimodality). *For every integer $k \geq 1$, the rank numbers $r_0, r_1, \ldots, r_{4k-2}$ of the domino flip lattice $L(3, 2k)$ satisfy $r_i = r_{4k-2-i}$ and*

$$r_0 \leq r_1 \leq \ldots \leq r_{2k-1} \geq r_{2k} \geq \ldots \geq r_{4k-2}.$$

More precisely, $S_k(q) := q^{-(2k-1)} \sum_{i=0}^{4k-2} r_i q^i$ decomposes as a nonnegative sum of centered flat rows:

$$S_k(q) = \sum_{j=0}^{2k-1} F_{k,j}\,[2j+1], \qquad F_{k,j} = \mathbf{1}_{j=1} + \sum_{i=2}^{k} f_{i,j} \geq 0,$$

with nonnegative integer multiplicities $F_{k,j}$ (where $f_{i,j} \geq 0$ is the multiplicity of $[2j+1]$ in $D_i = S_i - S_{i-1}$). Consequently, every tileable rectangle of width 3 (both $3 \times 2k$ and $2k \times 3$) is rank-symmetric and rank-unimodal.

The proof proceeds in two algebraic-combinatorial stages, stated below as Proposition 2 and Lemma 3. Throughout, set

$$u := q + q^{-1}, \qquad c := q^2 + q + q^{-1} + q^{-2} = u^2 + u - 2,$$

and define Chebyshev-type polynomials $V_n(c)$ by $V_0 = 1$, $V_1 = c$, and $V_{n+1} = cV_n - V_{n-1}$ ($n \geq 1$), so that $\sum_{n \geq 0} V_n(c)\, x^n = (1 - cx + x^2)^{-1}$.

Proposition 2 (Slice Lattice L_k, Exact Generating Function, and Hexagon Poset). *For every $k \geq 1$, let*

$$L_k := \{v \in \{-1, 0, 1\}^{2k-1} : \\ v_p \neq 0 \text{ at even } p \implies v_{p-1} = v_p = v_{p+1}\}.$$

1. *Sending a domino tiling T of $3 \times 2k$ to the vertical-line crossing code $v(T) \in L_k$ of Section 3 is a lattice isomorphism $L(3, 2k) \cong L_k$, under which upward flips correspond exactly to increasing a single coordinate by 1 within L_k, and rank equals $\rho(v) = (2k-1) + \sum_{p=1}^{2k-1} v_p$.*
2. *With $S_0 = 1$ and $D_k = S_k - S_{k-1}$ for $k \geq 1$,*

$$\sum_{k=0}^{\infty} S_k(q)\, x^k = \frac{(1 - xq^2)(1 - xq^{-2})}{(1 - x)(1 - cx + x^2)},$$

$$D_k(q) = u\, V_{k-1}(c).$$

Equivalently, $S_k = [5](S_{k-1} - S_{k-2}) + S_{k-3}$ for $k \geq 3$, $D_1 = u$, $D_2 = uc$, $D_{k+1} = cD_k - D_{k-1}$ for $k \geq 2$, and $S_k = (1,1,1)\, T^k (1,0,0)^{\mathsf{T}}$ for an explicit 3×3 transfer matrix T with characteristic polynomial $(t-1)(t^2 - ct + 1)$.

3. *L_k is isomorphic to the order-ideal lattice $J(P_k)$ of a $(4k-2)$-element poset P_k whose Hasse diagram ($k \geq 2$) is a chain of $k-1$ hexagons sharing vertical rungs.*

Lemma 3 (Admissible UM-Word Model and Reflection Involution for D_k). *A* UM-word *of size k is a tuple $y = (y_1, \ldots, y_{2k-1})$ with $y_p \in \{-1, 1\}$ at odd p and $y_p \in \{-1, 0, 1\}$ at even p, with partial heights $H_0 = 0$ and $H_p = y_1 + \cdots + y_p$. Call y* admissible *($y \in W_k$) if for every $1 \leq p \leq 2k-2$, neither $(y_p, y_{p+1}) = (1, -1)$ with $H_{p-1} \geq 0$ nor $(y_p, y_{p+1}) = (-1, 1)$ with $H_{p-1} \leq -1$ occurs.*

1. *For every $k \geq 1$, $D_k(q) = \sum_{y \in W_k} q^{H_{2k-1}(y)}$, proved via a sign-reversing edge-switching involution on signed path matchings.*
2. *For every $k \geq 2$, writing $D_k(q) = \sum_s a_s q^s$, a four-type reflection involution Φ on words of W_k carrying an event proves*

$$a_j - a_{j+1} = f_{k,j} \geq 0 \qquad (j \geq 0),$$

where $f_{k,j}$ counts the event-free words in W_k with $H_{2k-1}(y) = j$. Thus $D_k(q) = \sum_{j=0}^{2k-1} f_{k,j}[2j+1]$ is symmetric unimodal.

We also record two companion structural facts clarifying why log-concavity and real-rootedness cannot establish Theorem 1 uniformly across all $k \geq 1$, and when palindromy is forced on arbitrary rectangles.

Proposition 4 (Forced Palindromy and Small-Width Log-Concavity Failures). *Let $m, n \geq 2$ with mn even.*

1. *If m is odd, n is odd, or $m = n$, then the flip lattice $L(m, n)$ is self-dual and its rank sequence is palindromic ($r_k = r_{R-k}$).*
2. *For width 3, the rank sequence of $3 \times 2k$ fails log-concavity for $k \in \{2, 3, 4\}$ (3×4: $1^2 < 1 \cdot 2$; 3×6: $3^2 < 2 \cdot 5$; 3×8: $5^2 < 3 \cdot 9$) and fails real-rootedness for $k \in \{1, 2, 3, 4\}$.*

Alongside the written width-3 proof, exact computation verifies small worked boards, confirms Theorem 1–Lemma 3 across tens of thousands of tilings and words, exhibits small non-rectangular counterexamples, and tests thousands of wider rectangles without finding a single dip.

Proposition 5 (Exact Verifications, Staircase Dip, and Census). *Exact enumeration and archived transfer-matrix sweeps establish:*

1. *Standalone exact enumeration and transfer-matrix replay in `recompute.py` verify 2×3 (3 tilings; 1, 1, 1), 3×4 (11 tilings, 14 flips; 1, 1, 2, 3, 2, 1, 1), 4×4 (36 tilings, 70 flips; 1, 1, 2, 5, 6, 6, 6, 5, 2, 1, 1), 3×6 (41 tilings; 1, 2, 3, 5, 6, 7, 6, 5, 3, 2, 1), and 3×8 (153 tilings; 1, 3, 5, 9, 13, 16, 19, 21, 19, 16, 13, 9, 5, 3, 1); confirm Proposition 2 across all 40,544 tilings of 3×2 through 3×16 and Lemma 3 across all 29,678 admissible UM-words for $k = 2, \ldots, 8$ (plus DP event-free counts to $k = 60$); and confirm that the 40-square staircase Young diagram (8, 7, 6, 6, 6, 6, 1) (580 tilings, top rank 26) dips 49, 48, 49 at ranks 12–14, while the 76-square diagram (9, 8, 7, 6, 6, 6, 6, 6, 6, 5, 5, 3, 2, 1) (1,468 tilings) dips 77, 76, 77 at ranks 16–18.*
2. *Across archived multi-program exact transfer-matrix logs—covering 334 boards in an initial sweep (widths 2–3 to length 80, widths 4–10 to 40, widths 11–12 to 30), 33 further boards (including 3×200 and widths 13–14), 153 boards ($2 \leq m \leq 12$, $m \leq n \leq 24$) plus 9 boards past width 14 up to 17×18 (among them 16×16, with top rank $\binom{17}{3} = 680$ and exact OEIS `A004003` count*

$$2{,}444{,}888{,}770{,}250{,}892{,}795{,}802{,}079{,}170{,}816$$

tilings), and a blind re-review sweep of 5,097 boards with $m, n \geq 4$ up to 18×20—zero rectangular boards fail rank-unimodality.

Conjecture 6 (Rank-Unimodality of Rectangular Domino Flip Lattices for Width $m \geq 4$). *For every $m \times n$ rectangle with $m \geq 4$, $n \geq 4$, and mn even, the rank sequence $r_0, r_1, \ldots, r_R$ of its domino flip lattice $L(m, n)$ is unimodal.*

Figure 1 maps the proof and verification boundary. Note that a separate Lean 4 formalization addressing width-3 structure and cover-supported $\mathfrak{sl}_2$ Lefschetz obstructions exists in the companion repository under `domino-lefschetz`; this preprint itself presents the written width-3 proof and computational census without claiming Lean 4 certification.

Concept in Action: Width-3 Recurrence and Flat Rows on 3×6

```
c = {2: 1, 1: 1, -1: 1, -2: 1}
mul = lambda a, b: {e: sum(a.get(i, 0)*b.get(e - i, 0) for i in a)
                    for e in range(min(a)+min(b), max(a)+max(b)+1)}
D = [{0: 1}, {1: 1, -1: 1}]; D.append(mul(D[1], c))
for k in range(3, 5):
    cd = mul(c, D[-1])
    D.append({e: cd.get(e, 0) - D[-2].get(e, 0) for e in cd})
S = [{0: 1}]
for dk in D[1:]:
    S.append({e: S[-1].get(e,0)+dk.get(e,0) for e in S[-1] | dk})
assert [S[3][e] for e in range(-5, 6)] == [1,2,3,5,6,7,6,5,3,2,1]
F3 = [S[3].get(j,0) - S[3].get(j+1,0) for j in range(6)]
assert F3 == [1, 1, 2, 1, 1, 1]
```

Roadmap. Section 2 introduces notation, flip orientation, and worked examples (3×4, 3×6, 3×8, and the 40-square staircase). Section 3 proves Theorem 1, Propositions 2 and 4, and Lemma 3. Section 4 covers local replay via `recompute.py`, negative controls, the multi-program census, bulk log-concavity, and staircase dips for Proposition 5. Section 5 presents the generated Reader's Trust Boundary, Section 6 reviews prior work, logged searches, unread literature (`[to check]`), and open questions, and Section 7 records reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND WORKED EXAMPLES

For integers $m, n \geq 1$ with mn even, model the $m \times n$ rectangle as the cell set $\{(r,c) \in \mathbb{Z}^2 : 0 \leq r < m,\ 0 \leq c < n\}$, with row r increasing upward from 0 and column c increasing rightward from 0. Color cell (r,c) *blue* when $r + c$ is even (so the bottom-left cell $(0,0)$ is blue) and *white* when $r + c$ is odd. A *domino* covers two orthogonally adjacent cells; a *tiling* T partitions the region into dominoes.

Whenever a tiling T contains two parallel dominoes filling a 2×2 square with lower-left cell (r,c), replacing those two dominoes by the orthogonal pair inside the same 2×2 square produces a tiling T' differing from T by a *flip*. Call the flip $T \to T'$ *upward* if:

- (r,c) is blue and two horizontal dominoes turn into two vertical dominoes; or
- (r,c) is white and two vertical dominoes turn into two horizontal dominoes.

Otherwise call $T \to T'$ *downward*. As verified in [11, 3] (`[to check]`), on any simply connected tileable grid region (in particular any rectangle or Young diagram), this acyclic relation generates a graded distributive lattice L with a unique minimal tiling $T_{\min}$ (having no downward flips) and a unique maximal tiling $T_{\max}$ (having no upward flips). The *rank* $\mathrm{rk}(T)$ is the length of any saturated upward flip path from $T_{\min}$ to T, and $r_k := |\{T : \mathrm{rk}(T) = k\}|$.

For transfer-matrix enumeration on an arbitrary finite cell set, attach to each tiling T the linear score

$$\psi(T) := \sum_{\substack{\text{vertical domino} \\ \{(r,c),(r+1,c)\} \in T}} -(-1)^{r+c}\, c.$$

Rotating two horizontal dominoes at (r,c) into two vertical dominoes at columns c and $c+1$ changes ψ by $-(-1)^{r+c}c - (-1)^{r+c+1}(c+1) = (-1)^{r+c}$, which equals $+1$ when (r,c) is blue and -1 when (r,c) is white. Hence every upward flip raises $\psi(T)$ by $+1$, so $\mathrm{rk}(T) = \psi(T) - \psi(T_{\min})$. Notice also that transposing an $m \times n$ board across the main diagonal $(r,c) \mapsto (c,r)$ preserves cell colors while swapping horizontal and vertical dominoes, thereby reversing every upward flip into a downward flip; hence $L(n,m)$ is the order dual of $L(m,n)$ and its rank sequence is $(r_R, \ldots, r_1, r_0)$.

Worked examples. All five boards below are enumerated both by brute-force breadth-first search on the flip graph and by the transfer matrix for ψ in `recompute.py`:

- 2×3**:** 3 tilings, 2 flips, rank counts $1, 1, 1$ (top rank 2).
- 3×4**:** 11 tilings, 14 flips, rank counts $1, 1, 2, 3, 2, 1, 1$ (Figure 2, Panel A).
- 4×4**:** 36 tilings, 70 flips, rank counts 1, 1, 2, 5, 6, 6, 6, 5, 2, 1, 1 (top rank $10 = \binom{5}{3}$, with a three-rank flat peak at 6).
- 3×6 **and** 3×8**:** 41 tilings $(1, 2, 3, 5, 6, 7, 6, 5, 3, 2, 1)$ and 153 tilings $(1, 3, 5, 9, 13, 16, 19, 21, 19, 16, 13, 9, 5, 3, 1)$, drawn as stacks of centered flat rows in Figure 2, Panels B and C.
- **Non-unimodal staircase:** the 40-square Young diagram $(8, 7, 6, 6, 6, 6, 1)$ (bottom row length 8, top row length 1) has 580 tilings, 1,698 flips, top rank 26, and rank counts 1, 1, 2, 5, 8, 11, 16, 23, 29, 34, 40, 47, **49, 48, 49**, 47, 40, ..., 1, 1, dipping at rank 13.

3. PROOFS FOR WIDTH 3 AND STRUCTURAL COMPANION FACTS

We prove Proposition 2, Lemma 3, Theorem 1, and Proposition 4 in turn. For width-3 coordinates in this section, label cells of $3 \times 2k$ by (x,y) with column $x \in \{0, \ldots, 2k-1\}$ and row $y \in \{0,1,2\}$ (matching $c = x$ and $r = y$ from Section 2). For $i = 1, \ldots, k$, call columns $\{2i-2, 2i-1\}$ *slice* i. For $x \in \{1, \ldots, 2k-1\}$, let *line* x be the vertical grid segment separating column $x-1$ from column x; odd lines $2i-1$ split slice i, while even lines $2i$ separate slice i from slice $i+1$.

Proof of Proposition 2.

Proof of Proposition 2. Part (1): Bijection $L(3,2k) \cong L_k$. Fix a tiling T of $3 \times 2k$, and let n_x be the number of horizontal dominoes crossing line x. The $3x$ cells strictly left of line x are partitioned by dominoes lying wholly left of line x (contributing an even number of cells) plus one cell per crossing domino; hence $n_x \equiv 3x \equiv x \pmod 2$. Thus every odd line is crossed 1 or 3 times, and every even line is crossed 0 or 2 times.

We claim no even line $2i$ is crossed in rows $\{0,2\}$. Suppose it were. Then cells $(2i-1,0)$ and $(2i-1,2)$ are matched rightward across line $2i$, leaving $(2i-1,1)$ unable to pair vertically or rightward (which would create a third crossing of line $2i$). Hence $(2i-1,1)$ pairs leftward with $(2i-2,1)$, so line $2i-1$

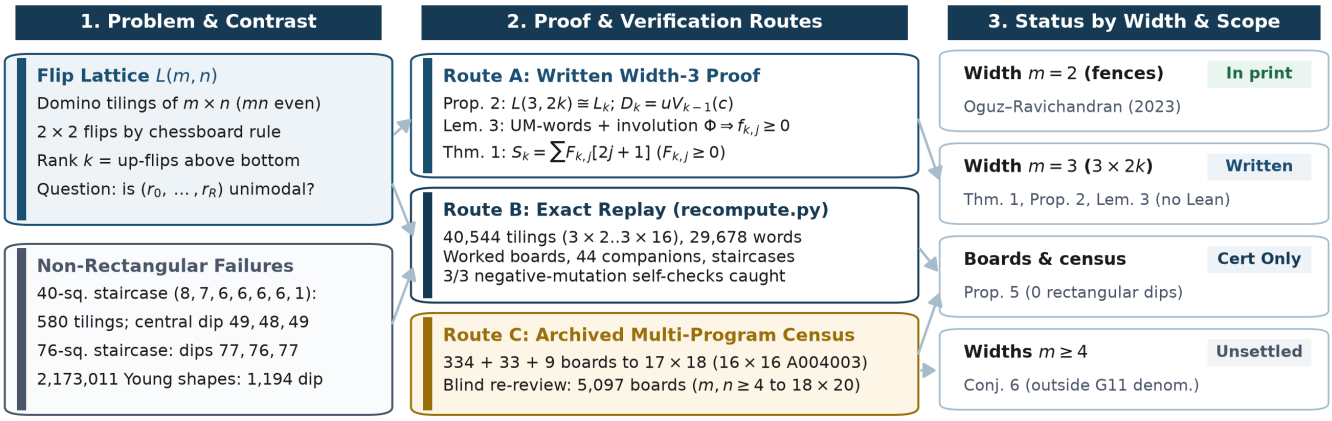


Figure 1. Problem scope, proof architecture, and verification boundary. Width 2 is in print [5]. Width 3 is proved in writing via Theorem 1, Proposition 2, and Lemma 3 (no Lean 4 certification claimed here; 0 of 5 proved main results in Lean 4, 4 of 5 by written proof, 1 of 5 by certificate). Local replay (`recompute.py`) and archived multi-program sweeps find 0 rectangular dips, while non-rectangular Young diagrams already dip at area 40. Widths $m \geq 4$ remain unsettled (Conjecture 6, excluded from the G11 denominator).

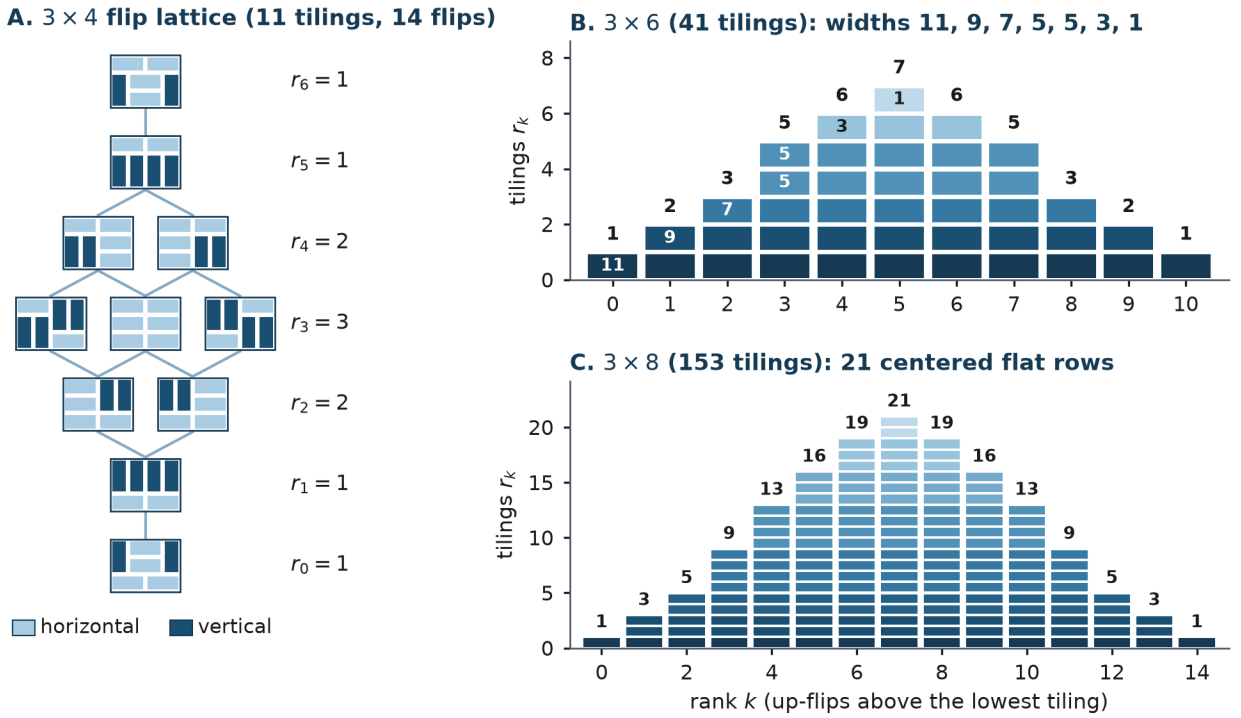


Figure 2. Worked width-3 examples. Panel A: all 11 domino tilings of the 3×4 board grouped by rank 0 to 6, with 14 upward flip edges and rank counts $1,1,2,3,2,1,1$ (horizontal dominoes light blue, vertical dominoes dark navy). Panel B: rank counts $1,2,3,5,6,7,6,5,3,2,1$ of 3×6 (41 tilings) stacked as 7 centered flat rows of odd widths $11,9,7,5,5,3,1$ ($F_{3,j} = 1,1,2,1,1,1$). Panel C: rank counts of 3×8 (153 tilings) stacked as 21 centered flat rows ($F_{4,j} = 2,3,3,4,4,2,2,1$). Because every row is flat and centered at rank $2k-1$, the column profile cannot dip.

is crossed only in row 1. Then $(2i-2, 0)$ and $(2i-2, 2)$ can pair neither rightward nor vertically, so both must pair leftward across line $2i-2$: necessarily $i \geq 2$ and line $2i-2$ is also crossed in rows $\{0, 2\}$. Descending by induction reaches slice 1, requiring columns left of $x = 0$, a contradiction.

Now let $R_i \subseteq \{0,1,2\}$ be the set of rows in which line $2i-1$ is crossed; $|R_i| \in \{1,3\}$. If $|R_i| = 3$, slice i consists of three horizontal dominoes and neither adjacent even line is crossed. The case $R_i = \{1\}$ is impossible because $(2i-2, 0)$ and $(2i-2, 2)$ would force line $2i-2$ to be crossed in rows $\{0, 2\}$. If $R_i = \{0\}$, then cells $(2i-2, 1)$ and $(2i-2, 2)$ either form a vertical domino (slice i is *closed on the left*) or cross line $2i-2$ in rows $\{1, 2\}$ (slice i is *open on the left*), and symmetrically on the right across line $2i$. The case $R_i = \{2\}$ is the vertical mirror with rows $\{0, 1\}$.

Define $v(T) = (v_1, \ldots, v_{2k-1}) \in \{-1, 0, 1\}^{2k-1}$ by:

- $v_{2i-1} = 0$ if $R_i = \{0,1,2\}$, $v_{2i-1} = -1$ if $R_i = \{0\}$, and $v_{2i-1} = +1$ if $R_i = \{2\}$;
- $v_{2i} = 0$ if line $2i$ is uncrossed, $v_{2i} = -1$ if line $2i$ is crossed in rows $\{1,2\}$, and $v_{2i} = +1$ if line $2i$ is crossed in rows $\{0,1\}$.

If $v_{2i} = -1$, cells $(2i-1,0)$ and $(2i,0)$ must pair outward across lines $2i-1$ and $2i+1$, forcing $R_i = R_{i+1} = \{0\}$, i.e., $v_{2i-1} = v_{2i+1} = -1$; symmetrically $v_{2i} = +1$ forces $v_{2i-1} = v_{2i+1} = +1$. Hence $v(T) \in L_k$. Conversely, any $v \in L_k$ uniquely reconstructs a tiling T: slice i gets three horizontal dominoes if $v_{2i-1} = 0$; if $v_{2i-1} = -1$ it gets a horizontal domino in row 0, a vertical domino in rows $\{1,2\}$ on each side whose neighboring even entry is 0 (or boundary), and two crossing horizontal dominoes in rows $\{1,2\}$ where the neighboring even entry is -1 (which agrees with the adjacent slice because $v \in L_k$); $v_{2i-1} = +1$ is the vertical mirror. Every cell is covered once, establishing a bijection $T \leftrightarrow v(T)$ onto L_k.

Flips and orientation. Any 2×2 block lies either inside one slice i (columns $2i-2, 2i-1$) or straddles an even line $2i$ (columns $2i-1, 2i$).

- Inside slice i: two horizontal dominoes in rows $\{r, r+1\}$ force $v_{2i-1} = 0$ (hence $v_{2i-2} = v_{2i} = 0$); turning them vertical leaves the third horizontal domino in row 2 ($r = 0$, changing $v_{2i-1} : 0 \to +1$) or row 0 ($r = 1$, changing $v_{2i-1} : 0 \to -1$). Because $2i-2$ is even, lower-left cell $(2i-2,0)$ is blue (so horizontal $\to$ vertical is upward, matching $0 \to +1$) and $(2i-2,1)$ is white (so vertical $\to$ horizontal is upward, matching $-1 \to 0$). Conversely, if $v, w \in L_k$ differ by 1 at odd $p = 2i-1$, the state with entry 0 forces both neighboring even entries to be 0 in both states, so slice i is closed on both sides and the change is this intra-slice flip.
- Across even line $2i$: two horizontal dominoes in rows $\{0,1\}$ represent $v_{2i} = +1$ (with $v_{2i-1} = v_{2i+1} = +1$), and lower-left cell $(2i-1,0)$ is white, so vertical $\to$ horizontal is upward ($0 \to +1$). Two horizontal dominoes in rows $\{1,2\}$ represent $v_{2i} = -1$ (with $v_{2i-1} = v_{2i+1} = -1$), and lower-left cell $(2i-1,1)$ is blue, so horizontal $\to$ vertical is upward ($-1 \to 0$). Conversely, if $v, w \in L_k$ differ by 1 at even $p = 2i$, the nonzero state forces both odd neighbors to share that sign, so columns $2i-1, 2i$ carry two parallel dominoes in the matching rows and the change is this cross-line flip.

Hence upward flips coincide exactly with unit coordinate increases $v \mapsto v + e_p$ inside L_k.

Distributivity and grading. For $v, w \in L_k$, let $m = \min(v, w)$ componentwise. At an even p, if $m_p = +1$ then $v_p = w_p = +1$, forcing $v_{p\pm1} = w_{p\pm1} = +1$ and hence $m_{p\pm1} = +1$; if $m_p = -1$ then say $v_p = -1$, forcing $v_{p\pm1} = -1$ and hence $m_{p\pm1} = -1$. Thus $m \in L_k$, and symmetrically $\max(v, w) \in L_k$. Being a sublattice of $\{-1,0,1\}^{2k-1}$, L_k is distributive. Moreover, whenever $v < w$ in L_k, some coordinate p satisfies $v_p < w_p$ and $v + e_p \in L_k$: pick any p with $v_p < w_p$; (i) if p is even and $v_p = -1$, setting coordinate p to 0 imposes no constraint, so $v + e_p \in L_k$; (ii) if p is odd and $v_p = 0$, both even neighbors are 0 in v, so $v + e_p \in L_k$; (iii) if p is odd and $v_p = -1$, either both even neighbors are 0 in v (giving $v + e_p \in L_k$) or an even neighbor p' has $v_{p'} = -1 < 0 \le w_{p'}$ (since $w_{p'} = -1$ would force $w_p = -1$), reducing to (i); (iv) if p is even with $v_p = 0, w_p = 1$, then $w_{p\pm1} = 1$, so either $v_{p\pm1} = 1$ (giving $v + e_p \in L_k$) or an odd neighbor p' has $v_{p'} < 1 = w_{p'}$, reducing to (ii) or (iii) (which cannot revisit p since $v_p = 0$). Hence the cover relations of L_k are precisely unit increases $v \mapsto v + e_p$, every maximal chain from $(-1,\dots,-1)$ to $(+1,\dots,+1)$ has length $4k-2$, and the rank is $\rho(v) = (2k-1) + \sum_{p=1}^{2k-1} v_p$.

Part (2): Generating function, increments, and transfer matrix. For $k \ge 1$, partition L_k by the last coordinate $s = v_{2k-1} \in \{0,+1,-1\}$, with corresponding Laurent polynomial sums S_k^0, S_k^+, S_k^- (weighted by $q^{\sum v_p}$). Deleting the last two coordinates $(b,s) = (v_{2k-2}, v_{2k-1})$ of $v \in L_k$ ($k \ge 2$) leaves $v' \in L_{k-1}$, and $(v', b, s) \in L_k$ if and only if $b = 0$, or $b = s = v'_{2k-3} \ne 0$. With $S_0 = 1$ and $S_0^+ = S_0^- = 0$, this yields for all $k \ge 1$:

$$\begin{aligned} S_k^0 &= S_{k-1}, \\ S_k^+ &= qS_{k-1} + q^2 S_{k-1}^+, \\ S_k^- &= q^{-1}S_{k-1} + q^{-2}S_{k-1}^-. \end{aligned}$$

In matrix form, $(S_k^0, S_k^+, S_k^-)^{\mathsf{T}} = T^k(1,0,0)^{\mathsf{T}}$, giving $S_k = (1,1,1)T^k(1,0,0)^{\mathsf{T}}$ with

$$T = \begin{pmatrix} 1 & 1 & 1 \\ q & q+q^2 & q \\ q^{-1} & q^{-1} & q^{-1}+q^{-2} \end{pmatrix}.$$

Subtracting q times row 1 from row 2 and q^{-1} times row 1 from row 3 shows $\det(T) = 1$; its trace is $1 + q + q^2 + q^{-1} + q^{-2} = [5]$, and the sum of its principal 2×2 minors is $q^2 + q^{-2} + (1 + q + q^{-1}) = [5]$, so its characteristic polynomial is $t^3 - [5]t^2 + [5]t - 1 = (t-1)(t^2 - ct + 1)$ since $c + 1 = [5]$.

Setting $F(x) = \sum_{k\ge0} S_k x^k$ and $F^{\pm}(x) = \sum_{k\ge0} S_k^{\pm} x^k$, the recurrences give $F^+ = xqF/(1-xq^2)$, $F^- = xq^{-1}F/(1-xq^{-2})$, and $F = 1 + xF + F^+ + F^-$. Using $xq(1-xq^{-2}) + xq^{-1}(1-xq^2) = xu(1-x)$ gives

$$F(x) \cdot \frac{(1-x)(1-cx+x^2)}{(1-xq^2)(1-xq^{-2})} = 1,$$

proving the rational formula for $\sum_{k\ge0} S_k x^k$. Multiplying by $1-x$ yields

$$\sum_{k=0}^{\infty} D_k x^k = \frac{1 - x(q^2+q^{-2}) + x^2}{1 - cx + x^2} = 1 + \frac{xu}{1-cx+x^2},$$

so $D_k = uV_{k-1}(c)$ for $k \ge 1$. Reading the denominators $(1-x)(1-cx+x^2) = 1 - [5]x + [5]x^2 - x^3$ and $1 - cx + x^2$ gives the stated recurrences for S_k and D_k.

Part (3): Hexagon-chain poset. Let P_k be the poset on $4k-2$ elements $\{b_p, t_p : 1 \le p \le 2k-1\}$ generated by $b_p < t_p$ for all p and, for every even p, $b_p < b_{p\pm1}$ and $t_{p\pm1} < t_p$. Sending $v \in L_k$ to $I(v) := \{b_p : v_p \ge 0\} \cup \{t_p : v_p = 1\}$ satisfies $t_p \in I(v) \implies b_p \in I(v)$ automatically, and closure under the remaining relations translates verbatim to $v_{p\pm1} \ge 0 \implies v_p \ge 0$ (i.e., $v_p = -1 \implies v_{p\pm1} = -1$) and $v_p = 1 \implies v_{p\pm1} = 1$. Thus $v \mapsto I(v)$ is a lattice isomorphism $L_k \cong J(P_k)$ with $|I(v)| = \sum_p (v_p + 1) = \rho(v)$. For even p, $b_p < b_{p+1} < t_{p+1} < t_p$ makes $b_p < t_p$ redundant as a cover, so for $k \ge 2$ the Hasse diagram of P_k consists of

$k-1$ hexagons $(b_{2i}, b_{2i-1}, t_{2i-1}, t_{2i}, t_{2i+1}, b_{2i+1})$ sharing the vertical rung $b_p < t_p$ at each interior odd p. □

Proof of Lemma 3.

Proof of Lemma 3. Part (1): Signed matchings to admissible UM-words. For $m \geq 0$, consider pairs (M, y) where M is a matching of disjoint adjacent edges $\{i, i+1\}$ on the path $\{1, \dots, m\}$ and y assigns each uncovered vertex i a step $y_i \in \{-1, 1\}$ (i odd) or $y_i \in \{-1, 0, 1\}$ (i even). Weight (M, y) by $(-1)^{|M|} q^{\sum_{i \notin V(M)} y_i}$, and let E_m be the signed sum over all pairs ($E_0 = 1, E_{-1} = 0$). Conditioning on whether vertex m is uncovered or covered by $\{m-1, m\}$ gives

$$E_m = w_m E_{m-1} - E_{m-2}, \qquad w_m = \begin{cases} u & (m \text{ odd}), \\ u+1 & (m \text{ even}). \end{cases}$$

Substituting $E_{2k} = (u+1)E_{2k-1} - E_{2k-2}$ into $E_{2k+1} = uE_{2k} - E_{2k-1}$ and using $uE_{2k-2} = E_{2k-1} + E_{2k-3}$ gives, for $k \geq 1$,

$$\begin{aligned} E_{2k+1} &= (u(u+1) - 2)E_{2k-1} - E_{2k-3} \\ &= c\, E_{2k-1} - E_{2k-3}. \end{aligned}$$

Since $E_{-1} = 0$ and $E_1 = u$, comparison with Proposition 2(2) yields $E_{2k-1} = uV_{k-1}(c) = D_k$.

Fix $m = 2k-1$. For a pair (M, y) and $1 \leq i \leq 2k-2$, set $h_i := \sum_{j<i, j \notin V(M)} y_j$. Call i a *trigger* of (M, y) if either $\{i, i+1\} \in M$, or both i and $i+1$ are uncovered and either $(y_i, y_{i+1}) = (1, -1)$ with $h_i \geq 0$ or $(y_i, y_{i+1}) = (-1, 1)$ with $h_i \leq -1$. If (M, y) has a trigger, let i be the least one and *switch* at i: replace $\{i, i+1\} \in M$ by uncovered values $(1, -1)$ if $h_i \geq 0$ or $(-1, 1)$ if $h_i \leq -1$, and conversely replace such an uncovered pair by the edge $\{i, i+1\}$. Because ± 1 are valid at both parities, the switch preserves $\sum y_j$, changes $|M|$ by ± 1 (reversing sign), and leaves h_j unchanged for $j \leq i$ and $j \geq i+2$. Hence no position $j \leq i-2$ changes trigger status. Position $i-1$ is a trigger in neither state: with $\{i, i+1\} \in M$, vertex i is covered by an edge other than $\{i-1, i\}$; with (y_i, y_{i+1}) uncovered, $\{i-1, i\} \notin M$, and if $i-1$ is uncovered then (y_{i-1}, y_i) cannot trigger because $h_i \geq 0$ implies $y_i = 1$ (and $(-1, 1)$ with $h_{i-1} \leq -1$ would force $h_i \leq -2$) while $h_i \leq -1$ implies $y_i = -1$ (and $(1, -1)$ with $h_{i-1} \geq 0$ would force $h_i \geq 1$). Thus i remains the least trigger, so switching is a weight-preserving, sign-reversing involution canceling all triggered pairs. The trigger-free pairs have $M = \emptyset$ and $h_i = H_{i-1}$, which are precisely the admissible UM-words W_k, proving $D_k = \sum_{y \in W_k} q^{H_{2k-1}(y)}$.

Part (2): Reflection involution Φ on W_k ($k \geq 2$). Fix $k \geq 2$ and $n = 2k-1$. Define an *event* of $y \in W_k$ to be a position $p \in \{1, \dots, n\}$ of one of four mutually exclusive types:

- **Type A:** p even, $H_{p-1} = 0$, and $y_p \in \{0, -1\}$;
- **Type B:** p even, $H_{p-1} = -1$, and $y_p \in \{0, 1\}$;
- **Type T:** p odd, $p \leq n-2$, $H_{p-1} = 0$, and $(y_p, y_{p+1}, y_{p+2}) \in \{(-1, -1, -1), (1, 0, 1)\}$;
- **Type T':** $p = n-1$ (even), $H_{p-1} = 1$, $y_{p-1} = y_{p+1} = -1$, and $y_p \in \{-1, 0\}$.

(Exclusivity is immediate: A, B, T' require distinct heights $0, -1, 1$ at an even p, while T requires odd p.) Call $y \in W_k$ *event-free* if it has no event.

For $y \in W_k$ carrying an event, let p be its least event position, and define $z = \Phi(y)$ by keeping $z_i = y_i$ for $i < p$ and setting:

- Type A: $z_p = -1 - y_p$, and $z_i = -y_i$ for $i > p$;
- Type B: $z_p = 1 - y_p$, and $z_i = -y_i$ for $i > p$;
- Type T: swap (y_p, y_{p+1}, y_{p+2}) between $(-1, -1, -1)$ and $(1, 0, 1)$, and $z_i = -y_i$ for $i > p+2$;
- Type T': $z_p = -1 - y_p$ and $z_n = y_n = -1$.

Set $m = p$ for types A, B and $m = p+2$ for type T. We verify four properties:

(i) *Mirrored endpoint:* $H_n(z) = -1 - H_n(y)$. For A, $H_p(y) = y_p$ and $H_p(z) = -1 - y_p$. For B, $H_p(y) = -1 + y_p$ and $H_p(z) = -y_p$. For T, H_{p+2} switches between -3 and 2, whose sum is -1. In all three cases $H_m(z) = -1 - H_m(y)$, and negating all steps for $i > m$ preserves $H_i(z) = -1 - H_i(y)$ through $i = n$. For T', $H_n(y) = 1 + y_p - 1 = y_p$ and $H_n(z) = -1 - y_p$.

(ii) *Admissibility:* $z \in W_k$. Odd positions remain in $\{-1, 1\}$. Every adjacent pair (z_i, z_{i+1}) with $i < p-1$ coincides with (y_i, y_{i+1}) at the same starting height. For $i \geq m+1$ (types A, B, T), $(z_i, z_{i+1}) = (-y_i, -y_{i+1})$ starts at height $H_{i-1}(z) = -1 - H_{i-1}(y)$, which is ≥ 0 if and only if $H_{i-1}(y) \leq -1$, so mirroring preserves admissibility. For the boundary pairs:

- Type A: at $i = p$ (existing because p is even and n is odd), $H_{p-1}(z) = 0$ and $z_p \leq 0$, so $(z_p, z_{p+1}) \neq (1, -1)$. At $i = p-1$, $H_{p-2}(z) = -y_{p-1}$; if $y_{p-1} = 1$ then $H_{p-2} = -1$ (only $(-1, 1)$ is forbidden, whereas the pair starts with $+1$), and if $y_{p-1} = -1$ then $H_{p-2} = 1$ (only $(1, -1)$ is forbidden, whereas the pair starts with -1).
- Type B: at $i = p$, $H_{p-1}(z) = -1$ and $z_p \geq 0$, avoiding $(-1, 1)$. At $i = p-1$, $H_{p-2}(z) = -1 - y_{p-1}$; if $y_{p-1} = 1$ it is -2 (pair starts with $+1$), and if $y_{p-1} = -1$ it is 0 (pair starts with -1).
- Type T: pairs at $i = p, p+1$ are $(-1, -1)$ or $(1, 0), (0, 1)$, never peaks or valleys. If $p \geq 3$, at $i = p-1$ we have $H_{p-2}(z) = -y_{p-1}$, safe for $y_{p-1} \in \{-1, 0, 1\}$ as above. If $p+3 \leq n$, at $i = p+2$: when z has $(1, 0, 1)$, $(1, -y_{p+3})$ starts at height 1 and is forbidden only if $y_{p+3} = 1$, which is impossible because $(-1, y_{p+3})$ in y started at height -2; when z has $(-1, -1, -1)$, $(-1, -y_{p+3})$ starts at height -2 and is forbidden only if $y_{p+3} = -1$, impossible because $(1, y_{p+3})$ in y started at height 1.
- Type T': pairs at $i = n-2$ and $n-1$ are $(-1, z_p)$ and $(z_p, -1)$ with $z_p \leq 0$, neither a peak nor a valley.

(iii) *Least event preserved and $\Phi(\Phi(y)) = y$.* An event at position p' depends only on $H_{p'-1}$ and $(y_{p'}, y_{p'+1}, y_{p'+2})$ (or (y_{n-2}, y_{n-1}, y_n) for T'). Since z agrees with y before p, an event of z strictly before p that was absent in y could only be type T at an odd $p' < p$ with $p' + 2 \geq p$, requiring $H_{p'-1}(y) = 0$. Checking all four cases: if p carries A or B, then $p' = p-1$ and $H_{p-1} = y_{p-1} \in \{-1, 1\}$ (contradicting $H_{p-1} = 0$ for A; for B giving $y_{p-1} = -1$, which cannot initiate $(-1, -1, -1)$ since $z_p \in \{0, 1\}$); if p carries T', then $p' = n-2$ and $H_{n-3} = H_{n-2} - y_{n-2} = 2 \neq 0$; if p carries T, then $p' = p-2$ and $H_{p-1} = y_{p-2} + y_{p-1} \in \{-2, 1\}$ (not 0). Thus z has no event before p, carries the same event type at p in its partner form, and $\Phi(z) = y$.

(iv) *Every event-free word stays nonnegative.* Suppose $y \in W_k$ attains a negative height, and let p be the least position

with $H_p = -1$; then $H_{p-1} = 0$ and $y_p = -1$. If p is even, p carries A. If p is odd with $p \le n-2$: if $y_{p+1} \in \{0,1\}$ then $p+1$ carries B (since $H_p = -1$); if $y_{p+1} = -1$, admissibility at height -1 prevents $(y_{p+1}, y_{p+2}) = (-1,1)$, so $y_{p+2} = -1$ and p carries T. Finally, if $p = n$ (here $k \ge 2$ ensures $n \ge 3$ and $n-1 \ge 2$ is even): $H_{n-1} = 0$; $y_{n-1} = 1$ would give $H_{n-2} = -1$, contradicting minimality of p; $y_{n-1} = 0$ gives $H_{n-2} = 0$ so $n-1$ carries A; and $y_{n-1} = -1$ gives $H_{n-2} = 1$, where $y_{n-2} = 1$ would create a forbidden peak $(1,-1)$ from height 0, forcing $y_{n-2} = -1$ and event T' at $n-1$. In every case y has an event.

Conclusion. By (i)–(iii), Φ pairs words in W_k with an event ending at $j \ge 0$ bijectively with those ending at $-1-j < 0$. By (iv), no event-free word ends below 0. Hence $a_j - a_{-1-j} = f_{k,j} \ge 0$ for all $j \ge 0$. Because $D_k(q) = uV_{k-1}(c)$ is invariant under $q \mapsto q^{-1}$, we have $a_{-1-j} = a_{j+1}$, giving $a_j - a_{j+1} = f_{k,j} \ge 0$ and $D_k(q) = \sum_{j=0}^{2k-1} f_{k,j}[2j+1]$.

For illustration at $k = 2$, W_2 has 8 words: Φ pairs $(1,0,1) \leftrightarrow (-1,-1,-1)$ (type T), $(-1,0,1) \leftrightarrow (-1,1,-1)$, and $(-1,0,-1) \leftrightarrow (-1,1,1)$ (type B), leaving the two event-free words $(1,0,-1)$ and $(1,1,1)$ at heights 0 and 3, so $D_2 = [1] + [7]$. In general, event-free words are explicitly characterized by four conditions: (1) all heights $H_p \ge 0$ and no $(1,-1)$ occurs; (2) every even-position step from height 0 is $+1$; (3) no $(1,0,1)$ starts at an odd position from height 0; and (4) the word does not end with $(-1,0,-1)$ at height 0. □

Proofs of Theorem 1 and Proposition 4.

Proof of Theorem 1. By Proposition 2, the rank polynomial of $L(3,2k)$ is $q^{2k-1}S_k(q)$, with $S_1(q) = 1 + D_1(q) = q^{-1} + 1 + q = [3]$ and $S_k(q) = S_1(q) + \sum_{i=2}^{k} D_i(q)$. By Lemma 3(2), each increment $D_i(q)$ $(i \ge 2)$ equals $\sum_{j=0}^{2i-1} f_{i,j}[2j+1]$ with $f_{i,j} \ge 0$. Summing over $i = 2, \dots, k$ yields

$$S_k(q) = \sum_{j=0}^{2k-1} F_{k,j}\,[2j+1], \qquad F_{k,j} = \mathbf{1}_{j=1} + \sum_{i=2}^{k} f_{i,j} \ge 0.$$

Every basis polynomial $[2j+1]$ is symmetric about exponent 0 and weakly decreases away from 0, so any nonnegative linear combination $S_k(q)$ satisfies $r_i = r_{4k-2-i}$ and $r_0 \le \dots \le r_{2k-1} \ge \dots \ge r_{4k-2}$. For $2k \times 3$, transposition reverses the rank sequence, preserving both symmetry and unimodality. □

Proof of Proposition 4. Part (1): On an $m \times n$ board with m odd, vertical reflection $(r,c) \mapsto (m-1-r, c)$ maps any 2×2 block with lower-left cell (r,c) to the block with lower-left cell $(m-2-r, c)$, whose parity $(m-2-r)+c \equiv r+c+1 \pmod 2$ is opposite while horizontal and vertical domino orientations are preserved; hence every upward flip becomes downward, inducing an anti-automorphism of $L(m,n)$ and giving $r_k = r_{R-k}$. Symmetrically, horizontal reflection reverses flip directions when n is odd. When $m = n$, diagonal transposition $(r,c) \mapsto (c,r)$ preserves cell parity while swapping horizontal and vertical dominoes, again reversing every upward flip and giving $L(m,m) \cong L(m,m)^{\mathrm{op}}$ and $r_k = r_{R-k}$.

Part (2): From Proposition 2(2), $S_1 = [3]$ (counts 1, 1, 1, so $q^2 + q + 1$ has no real root), S_2 has counts 1, 1, 2, 3, 2, 1, 1 ($r_1^2 = 1 < 2 = r_0 r_2$), S_3 has counts 1, 2, 3, 5, 6, 7, 6, 5, 3, 2, 1 ($r_2^2 = 9 < 10 = r_1 r_3$), and S_4 has counts 1, 3, 5, 9, 13, 16, 19, 21, 19, 16, 13, 9, 5, 3, 1 ($r_2^2 = 25 < 27 = r_1 r_3$). Since Newton's inequalities imply every real-rooted polynomial with positive coefficients is log-concave, real-rootedness fails for $k \in \{1,2,3,4\}$. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Standalone Local Replay and Negative-Mutation Controls. Every claim in Proposition 5(1) is verified deterministically in under 9 seconds by `recompute.py` (output saved in `recompute.out`):

1. *Worked boards and dual-engine agreement:* Brute-force BFS enumeration of the full flip graph matches the column-sweep transfer matrix on 2×3, 3×4, 4×4, 3×6, and 3×8, as well as on both non-unimodal staircase Young diagrams $(8,7,6,6,6,6,1)$ (580 tilings, 1,698 flips, dip 49, 48, 49 at ranks 12–14) and $(9,8,7,6,6,6,6,6,6,5,5,3,2,1)$ (1,468 tilings, 4,421 flips, dip 77, 76, 77 at ranks 16–18).
2. *Width-3 theorem verification:* Across 3×2 through 3×16 ($k = 1, \dots, 8$: 3, 11, 41, 153, 571, 2,131, 7,953, 29,681 tilings; 40,544 tilings and 196,228 flips in total), `recompute.py` checks the bijection $T \leftrightarrow v(T) \in L_k$, the exact flip-edge correspondence, rank $\rho(v) = (2k-1) + \sum v_p$, and agreement with $S_k(q)$, and extends the transfer-matrix match through 3×20 ($k = 10$). The signed-matching cancellation of Lemma 3(1) is checked exhaustively for $k = 1, \dots, 5$ (up to 7,810 pairs), the four-type reflection involution Φ is checked on all 29,678 admissible UM-words for $k = 2, \dots, 8$, and exact dynamic programming checks $f_{k,j} = a_j - a_{j+1} \ge 0$ and global log-concavity for $k = 5, \dots, 60$ (with failures solely at $k \in \{2,3,4\}$ through $k = 60$).
3. *Companion checks and Sylvester/Stanton contrast:* Across all 44 tileable rectangles with $2 \le m \le 8$ and $m \le n \le 12$, `recompute.py` confirms 44/44 unimodal, 30/30 odd-or-square boards palindromic, 0/14 even-unequal boards palindromic, and 44/44 obeying the classical alternating-sum rule ($\sum_k (-1)^k r_k \in \{0, \pm 1\}$, nonzero if and only if $\gcd(m+1, n+1) = 1$). It also confirms top rank $\binom{2k+1}{3}$ for 2×2, 4×4, 6×6, and 8×8, Sylvester box-partition counts for 2×3 (10 partitions) and 4×4 (70 partitions), Stanton's non-unimodal shape $(8,8,4,4)$ (dip 31, 30, 31 at sizes 14–16), and the Section 1 Python snippet verbatim.
4. *Negative-mutation self-checks (3/3 caught):* Lowering the peak of 3×6 from 7 to 5 triggers non-unimodality and $F_{3,0} = -1 < 0$; omitting event type T' from Φ leaks 2 negative-height words at $k = 5$ (the least k where T' is a first event, starting with $(1,0,-1,1,1,0,-1,-1,-1)$); and flipping the chessboard sign at cell $(0,0)$ on 3×4 breaks unique-bottom grading (2 bottom tilings).

Table 1 compiles the worked rectangle statistics and the exact increment multiplicities $f_{k,j}$ of $D_2, \dots, D_7$.

Archived Multi-Program Census and Empirical Picture. For boards beyond the fast local runtime of `recompute.py`, Ta-

Table 1. Top: exact rank statistics of representative rectangles verified by `recompute.py` (and OEIS `A004003` for squares). Bottom: exact flat-row multiplicities $f_{k,j} = a_j - a_{j+1}$ of the width-3 increments $D_k = \sum_{j=0}^{2k-1} f_{k,j}[2j+1]$ for $k = 2, \dots, 7$.

Board	Tilings	Top rank R	Peak max r_k	Palindromic?	Unimodal?	Central counts or notes
2×3	3	2	1	Yes	Yes	1, 1, 1 (fence)
3×4	11	6	3	Yes	Yes	1, 1, 2, 3, 2, 1, 1 (14 flips)
3×6	41	10	7	Yes	Yes	1, 2, 3, 5, 6, 7, 6, 5, 3, 2, 1
3×8	153	14	21	Yes	Yes	1, 3, 5, 9, 13, 16, 19, 21, 19, ..., 1
4×4	36	10	6	Yes	Yes	1, 1, 2, 5, **6**, **6**, **6**, 5, 2, 1, 1 (70 flips)
4×5	95	14	15	Yes	Yes	..., 10, 13, 15, 13, 10, ...
4×6	281	18	35	No	Yes	..., 31, **35**, **35**, 34, 29, ...
5×6	1,183	26	119	Yes	Yes	..., 106, 116, 119, 116, 106, ...
6×6	6,728	35	571	Yes	Yes	..., 548, **571**, **571**, 548, ...
8×8	12,988,816	84	646,036	Yes	Yes	$R = \binom{9}{3} = 84$; min bulk ratio 1.0144

Width-3 increment multiplicities: $f_{k,0}, f_{k,1}, \dots, f_{k,2k-1}$ for $D_k = S_k - S_{k-1}$ $(k = 2, \dots, 7)$

$k = 2$	$1, 0, 0, 1 \quad (D_2 = [1] + [7])$
$k = 3$	$0, 0, 2, 0, 1, 1 \quad (D_3 = 2[5] + [9] + [11])$
$k = 4$	$1, 2, 1, 3, 3, 1, 2, 1$
$k = 5$	$0, 4, 7, 7, 6, 8, 5, 3, 3, 1$
$k = 6$	$6, 12, 16, 22, 24, 20, 18, 16, 9, 6, 4, 1$
$k = 7$	$10, 34, 57, 65, 70, 72, 64, 50, 41, 29, 16, 10, 5, 1$

ble 2 summarizes the archived exact transfer-matrix sweeps carried out by four independent programs (Python exact integers, custom C multi-limb integers, C 128-bit integers, and C/GMP exact integers in a shielded blind re-review). Across all sweeps, zero rectangular boards dip. As an external sanity check on the largest square, the 16×16 run produced top rank $680 = \binom{17}{3}$ (matching Theorem C of Parlier and Zappa [7]) and exact total tiling count

$$2{,}444{,}888{,}770{,}250{,}892{,}795{,}802{,}079{,}170{,}816,$$

agreeing digit-for-digit with OEIS `A004003`. The independent width-3 checker also confirmed the involution on all 299,303,198 admissible UM-words up to $k = 15$ (3×30) and verified the closed form and log-concavity $(k \geq 5)$ through $k = 2{,}000$ $(3 \times 4{,}000)$.

Two empirical features explain why Conjecture 6 appears far from any counterexample in the data, in sharp contrast to general Young diagrams (Figure 3):

- *Bulk log-concavity and shared rising tails:* Define the *bulk* as ranks with $r_k \geq \frac{1}{4} \max_j r_j$. Among all 153 boards with $2 \leq m \leq 12$ and $m \leq n \leq 24$, only six small boards (2×4, 3×4, 3×6, 4×4, 4×5, 4×6) fail log-concavity inside the bulk; every board from 4×7 upward is log-concave throughout its bulk (Figure 3, Panel B: minimum bulk curvature $r_k^2/(r_{k-1}r_{k+1})$ is 1.0186, 1.0144, 1.0064, and 1.0032 on 6×6, 8×8, 10×10, and 12×12). In the blind re-review, 4,963 of 5,095 boards computed by `vstat.c` are globally log-concave, and all 132 failures occur within 10 ranks of an endpoint. Meanwhile, large even squares share a stabilizing, strictly increasing opening prefix (6×6 and 8×8 share their first 6 terms; 8×8 and 10×10 share 8; 10×10 and 12×12 share 10: 1, 1, 2, 5, 10, 18, 32, 59, 106, 181), leaving no place for a dip to form at either the ends or the bulk.
- *Sharp boundary on Young diagrams:* Across an archived census of all 2,173,011 tileable Young diagrams of area 40 through 60, exactly 1,194 fail unimodality (2, 4, 10, 18, 34, 54, 88, 130, 194, 274, and 386 at even areas 40 to 60), starting at area 40 with the staircase (8, 7, 6, 6, 6, 6, 1) and its transpose (Figure 3, Panel A). General simply connected regions fail even earlier (9 of 1,248 sampled 16-square regions). Thus rectangularity plays an essential structural role.

Table 2. Archived multi-program rectangular unimodality census (zero non-unimodal rectangles found across all four independent programs; logged in `docs/confirmed-strong-and-novel/domino-flips/`). Note that none of these sweeps is exhaustive in m and n jointly.

Program / Sweep	Arithmetic	Boards	Range covered	Dips
Sweep 1 (`qdp.py`)	Python exact	334	$m \in \{2,3\}$ to $n \le 80$; $4 \le m \le 10$ to 40; $11 \le m \le 12$ to 30	0
Sweep 2 (`rgf.c`)	C multi-limb	33 added*	Widths 13, 14 to 13×20, 14×16; 3×200; 4×80; 12×36	0
Sweep 3 (`phi.c`)	C 128-bit	153 + 9	All $2 \le m \le 12$, $m \le n \le 24$, plus 9 boards to 17×18 (16×16 `A004003`)	0
Re-review (`vstat.c`)	C + GMP	5,097	$m, n \ge 4$: width 4 to 2,000, 5 to 1,500, 6 to 1,000, all $4 \le m, n \le 18$, to 18×20	0

*Sweep 2 ran 51 executions (50 distinct boards), of which 33 lay outside Sweep 1's 334-board range; Sweep 3's 11 runs past width 14 cover 9 distinct boards.

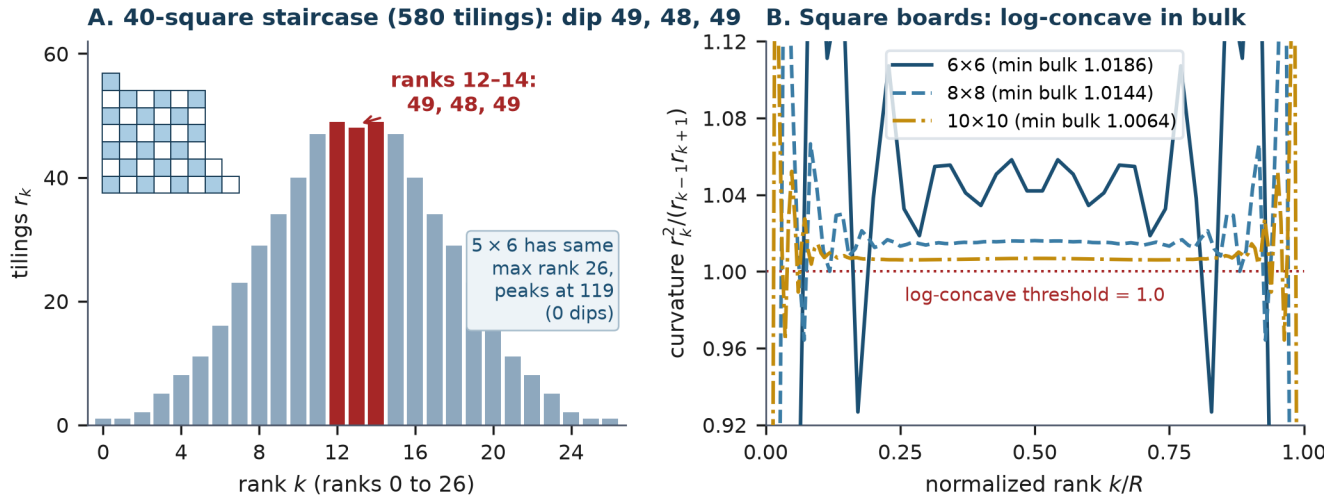


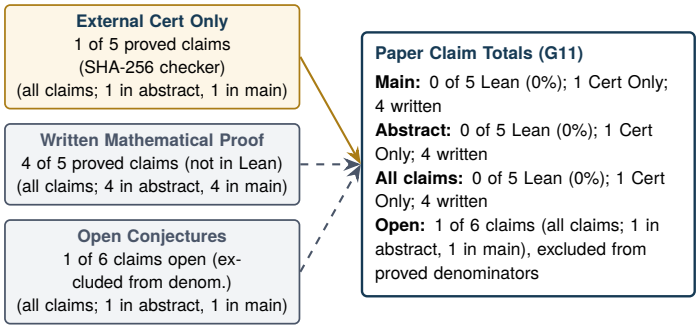
Figure 3. Boundary contrast between non-rectangular Young diagrams and rectangular boards. Panel A: rank distribution of the 40-square staircase $(8,7,6,6,6,6,1)$ (580 tilings, inset checkerboard), showing its central dip 49, 48, 49 at ranks 12–14, contrasted with the 5×6 rectangle (same top rank 26, 1,183 tilings, single peak 119). Panel B: curvature ratios $r_k^2/(r_{k-1}r_{k+1})$ versus normalized rank k/R for 6×6, 8×8, and 10×10 squares; values drop below 1.0 only near the ends and remain strictly above 1.0 throughout the bulk ($r_k \ge \frac{1}{4}\max r$).

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Every width-3 tileable rectangle 3 x 2k (and 2k x 3) is rank-symmetric and rank-unimodal, with S_k(q) = sum_{j=0}^{2k-1} F_{k,j}[2j+1] and F_{k,j} >= 0	Written	Written proof	Lean: none; fidelity: n/a	Written combinatorial proof deduced from Proposition 2 and Lemma 3 (Section 3); independently cross-checked on all 40,544 tilings of 3x2..3x16 and up to k=60 in recompute.py. No Lean 4 formalization claimed in this draft.
Prop. 2: Lattice isomorphism L(3, 2k) = L_k = J(P_k) (chain of k-1 hexagons), exact GF sum S_k x^k = (1-x q^2)(1-x q^{-2})/((1-x)(1-c x+x^2)), D_k = u V_{k-1}(c), order-3 recurrence, and 3x3 matrix T	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof via parity of vertical-line crossings, slice reconstruction, sublattice distributivity, and transfer recurrences (Section 3.1); verified through 3x16 (40,544 tilings) and k=60 in recompute.py. No Lean 4 formalization claimed.
Lem. 3: D_k counts admissible UM-words W_k (via signed-matching cancellation), and for k >= 2 the four-type reflection involution Phi proves a_j - a_{j+1} = f_{k,j} >= 0	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof via edge-switching involution on signed path matchings and four-type reflection involution Phi (Section 3.2); exhaustively tested on 29,678 admissible words (k=2..8) and DP to k=60 in recompute.py. No Lean 4 formalization claimed.
Prop. 4: Forced palindromy of L(m, n) when m or n is odd or m = n, and failure of log-concavity for 3x4, 3x6, 3x8 (and real-rootedness for k in {1,2,3,4})	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof via board reflections/transposition reversing flip orientation and explicit small-k coefficient inequalities (Section 3.3). No Lean 4 formalization claimed.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 5: Exact verification of worked boards, 40,544 width-3 tilings, 40-square staircase dip (49, 48, 49) and 76-square dip (77, 76, 77), plus multi-program rectangular census (0 dips to 17x18 including 16x16 A004003, and 5,097 re-review boards to 18x20)	Cert Only	recompute.py (fa6e57b3021d) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Part (1) replayed deterministically in recompute.py (<9s, 3/3 negative mutations caught); Part (2) cites archived exact multi-program transfer-matrix logs catalogued in docs/confirmed-strong-and-novel/domino-flips/.
Conj. 6: Every rectangle m x n with m >= 4, n >= 4, and mn even has a rank-unimodal domino flip lattice L(m, n)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open conjecture for widths m >= 4 (excluded from G11 denominators).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 66746caed16b | Fidelity SHA-256: 69805dd45769.

6. PRIOR WORK AND OPEN QUESTIONS

The distributive lattice structure of domino tilings under 2×2 flips goes back to Thurston's height-function framework [11] (`[to check]`) and was formalized systematically for planar regions by Desreux, Matamala, Rapaport, and Rémila [3] (`[to check]`). Parlier and Zappa [7] determined the flip-graph diameter of rectangular boards (Theorem C gives top rank $\binom{2k+1}{3}$ for $2k \times 2k$ squares), without addressing rank distributions or unimodality. On the unimodality side, Kantarcı Oğuz and Ravichandran [5] proved rank-unimodality for all fence posets (settling width $m = 2$, building on McConville, Sagan, and Smyth [6]). The broader genre parallels Sylvester's theorem on partitions in a rectangle and Stanton's analysis of non-unimodal subshapes in Young's lattice [10].

We explicitly credit two external results that arose during follow-up work and are not claimed here: (i) width 2 belongs to Kantarcı Oğuz and Ravichandran [5]; and (ii) a later observation connecting log-concavity of the flat-row multiplicities $F_{k,j}$ to log-concavity of the rank sequence r_k is an instance of the classical probability fact that a log-concave distribution has a log-concave survival sequence (increasing failure rate), stated in Ai, Karagiannis, Kuželka, and Pelekis [1] (`[to check]`). Similarly, Proctor's Dynkin-diagram classification of vertex-labelable posets [8] (`[to check]`, read via Zbl `0639.06006`) does not cover P_k for $k \geq 2$: solving Proctor's vertex-labeling equations on a single hexagon P_2 yields a contradiction ($\pi(a_2) + \pi(c_2) = 4 \neq 10$).

Prior-work searches logged. Four gated literature-search logs (all passing `litsearch.py check`, exit 0) were conducted across the conjecture and width-3 theorem: 52 and 40 structured query rows across OEIS, arXiv, and zbMATH (plus self-archives, catalogues, and web searches); 7 keyed CORE requests on 2026-11-19 (12 records: 3 relatives, 0 matching claims); two targeted width-3 sweeps on 2026-11-20 covering OEIS term queries for 3×6, 3×8, 3×10, and $f_{k,j}$, arXiv, zbMATH, Zenodo, DataCite, HAL, 104 GitHub repositories matching `"domino tilings"`, the 40 papers citing [5], and the 6 papers citing [3]; and a 5-query Google Scholar sweep on 2026-11-22 (50 records inspected). None of the rectangular rank sequences for $m \geq 3$ nor the multiplicity rows $f_{k,j}$ appears in the OEIS (only tiling totals `A001835`, `A004003`, differences `A052530`, and square diameters `A000447`). No source stating or proving rank-unimodality for rectangles of width $m \geq 3$, or deriving the $3 \times 2k$ rank formula, was found in the searches logged.

Unread and unsearched items (`[to check]`). The following sources remain unread in full or unsearched and are flagged `[to check]` in `references-checked.md`: Stanley's 1989 unimodality survey [9]; Brenti's 1994 update [2]; Proctor's full paper [8] (only the zbMATH review was read); Gansner's 1982 up-down poset paper [4]; full texts of [11, 3, 1]; and print-only books (including exercises on distributive lattices in Stanley's *Enumerative Combinatorics*, Vol. 1, and Croft–Falconer–Guy).

Open questions. Beyond Conjecture 6 (widths $m \geq 4$), three concrete width-3 questions remain open: (1) prove that the $3\times2k$ rank sequence is globally log-concave for every $k \geq 5$ (verified to $k = 2{,}000$; the minus sign in $S_k = [5](S_{k-1} - S_{k-2}) + S_{k-3}$ blocks standard preservation theorems); (2) give a second-reader check and closed form for strict positivity $f_{k,j} > 0$ at $k = 4$ and all $k \geq 6$; and (3) determine whether $L(3, 2k)$ admits a symmetric chain decomposition or the strong Sperner property (Lemma 3 counts by an involution and constructs no chains).

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, receipts, and the standalone exact verifier live in `papers/domino-flips/draft1/`. Running `python3 papers/domino-flips/draft1/recompute.py` finishes in under 9 seconds and deterministically verifies all worked

boards (2×3, 3×4, 4×4, 3×6, 3×8), both non-unimodal staircase counterexamples, all 40,544 tilings of 3×2 through 3×16, all 29,678 admissible UM-words for $k = 2, \ldots, 8$, dynamic-programming multiplicity checks to $k = 60$, all 44 companion boards, Sylvester and Stanton checks, the Section 1 Python snippet, and 3/3 negative-mutation self-checks. Archived logs for the larger multi-program census ($334 + 33 + 9$ boards to 17×18 and the 5,097-board re-review) are catalogued in `docs/confirmed-strong-and-novel/domino-flips/`. No Lean 4 formalization is claimed in this draft (a separate Lean 4 development covering width-3 and cover-supported Lefschetz obstructions resides under `domino-lefschetz` in the companion repository).

AI research agents built and directed by Tyler B. Hughes proposed the conjecture, discovered and independently re-derived the width-3 proof, executed the computational sweeps and verification scripts, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 24, 2026

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