MATHDISCOVERIES
Mathematical Preprint

Working Draft

Erdős Matching Conjecture, $s = 2$ (Erdős #1020): Conditional $D(5) = 2{,}366$, All-r Bounds, and a Conjecture H Refutation

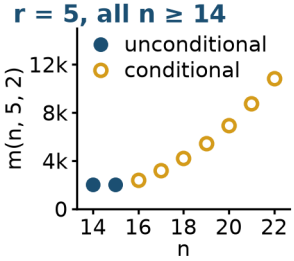
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Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.

The question

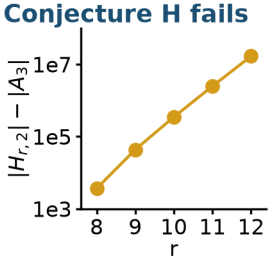
Largest family of r-sets of [n], n ≥ 3r − 1, with no three pairwise disjoint members?

Guess: the larger of all r-sets of [3r − 1] and the 2-star on {p, q}



Diagonal n = 3r + 1

J ≤ 1	closed, Thm 7
2 ≤ J ≤ r − 3	open
J ≥ r − 2	closed, r ≥ 5



ABSTRACT: Let $m(n,r,2)$ be the largest size of a family of r-subsets of $[n]$ with no three pairwise disjoint members; the case $s = 2$ of the Erdős Matching Conjecture (Erdős Problem #1020) predicts $m(n,r,2) = \max\{\binom{3r-1}{r}, \binom{n}{r} - \binom{n-2}{r}\}$ for $n \geq 3r-1$. We prove the upper bound for $1 \leq r \leq 3$ and all $n \geq 3r-1$, and $m(15,5,2) = 2{,}002$. If the Lean checker for reverse unit propagation (RUP) accepts clause lists for two explicit CNFs, then $m(n,5,2) = \max\{2{,}002, \binom{n}{5} - \binom{n-2}{5}\}$ for every $n \geq 14$, so $m(16,5,2) = 2{,}366$; if it also accepts one for a third formula, the complete 2-stars are the only extremal families for $n \geq 16$. No such list is part of this paper's checked material. On $3r+1$ points we prove a Katona-cycle bound, a Hilton–Milner separation, a slice bound for $r \geq 5$, refutations of Conjecture H at level $j = 2$ for $r \geq 8$ and $j = 1$ for $r \geq 6$, and weights on the r-subsets with totals above the 2-star size: Raney–star for $r \geq 4$, linear ramp for $r \geq 7$. Main results: 14 of 14 are Lean 4 kernel theorems, 2 of them conditional on the RUP hypotheses. No human mathematician has yet refereed the proof.

Verification Scope (G11): WARN: Human referee: pending
Main results: 14 of 14 in Lean Kernel (100%) · 1 open (not counted)
Abstract claims: 11 of 11 in Lean Kernel (100%)

Full Trust Boundary: Section 5.

MSC 2020: 05D05, 05C65, 05C70, 68V20 **Keywords:** Erdős matching conjecture, uniform hypergraphs, shifting, Hilton–Milner, Raney cycle lemma, RUP certificates, Lean 4

1. INTRODUCTION AND MAIN RESULTS

How many committees of r people can be drawn from n people if no three may be pairwise disjoint? Erdős posed this question in 1965 for any forbidden matching size [6]; his threshold $f(n;r,3)$ is one more than the maximum size $m(n,r,2)$ of an r-uniform family $\mathcal{F}$ on $[n] = \{1,\dots,n\}$ with no three pairwise disjoint members. Two families compete for the maximum: the clique $\binom{[3r-1]}{r}$ and the complete 2-star of all r-subsets meeting a fixed pair $\{p,q\}$. The case $s = 2$ of the Erdős Matching Conjecture predicts, for $r \geq 1$ and $n \geq 3r-1$,

$$m(n,r,2) = \max\left\{\binom{3r-1}{r},\ \binom{n}{r} - \binom{n-2}{r}\right\}. \tag{1}$$

The graph case $r = 2$ goes back to Erdős and Gallai [7], large-n bounds to [2, 18, 8, 9], and Frankl's link induction [10] reduces (1) for all $r \leq K$ and $n \geq 3r-1$ to the diagonal bounds

$$D(r): \quad m(3r+1,r,2) \leq \binom{3r+1}{r} - \binom{3r-1}{r}, \qquad 1 \leq r \leq K,$$

whose right-hand sides are 2, 11, 64, 385, and 2,366 for $r = 1,\dots,5$. We write $D(5) = 2{,}366$ for the statement $D(5)$, which together with the 2-star construction yields $m(16,5,2) = 2{,}366$.

Scope and verification. All fourteen main theorems are Lean 4 kernel theorems of `ErdosMatching5` (Lean 4 [21], mathlib [24]; Section 5). 12 of the 14 main results and 9 of the 11 abstract claims are unconditional; Theorems 1 and 2 are kernel-checked implications whose hypotheses are not discharged (requiring reverse unit propagation (RUP) clause lists for three Lean CNFs accepted by a kernel-verified checker). An earlier research record cites `drat-trim` [27, 16] refutations for related encodings, but no refutations are in the checked material (Section 4), so both $r = 5$ results stay conditional.

1.1 Main results.

Theorem 1 ($r = 5$ for all $n \geq 14$, conditional). *Suppose the CNF formulas Ψ_4 and Ψ_5 of Definition 17 each have a proof accepted by the RUP checker* check *of Definition 17. Then for every $n \geq 14$, every 5-uniform family $\mathcal{F}$ of subsets of $[n]$ with no three pairwise*

disjoint members satisfies $|\mathcal{F}| \le \max\{2002, \binom{n}{5} - \binom{n-2}{5}\}$, *and some such family attains equality.*

Theorem 2 (Unshifted uniqueness for $r = 5$, conditional). *Suppose* Ψ_4, Ψ_5 *and* $\Psi_5^\star$ *of Definition 17 each have a proof accepted by* check. *Then for every* $n \ge 16$ *and every* 5*-uniform family* $\mathcal{F}$ *of subsets of* $[n]$ *with no three pairwise disjoint members,* $|\mathcal{F}| = \binom{n}{5} - \binom{n-2}{5}$ *holds exactly when* $\mathcal{F}$ *is a complete* 2*-star, that is,* $\mathcal{F} = \{E \in \binom{[n]}{5} : p \in E \text{ or } q \in E\}$ *for some distinct* $p, q \in [n]$.

Theorem 3 ($r = 5$, $n = 15$). *Every* 5*-uniform family of subsets of* $[15]$ *with no three pairwise disjoint members has at most* 2002 *members, and some such family has exactly* 2002 *members.*

Theorem 4 ($r \le 3$, all n). *For* $1 \le r \le 3$ *and* $n \ge 3r - 1$, *every* r*-uniform family* $\mathcal{F}$ *of subsets of* $[n]$ *with no three pairwise disjoint members satisfies* $|\mathcal{F}| \le \max\{\binom{3r-1}{r}, \binom{n}{r} - \binom{n-2}{r}\}$.

Theorem 5 (Unshifted uniqueness for $r = 2$). *For* $n \ge 7$ *and every* 2*-uniform family* $\mathcal{F}$ *of subsets of* $[n]$ *with no three pairwise disjoint members,* $|\mathcal{F}| = \binom{n}{2} - \binom{n-2}{2}$ *holds exactly when* $\mathcal{F}$ *is a complete* 2*-star.*

Theorem 6 (Unshifted uniqueness for $r = 3$). *For* $n \ge 10$ *and every* 3*-uniform family* $\mathcal{F}$ *of subsets of* $[n]$ *with no three pairwise disjoint members,* $|\mathcal{F}| = \binom{n}{3} - \binom{n-2}{3}$ *holds exactly when* $\mathcal{F}$ *is a complete* 2*-star.*

On the diagonal $n = 3r+1$, write $A_1 = A_1(3r+1, r)$ for the complete 2-star centred at $\{1, 2\}$, so $|A_1| = \binom{3r+1}{r} - \binom{3r-1}{r}$.

Theorem 7 (Katona-cycle bound). *Let* $r \ge 2$, *let* $\mathcal{F}$ *be an* r*-uniform family of subsets of* $[3r+1]$ *with no three pairwise disjoint members, and let* $p \ne q$ *be points of* $[3r+1]$. *If every two members of* $\mathcal{F}$ *that avoid both* p *and* q *intersect, then* $|\mathcal{F}| \le |A_1|$.

Theorem 8 (Hilton–Milner gap). *Let* $r \ge 2$, *let* $\mathcal{F}$ *be an* r*-uniform family of subsets of* $[3r + 1]$ *with no three pairwise disjoint members, let* $p \ne q$ *be points of* $[3r + 1]$, *and suppose* $\mathcal{F}_0 = \{E \in \mathcal{F} : p \notin E,\ q \notin E\}$ *is nonempty (two members of* $\mathcal{F}_0$ *may be disjoint). Then* $|\mathcal{F}| + \binom{2r-1}{r-1} \le |A_1| + |\mathcal{F}_0|$.

Theorem 9 (Hilton–Milner separation). *Let* $r \ge 4$, *let* $p \ne q$ *be points of* $[3r + 1]$ *and let* E_0 *be an* r*-subset of* $[3r + 1]$ *containing neither* p *nor* q. *Then* $|\mathcal{H}_{r,1}| < |\mathcal{H}^{(r)}(3r + 1, 2; p, q, E_0)|$.

For a left-shifted r-uniform family $\mathcal{F}$ on $[3r + 1]$ with no three pairwise disjoint members and $J_r(\mathcal{F}) \ge j$ $(1 \le j \le r - 1)$, the natural inductive bound $|\mathcal{F}| \le |A_{j+1}(3r + 1, r)|$ is our own working hypothesis, which we call *Conjecture H* (it is not taken from the literature). The next two theorems refute it.

Theorem 10 (Conjecture H fails at $j = 2$). *For every* $r \ge 8$, *the family* $\mathcal{H}_{r,2}$ *is left-shifted,* r*-uniform, has no three pairwise disjoint members, satisfies* $J_r(\mathcal{H}_{r,2}) = 2$, *and* $|A_3(3r + 1, r)| < |\mathcal{H}_{r,2}| < |A_2(3r + 1, r)|$ *and* $|\mathcal{H}_{r,2}| < |A_1|$.

Theorem 11 (Conjecture H fails at $j = 1$). *For every* $r \ge 6$, *the family* $\mathcal{H}_{r,1}$ *is left-shifted,* r*-uniform, has no three pairwise disjoint members, satisfies* $J_r(\mathcal{H}_{r,1}) = 1$, *and* $|A_2(3r + 1, r)| < |\mathcal{H}_{r,1}| < |A_1|$.

Theorem 12 (Slice bound). *Let* $r \ge 5$ *and let* $\mathcal{F}$ *be a left-shifted* r*-uniform family of subsets of* $[3r + 1]$ *with no three pairwise disjoint members that contains two disjoint members avoiding* $[r-2]$. *Then* $|\mathcal{F}| < |A_1|$.

Theorem 13 (Raney–star weight total). *For every* $r \ge 4$, $|A_1| < \sum_E \hat{y}_E$, *the sum taken over all* r*-subsets* E *of* $[3r + 1]$, *with* $\hat{y}$ *as in Definition 16.*

Theorem 14 (Linear-ramp weight total). *For every* $r \ge 7$, $|A_1| < \sum_E y^*_E$, *the sum taken over all* r*-subsets* E *of* $[3r + 1]$, *with* y^* *as in Definition 16.*

Theorems 7 and 12 settle shifted families with $J_r \le 1$ and $J_r \ge r - 2$ $(r \ge 5)$, narrowing $D(r)$ for $r \ge 6$ to $2 \le J_r \le r - 3$. Our working hypothesis Conjecture H (above; not from the literature) fails at $j = 2$ $(r \ge 8)$ and $j = 1$ $(r \ge 6)$, while Theorems 13–14 (with feasibility in Appendix A) show that two natural fractional relaxations have optimum above $|A_1|$ and cannot prove $D(r)$.

Conjecture 15 (Erdős, diagonal case $s = 2$). *$D(r)$ holds for every $r \ge 4$.*

Babanskyy [1] reports a computer-assisted proof of the 4-uniform conjecture for all $s \ge 1$, which would give $D(4)$ (not replayed in Lean here; Section 6).

Figure 1 summarizes the architecture. The box below checks the refutation of Conjecture H at $r = 8$ and $|A_1(16, 5)| = 2{,}366$; `recompute.py` runs it verbatim and confirms the counts by direct enumeration over $\binom{[25]}{8}$.

Instant Witness Check (verbatim in `recompute.py`)

```
from math import comb as C
def A(n, r, l):  # |A_l(n, r)|
    return sum(C(3*l-1, i) * C(n-3*l+1, r-i)
               for i in range(l, r + 1))
def H(r, j):     # |H_{r,j}|, j >= 2
    q, m = r - j, 3*j - 1
    return (A(3*r+1, r, j) - C(m, j) * C(2*q, q)
            + C(m, j-1) * (q+2) + C(m, j-2))
assert A(25, 8, 3) < H(8, 2) < A(25, 8, 2)
print(H(8, 2) - A(25, 8, 3), A(16, 5, 1))
```

The box prints `3653 2366`: $\mathcal{H}_{8,2}$ beats $A_3(25, 8)$ by 3,653 edges and $|A_1(16, 5)| = 2{,}366$.

Organization. Section 2 fixes notation; Section 3 proves Theorems 1–14; Sections 4–5 give replay, RUP status, and the trust boundary; Sections 6–7 cover prior work, open questions, and data; Appendices A–B give uniform proofs and companion windows.

2. DEFINITIONS, NOTATION, AND SETUP

Indexing and binomial conventions. Lean vertices live in $\mathrm{Fin}\, n = \{0, \dots, n-1\}$, identified with $[n]$ via $x \mapsto x+1$. For an r-set E and $m \ge 0$, put $c_E(m) = |E \cap [m]|$ (Lean: `prefixCount`), with $[0] = \emptyset$. Natural subtraction is exact under all theorem hypotheses (Table 2), except $3r - 1 \mapsto 0$ at $r = 0$ in $D(r)$ and Ψ_r; in counting proofs, $\binom{a}{b} = 0$ for $b < 0$ (Lean: `franklSub2`, so $\binom{3j-1}{j-2} = 0$ at $j = 1$).

Families, shifting, and J_r. A family $\mathcal{F}$ on $[n]$ is r-uniform if every member has size r, and has *no three pairwise disjoint members* (`NoThreeMatching`) if no $A, B, C \in \mathcal{F}$ are pairwise

(i) What was found	(ii) How it was found	(iii) How it is proved
r = 5: m(n, 5, 2) for all n ≥ 14; unique 2-stars for n ≥ 16	shifting, link reduction, shifted CNF; upstream SAT runs (files not here)	**Lean kernel, conditional** Thms 1–2; RUP hypotheses open
m(15, 5, 2) = 2,002, the case n = 3r at r = 5	double count over ordered partitions of [3r] into three r-sets	**Lean kernel** Thm 3, unconditional
1 ≤ r ≤ 3 for all n ≥ 3r − 1; unique 2-stars: r = 2 for n ≥ 7, r = 3 for n ≥ 10	integer dual certificates on the [3r]-board; complementary slackness	**Lean kernel** Thms 4–6, unconditional
Katona-cycle bound (r ≥ 2), Hilton–Milner gap (r ≥ 2) and separation (r ≥ 4), slice bound (r ≥ 5)	two-point decomposition, circle method, double counting	**Lean kernel** Thms 7–9 and 12
Conjecture H fails: j = 2 for r ≥ 8, j = 1 for r ≥ 6	two-part Gale down-sets $H_{r,j}$, exact counting and an induction	**Lean kernel** Thms 10–11
Weight totals above the 2-star: Raney-star (r ≥ 4), linear ramp (r ≥ 7)	Raney's cycle lemma and explicit rational weights	**Lean kernel** Thms 13–14
D(r) unproved here for r ≥ 4 (r = 4 claimed by Babanskyy; for r ≥ 6 open at 2 ≤ J ≤ r − 3)	small ranks: Thm 4; outer slices: Thms 7 and 12	**Unsettled** Conjecture 15, open

Figure 1. Discovery and proof architecture: (i) what was found, (ii) how it was found, and (iii) how each part is proved. Conditional rows are Lean 4 kernel theorems whose RUP hypotheses are not discharged here; the final row is unsettled.

disjoint (automatically distinct for $r \geq 1$). For $\ell \geq 1$, Frankl's families are $A_\ell(n,r) = \{E \in \binom{[n]}{r} : c_E(3\ell-1) \geq \ell\}$, so $A_1(n,r)$ is the complete 2-star centred at $\{1,2\}$ (`starTwoFamily`). A *complete* 2-*star* (`IsStarTwo`) is $\{E \in \binom{[n]}{r} : p \in E \text{ or } q \in E\}$ (`starTwoOn`) for distinct $p,q \in [n]$. For $i < j$, the shift $s_{ij}(A)$ replaces j by i if $j \in A$ and $i \notin A$; $S_{ij}(\mathcal{F})$ applies s_{ij} to each $A \in \mathcal{F}$ whose image is not already in $\mathcal{F}$, and $\mathcal{F}$ is *left-shifted* (`IsShifted`) if $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $i < j$. In the Gale order [13] [to check], $\{a_1 < \cdots < a_r\} \preceq \{b_1 < \cdots < b_r\}$ means $a_k \leq b_k$ for all k; left-shiftedness is downward closure under $\preceq$. On $[3r+1]$, let $J_r(\mathcal{F})$ (`jInvariant`) be the largest $j \in \{0,\dots,r-1\}$ such that $\mathcal{F}$ has two disjoint members avoiding $[j]$ (`HasDisjointAvoiders`), or -1 if none exists.

Extremal constructions. For $r \geq 2$ and $1 \leq j \leq r-1$, set $K = r + 2j + 1$ and define (`hybridH`) $\mathcal{H}_{r,j} = \{E \in A_j(3r+1,r) : c_E(K) \geq j+1\} \cup \binom{[K]}{r}$. For distinct $p,q \in [n]$ and $E_0 \in \binom{[n]\setminus\{p,q\}}{r}$, the Hilton–Milner family [2, 11] (`hiltonMilnerTwo`) is $\mathcal{H}^{(r)}(n,2;p,q,E_0) = \{E \in \binom{[n]}{r} : p \in E,\ E = E_0, \text{ or } (q \in E,\ E \cap E_0 \neq \emptyset)\}$.

Definition 16 (Raney sets and weights). $\mathcal{Z}_r$ is the set of r-subsets E of $[3r+1]$ with $3c_E(m) \leq m$ for every $0 \leq m \leq 3r+1$. For an r-subset E of $[3r+1]$ let $\hat{y}_E = 1$ if $1 \in E$, $\hat{y}_E = 0$ if $1 \notin E$ and $E \in \mathcal{Z}_r$, and $\hat{y}_E = \frac{1}{2}$ otherwise. Let $S(E) = c_E(2r+2)$ and let $y^*_E = 0$ if $E \in \mathcal{Z}_r$ or $S(E) \leq 2$, and $y^*_E = (S(E)-2)/(r-2)$ otherwise (a rational number).

Definition 17 (CNF formulas and the RUP checker). A clause c is *RUP* over a CNF G if negating every literal of c either yields an immediate contradiction or reaches a conflict within $|G|+1$ passes of unit propagation (`isRUP`, `SATCert5.lean` lines 285–324). The checker check(F,π) (`checkRUPProof`, lines 358–364) scans π in order starting with $G = F$, verifying each clause is RUP over G, adding nonempty accepted clauses, returning true at the first accepted empty clause, and false on any failed step or exhausted list. For n,r,T, the formula $\Phi(n,r,T)$ (`shiftedFamilyAtLeastCNF`) has edge variables x_E ($E \in \binom{[n]}{r}$) and sequential counter variables [26] forcing left-shiftedness, no three disjoint members, and size at least T. Put $\Psi_r = \Phi(3r+1, r, \binom{3r+1}{r} - \binom{3r-1}{r} + 1)$ (so Ψ_4 and Ψ_5 ask for 386 and 2,367 members), and let $\Psi^\star_5 = \Phi(16,5,2366) \wedge x_{\{3,4,5,6,7\}}$. Box 1 (Section 5) prints the core definitions verbatim, indexes helpers, and lists two Lean sanity checks (`checkBoardDualCert_r2`, `rup_check_emc1`).

3. PROOF OF THE MAIN RESULTS

3.1 Shifting, $n = 3r$, and Frankl's reduction.

Extremal lower bound. For $r \geq 1$ and $n \geq 3r-1$, $\binom{[3r-1]}{r}$ has no three disjoint members (needing $3r$ points), and $A_1(n,r)$ has none (at most two disjoint sets meet $\{1,2\}$), giving size $\max\{\binom{3r-1}{r}, \binom{n}{r} - \binom{n-2}{r}\}$ (`exists_extremal_lower_bound`).

Kleitman's uniform $n = 3r$ bound and Theorem 3. While Kleitman treated nonuniform families [20] [to check], on $[3r]$ every ordered partition into three r-sets meets $\mathcal{F}$ in at most two blocks. Counting across all $\binom{3r}{r}\binom{2r}{r}$ ordered partitions (each r-set appears in $3\binom{2r}{r}$) yields $3|\mathcal{F}| \leq 2\binom{3r}{r} = 3\binom{3r-1}{r}$, with equality for $\binom{[3r-1]}{r}$ (`kleitman_3r_exact`). Setting $r = 5$ gives $\binom{14}{5} = 2{,}002$, proving Theorem 3.

Shifting and diagonal reduction. Each shift S_{ij} preserves uniformity, cardinality, and the absence of three pairwise disjoint members: if $A, B, C \in S_{ij}(\mathcal{F})$ were pairwise disjoint, at most one contains i, and undoing that shift (and, if j lies in one of the other two sets, using that this set survived the shift because its image is already in $\mathcal{F}$) gives three pairwise disjoint members of $\mathcal{F}$. A nontrivial shift strictly reduces $\sum_{A\in\mathcal{F}}\sum_{x\in A} x$, so repeated shifting yields a left-shifted family of equal size. By Frankl's link induction [10] (`diagonal_implies_all_n`), if $D(1),\dots,D(K)$ hold, then (1) holds for $1 \le r \le K$ and $n \ge 3r-1$: $r=1$ is trivial, $n = 3r-1, 3r, 3r+1$ follow from $\binom{[3r-1]}{r}$, the $n = 3r$ count, and $D(r)$, and for $n = m+1 \ge 3r+2$ splitting a shifted $\mathcal{F}$ into $\mathcal{F}_0 = \{E \in \mathcal{F} : m+1 \notin E\}$ and the link $\mathcal{F}_1 = \{E \setminus \{m+1\} : m+1 \in E \in \mathcal{F}\}$ gives no three disjoint members in $\mathcal{F}_1$ (else adding two free points of $[m]$ and $m+1$ contradicts shiftedness). Because $\binom{m}{r} - \binom{m-2}{r} = \binom{m-1}{r-1} + \binom{m-2}{r-1} \ge 2\binom{3r-1}{r-1} = \binom{3r-1}{r}$ for every $m \ge 3r+1$ (`choose_clique_le_starTwo`), the 2-star size is the larger term of the maximum for both (m, r) and $(m, r-1)$; induction and Pascal's rule then complete the proof.

3.2 Dual certificates, CNF bridge, and Theorems 1, 4.

Proof of Theorem 4. By the diagonal reduction it suffices to prove $D(1)$, $D(2)$, and $D(3)$. $D(1)$ is immediate. For a left-shifted $\mathcal{F}$ on $[3r+1]$, set $\mathcal{F}' = \mathcal{F} \cap \binom{[3r]}{r}$. Mapping each $S \cup \{3r+1\} \in \mathcal{F}$ to $S \cup \{u\}$ ($u = \max([3r] \setminus S)$) yields $|\mathcal{F}| \le \sum_{A\in\mathcal{F}'}(1 + t(A))$, where $t(A)$ is the length of the trailing run of A ending at $3r$. Shiftedness forces $x_B = 0$ for every Raney set $B \in \binom{[3r]}{r}$ whose k-th element is at least $3k$ for all k, since $\{q, q+3, \dots, q+3r-3\}$ ($q = 1, 2, 3$) all lie $\le B$. Indexing $\binom{[3r]}{r}$ lexicographically as $B_0, B_1, \dots$ with $B_0 = [r]$ and indicators $x_i \in \{0,1\}$, valid rows are $x_i \le 1$, $x_v \le x_u$ for $B_u \le B_v$, and $x_a + x_b + x_c \le 2$ for pairwise disjoint triples. Any nonnegative integer dual combination whose net coefficient on every non-Raney x_i is at least $1 + t(B_i)$ bounds $|\mathcal{F}|$ by its right-hand side (`card_le_of_boardDualCert`); kernel evaluation checks `dualCertR2` $(11, r=2)$ and `dualCertR3` $(64, r=3)$, replayed in Section 4. □

CNF bridge and RUP soundness. For every $r \ge 0$, Ψ_r is unsatisfiable exactly when $D(r)$ holds (`diag_iff_unsat_shiftedEMCCNF`): any counterexample shifts to a model of Ψ_r with sequential counter values, and conversely any model decodes to an oversized shifted family. Moreover, if σ satisfies a CNF G, then σ satisfies every clause c that is RUP over G (else unit propagation from the negated literals of c could never conflict); hence acceptance of π by check(F, π) implies F is unsatisfiable (`checkRUPProof_sound`).

Proof of Theorem 1. RUP soundness and the CNF bridge turn the hypotheses on Ψ_4, Ψ_5 into $D(4)$ and $D(5)$. Combining $D(1)$–$D(3)$ (Theorem 4) with the reduction and lower bound ($\binom{14}{5} = 2{,}002$) yields all $n \ge 14$. □

3.3 Unshifted uniqueness (Theorems 2, 5, and 6).

Kneser lifting and propagation. If $|U| \ge 2k+1$, any two-colouring of $\binom{U}{k}$ constant on disjoint pairs is constant everywhere (the Kneser graph is connected via single-element swaps [22]). Suppose $r \ge 2, n \ge 3r$, $\mathcal{F}$ is an r-uniform family on $[n]$ with no three pairwise disjoint members, and $S_{ij}(\mathcal{F})$ is the complete 2-star centred at $\{p, q\}$. If $\{i, j\} = \{p, q\}$ or $\{i, j\} \cap \{p, q\} = \emptyset$, then $\mathcal{F} = S_{ij}(\mathcal{F})$; $j \in \{p, q\}$, $i \notin \{p, q\}$ is impossible; and when $p = i$, $q \notin \{i, j\}$, colouring $X \in \binom{U}{r-1}$ on $U = [n] \setminus \{i, j, q\}$ ($|U| \ge 2r-1$) by $\{i\} \cup X \in \mathcal{F}$ forbids disjoint bichromatic pairs (else a third disjoint set $\{q\} \cup Z$ appears), so the colouring is constant and $\mathcal{F}$ is the 2-star on $\{i, q\}$ or $\{j, q\}$. Inducting on shift weight lifts shifted uniqueness to all families. Finally, if $D(1),\dots,D(r)$ hold and $A_1(3r+1, r)$ is the only left-shifted extremal family on $[3r+1]$, then for every $n \ge 3r+1$ the only left-shifted extremal family on $[n]$ is $A_1(n, r)$: in Frankl's link split both parts are extremal, induction gives $\mathcal{F}_0 = A_1(m, r)$, and a link member avoiding $\{1, 2\}$ would shift to a member of $\mathcal{F}_0$ avoiding $\{1, 2\}$.

Proof of Theorems 5 and 6. If a shifted extremal family $\mathcal{F}$ on $[3r+1]$ ($r \in \{2, 3\}$) avoids $\{1, 2\}$ at some edge, shiftedness puts $\{3, \dots, r+2\}$ into $\mathcal{F}'$. Since $|\mathcal{F}|$ matches the dual value, every positively weighted row of `dualCertR2` or `dualCertR3` is tight. For $r = 2$, $B_9 = \{3, 4\}$ has $x_9 = 1$; tight rows $B_4 = \{1, 6\} \le B_8 = \{2, 6\}$, $B_7 = \{2, 5\} \le B_8$, and disjoint triple (B_4, B_7, B_9) give $x_4 = x_8 = x_7$ and $2x_4 + 1 = 2$, impossible in $\{0, 1\}$. For $r = 3$, $B_{49} = \{3, 4, 5\}$ gives $x_{49} = 1$, and two tight branches on x_{27} yield a contradiction (detailed in Appendix A). Hence $\mathcal{F} = A_1(3r+1, r)$; propagation and lifting finish the proof. □

Proof of Theorem 2. RUP soundness gives $\Psi_5^\star$ unsatisfiable. Any shifted 2,366-edge family on $[16]$ with a $\{1, 2\}$-avoider must contain $\{3, 4, 5, 6, 7\}$, yielding a model of $\Psi_5^\star$, impossible. Thus $\mathcal{F} = A_1(16, 5)$; propagation from $D(1)$–$D(5)$ and lifting finish all $n \ge 16$. □

3.4 Diagonal bounds, Conjecture H, and LP barriers (Theorems 7–14).

Fix $r \ge 2$, distinct $p, q \in [3r+1]$, and $Y = [3r+1] \setminus \{p, q\}$. Partition $\mathcal{F}$ into members containing $\{p, q\}$ (at most $\binom{3r-1}{r-2}$), the traces $\mathcal{G}_p, \mathcal{G}_q \subseteq \binom{Y}{r-1}$ of members meeting $\{p, q\}$ once, and $\mathcal{F}_0 = \{E \in \mathcal{F} : p, q \notin E\}$. No $E_0 \in \mathcal{F}_0$, $G \in \mathcal{G}_p$, $G' \in \mathcal{G}_q$ are pairwise disjoint.

Proof of Theorem 7. Put $M = 3r-1$ and order Y cyclically [19]. Let a_j, b_j be the arcs of length $r-1$ and r starting at $j \in \mathbb{Z}/M$, and set $g(j) = [a_j \in \mathcal{G}_p] + [a_j \in \mathcal{G}_q] \le 2$ and $W = \sum_j (g(j) + 2[b_j \in \mathcal{F}_0])$. If $\mathcal{F}_0$ has no arc, $W \le 2M$. Otherwise pairwise intersection of $\mathcal{F}_0$ confines all $\mathcal{F}_0$-arc starts, after rotation, to $\{0, \dots, r-1\}$. For $0 \le d \le r-1$, set $(u_d, v_d) = (r, 2r-1)$ for $d = 0$, $(d+r, d+2r)$ for $1 \le d \le r-2$, and $(0, 2r)$ for $d = r-1$. Each triple (b_d, a_{u_d}, a_{v_d}) is pairwise disjoint, forcing $g(u_d) + g(v_d) + 2[b_d \in \mathcal{F}_0] \le 4$. The $2r$ positions u_d, v_d are distinct and complement $\{1, \dots, r-1\}$ in $\mathbb{Z}/M$, giving $W \le 4r + 2(r-1) = 2M$. Averaging over cyclic orders with $\binom{M}{r} = 2\binom{M}{r-1}$ yields $|\mathcal{G}_p| + |\mathcal{G}_q| + |\mathcal{F}_0| \le 2\binom{M}{r-1}$, so $|\mathcal{F}| \le |A_1|$. □

Proof of Theorems 8 and 9. For Theorem 8, fix $E_0 \in \mathcal{F}_0$ and $Z = Y \setminus E_0$ ($|Z| = 2r-1$). No member of $\mathcal{G}_p \cap \binom{Z}{r-1}$ is

disjoint from one of $\mathcal{G}_q \cap \binom{Z}{r-1}$; since each $(r-1)$-set in Z has r disjoint partners, these two traces together contribute at most $\binom{2r-1}{r-1}$, leaving at least $\binom{2r-1}{r-1}$ missing edges from A_1. For Theorem 9, disjoint union of the three clauses defining $\mathcal{H}^{(r)}(n,2;p,q,E_0)$ gives exact size $\binom{n}{r} - \binom{n-2}{r} - \binom{n-r-2}{r-1} + 1$ (`hiltonMilnerTwo_card`); at $n = 3r+1$, subtracting $|\mathcal{H}_{r,1}|$ below leaves $C_{r-1} - r = (\binom{2r-2}{r-1} - r^2)/r$, which is 1 at $r = 4$ and stays positive for all $r \geq 4$ by induction because $C_r = \frac{4r-2}{r+1} C_{r-1} > 2C_{r-1}$ while $2r > r+1$ for $r \geq 4$. □

Proof of Theorem 12. Let $X = [3r-2]$, $T = \{3r-1, 3r, 3r+1\}$, and $N = \binom{3r-2}{r-1}$. Shifting the two disjoint $[r-2]$-avoiders produces members $B, C \in \mathcal{F}$ partitioning $\{r-1, \dots, 3r-2\}$. No $E \in \mathcal{F}$ meets T in at least two points, else keeping two points of $E \cap T$ and shifting the other $r-2$ points of E (including the third point of T when $|E \cap T| = 3$) into $[r-2]$ yields a member disjoint from B and C. Thus $|\mathcal{F}| = f + g_1 + g_2 + g_3$, where $f = |\mathcal{F} \cap \binom{X}{r}|$ and $g_k = |\mathcal{G}_k|$ counts the link of $t_k = 3r-2+k$ inside X, with $g_1 \geq g_2 \geq g_3$ by shiftedness. Over all $N\binom{2r-1}{r-1}$ ordered disjoint pairs (A, A') of $(r-1)$-subsets of X with complement $Z = X \setminus (A \cup A')$, at most two of $A \in \mathcal{G}_1$, $A' \in \mathcal{G}_2$, $Z \in \mathcal{F}$ hold; double counting gives $rf + (2r-1)(g_1+g_2) \leq 2(2r-1)N$. Since $rf \leq r\binom{3r-2}{r} = (2r-1)N$ and $2(g_1+g_2+g_3) \leq 3(g_1+g_2)$, for $r \geq 3$:

$$\begin{aligned}(2r-1)\cdot 2r|\mathcal{F}| &\leq (r-2)\,rf + 3r(rf + (2r-1)(g_1+g_2)) \\ &\leq (r-2)(2r-1)N + 6r(2r-1)N \\ &= (2r-1)(7r-2)N,\end{aligned}$$

so $2r|\mathcal{F}| \leq (7r-2)N$. With $(2r+1)\cdot 2r|A_1| = (5r+1)(3r-1)N$ and $(5r+1)(3r-1) - (2r+1)(7r-2) = r^2 - 5r + 1 > 0$ $(r \geq 5)$, $|\mathcal{F}| < |A_1|$. □

Proof of Theorems 10 and 11. Write $q = r - j$ and $\mathcal{H}_{r,j} = \mathcal{M}_1 \cup \mathcal{M}_2$ with $\mathcal{M}_1 = \{E \in A_j(3r+1, r) : c_E(K) \geq j+1\}$ and $\mathcal{M}_2 = \binom{[K]}{r}$. Both parts are Gale down-sets, so $\mathcal{H}_{r,j}$ is left-shifted. Three disjoint members cannot all lie in $\mathcal{M}_1$ (requiring $3j$ points in $[3j-1]$), nor can one lie in $\mathcal{M}_2$ (putting $r + 2(j+1) = K+1$ points in $[K]$). The disjoint $[j]$-avoiders $B = \{j+1, \dots, 2j\} \cup \{K\} \cup \{2r+j+3, \dots, 3r+1\} \in \mathcal{M}_1$ and $C = \{2j+1, \dots, r+2j\} \in \mathcal{M}_2$ show $J_r(\mathcal{H}_{r,j}) \geq j$, while no two disjoint members avoid $[j+1]$ (comparing prefix-count sums at $m = 3j-1$ and $m = K$ against $m-j-1$), so $J_r(\mathcal{H}_{r,j}) = j$. Counting gives $|\mathcal{H}_{r,j}| = |A_j| - L_j(q) + G_j(q)$, where $L_j(q) = \binom{3j-1}{j}\binom{2q}{q}$ and $G_j(q) = \binom{3j-1}{j-1}(q+2) + \binom{3j-1}{j-2} > 0$. Put $\Delta_j(q) = |A_j| - |A_{j+1}|$. By Frankl's identity (Appendix A), for $q \geq 2$, $4\Delta_j(q) \leq \Delta_j(q+1)$ since $24(q+1)(2q+3)(j+q+1) \leq 9(3q+2)(3q+1)(j+q+2)$, while $L_j(q+1) \leq 4L_j(q)$. Base checks at $(8,2)$ $(\Delta = 12{,}852 \geq 9{,}240 = L$, surplus 3,653) and $(6,1)$ $(\Delta = 637 \geq 504 = L$, surplus 140) give $\Delta_j(q) \geq L_j(q)$, hence $|\mathcal{H}_{r,j}| > |A_{j+1}|$. Finally $G_j(q) < L_j(q)$ for $q \geq 1$ $(q+3 \leq 2\binom{2q}{q})$, so $|\mathcal{H}_{r,j}| < |A_j| \leq |A_1|$. □

Proof of Theorems 13 and 14. Raney's cycle lemma gives $(r+1)|\mathcal{Z}_r| = \binom{3r+1}{r}$ (Appendix A, with LP feasibility of $\hat{y}$ and y^*). For Theorem 13, $\sum_E \hat{y}_E = \frac{1}{2}\binom{3r+1}{r} + \frac{1}{2}\binom{3r}{r-1} - \frac{1}{2}|\mathcal{Z}_r| = \frac{r(2r+1)}{(r+1)(3r+1)}\binom{3r+1}{r}$, exceeding $|A_1| = \frac{5r+1}{3(3r+1)}\binom{3r+1}{r}$ by $\frac{r^2-3r-1}{3(r+1)(3r+1)}\binom{3r+1}{r} > 0$ for $r \geq 4$. For Theorem 14, every r-subset meets $[2r+2]$ and exactly $2r+2$ meet it once, so the unfiltered ramp has total $\frac{2r^2-4r-2}{(r-2)(3r+1)}\binom{3r+1}{r} + \frac{2r+2}{r-2}$. On $\mathcal{Z}_r$ one has $3S(E) \leq 2r+2$, capping the raw ramp at $\frac{2}{3}$ there; hence $\sum_E y^*_E - |A_1| \geq \frac{r(r^2-8r+3)}{3(r-2)(r+1)(3r+1)}\binom{3r+1}{r} + \frac{2r+2}{r-2} > 0$ for $r \geq 8$. At $r = 7$, every $E \in \mathcal{Z}_7$ satisfies $3S(E) \leq 16$, so $S(E) \leq 5$ and the raw ramp is at most $\frac{2}{5}$ unless $S(E) = 5$ (where it is $\frac{3}{5}$). If $E \in \mathcal{Z}_7$ has $S(E) = 5$, then $17 \notin E$ (else $3c_E(17) \geq 18 > 17$), so $E \cap [16] \in \mathcal{Z}_5$ $(|\mathcal{Z}_5| = 728)$ and $E \cap \{18, \dots, 22\}$ has $\binom{5}{2} = 10$ choices, giving at most 7,280 such sets. Since $|\mathcal{Z}_7| = 21{,}318$, the $\mathcal{Z}_7$-subtraction is at most $\frac{2}{5} \cdot 21{,}318 + \frac{1}{5} \cdot 7{,}280 = \frac{49916}{5}$, against the unfiltered total $\frac{527152}{5}$, giving $\sum_E y^*_E \geq \frac{527152}{5} - \frac{49916}{5} = \frac{477236}{5} > 93{,}024 = |A_1(22,7)|$. □

4. COMPUTATIONAL VERIFICATION

Certificate parser and exact replay. The standalone zero-float script `certificate.py` parses the serialized certificates from `ShiftingAndReduction.lean` and `SATCert5.lean`: (i) `dualCertR2` ($r = 2$, 15 sets, 3 Raney zeros, 12/12 active columns tight, $5 + 4 + 2 = 11$ terms, objective 11, 0 non-star solutions); (ii) `dualCertR3` ($r = 3$, 84 sets, 12 Raney zeros, 72/72 active columns tight, $9 + 18 + 22 = 49$ terms, objective 64, 0 non-star solutions); (iii) `rupProofEMC1` (2 clauses parsed into literals and replayed on the 54-clause Ψ_1, 4 edge variables, $T = 3$). It parses $\{3,4,5,6,7\}$ and recomputes the dimensions of Ψ_4 (715, $T = 386$) and Ψ_5 (4,368, $T = 2{,}367$). In exact arithmetic, `recompute.py` runs the Section 1 box verbatim, runs `certificate.py`, and replays $|A_1(16,5)| = 2{,}366$, $|\mathcal{H}^{(5)}(16,2)| = 2{,}241$, the $(8,2)/(6,1)$ refutations, $|\mathcal{Z}_r|$ $(0 \leq r \leq 5)$, weights $(r \in \{4,5,7\})$, identities to $r = 40$, and Table 1.

n	bound	status	n	bound	status
14	2,002	uncond.	19	5,440	cond.
15	2,002	uncond.	20	6,936	cond.
16	2,366	cond.	21	8,721	cond.
17	3,185	cond.	22	10,830	cond.
18	4,200	cond.			

Table 1. Values of $\max\{2002, \binom{n}{5} - \binom{n-2}{5}\}$ in Theorem 1: $n \in \{14,15\}$ unconditional (`erdos_1020_r5_s2_n14`, Theorem 3); $16 \leq n \leq 22$ conditional ($n \geq 23$ by [9]).

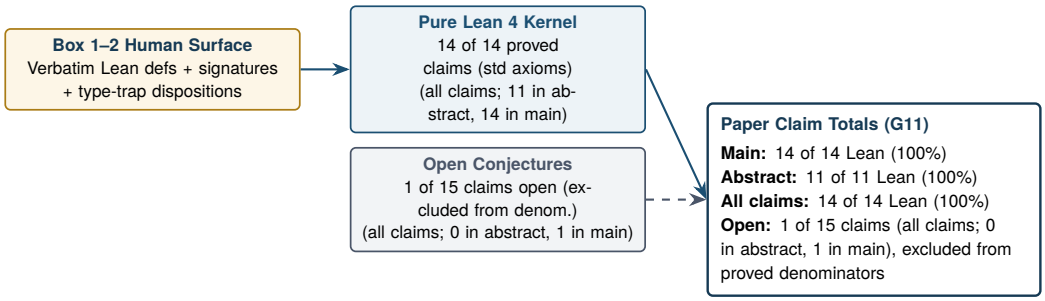
RUP status, Lean build, and Section 5. Discharging Theorems 1–2 in Lean needs accepted lists for $\Psi_4, \Psi_5, \Psi_5^\star$; with no completeness converse to RUP soundness, external refutations of Ψ_4 or Ψ_5 yield $D(4)$ or $D(5)$ via the CNF bridge (certificate-dependent tier) without supplying the RUP hypotheses. None (nor `drat-trim` files) is bundled, so both stay conditional. Rebuilding `ErdosMatching5` (Lean 4/Mathlib `v4.35.0`) reproduces the 237-declaration report (standard three axioms, zero `sorry`/`native_decide`; Appendix B covers companion windows). Section 5 totals (14 of 14 main, 11 of 11 abstract) count Theorems 1–2 as implications (12 of 14 main, 9 of 11 abstract unconditional), and "no external certificate artifacts" means none is supplied; deriving $m(16,5,2) = 2{,}366$ from Theorem 1 needs the Ψ_4, Ψ_5 lists.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 89f435d9ac11 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Artifacts & Remaining Claims:** No external certificate artifacts are required; any claims marked Written or Unsettled rely on standard written proofs or remain explicitly unproved.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)

PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)

PASS: Coverage: 237/237 public theorems (source-level approx)

WARN: Fidelity: 14/14 exact (draft; auditor/formalizer relationship unknown) | WARN: Witnesses: pending gate replay

WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

ErdosMatching5.IsUniform (ErdosMatching5/ShiftingAndReduction.lean:55-56) — Section 2 (Families: r-uniform)

```
def IsUniform (r : ℕ) {n : ℕ} (F : Finset (Finset (Fin n))) : Prop :=
  ∀ A ∈ F, A.card = r
```

ErdosMatching5.HasThreeMatching (ErdosMatching5/ShiftingAndReduction.lean:59-60) — Section 2 (Families)

```
def HasThreeMatching {n : ℕ} (F : Finset (Finset (Fin n))) : Prop :=
  ∃ A ∈ F, ∃ B ∈ F, ∃ C ∈ F, Disjoint A B ∧ Disjoint A C ∧ Disjoint B C
```

ErdosMatching5.NoThreeMatching (ErdosMatching5/ShiftingAndReduction.lean:63-64) — Section 2 (Families: no three pairwise disjoint members)

```
def NoThreeMatching {n : ℕ} (F : Finset (Finset (Fin n))) : Prop :=
  ¬ HasThreeMatching F
```

ErdosMatching5.starTwoFamily (ErdosMatching5/ShiftingAndReduction.lean:184-185) — Section 2 (A_1, the complete 2-star centred at the first two points)

```
def starTwoFamily (n r : ℕ) : Finset (Finset (Fin n)) :=
  ((Finset.univ : Finset (Fin n)).powersetCard r).filter (fun A => ∃ x ∈ A, x.val < 2)
```

ErdosMatching5.franklA (ErdosMatching5/ShiftingAndReduction.lean:246-248) — Section 2 (Frankl's families A_l)

```
def franklA (n r l : ℕ) : Finset (Finset (Fin n)) :=
  ((Finset.univ : Finset (Fin n)).powersetCard r).filter
    (fun A => l ≤ (A.filter (fun x => x.val < 3 * l - 1)).card)
```

ErdosMatching5.starTwoOn (ErdosMatching5/UnshiftedUniqueness.lean:152-153) — Section 2 (complete 2-star with centre p, q)

```
def starTwoOn (n r : ℕ) (p q : Fin n) : Finset (Finset (Fin n)) :=
  ((Finset.univ : Finset (Fin n)).powersetCard r).filter (fun E => p ∈ E ∨ q ∈ E)
```

ErdosMatching5.IsStarTwo (ErdosMatching5/UnshiftedUniqueness.lean:156-157) — Section 2 (is a complete 2-star)

```
def IsStarTwo (n r : ℕ) (F : Finset (Finset (Fin n))) : Prop :=
  ∃ p q : Fin n, p ≠ q ∧ F = starTwoOn n r p q
```

ErdosMatching5.shiftSet (ErdosMatching5/ShiftingAndReduction.lean:442-443) — Section 2 (Shifting: s_ij)

```
def shiftSet {n : ℕ} (i j : Fin n) (A : Finset (Fin n)) : Finset (Fin n) :=
  if j ∈ A ∧ i ∉ A then insert i (A.erase j) else A
```

ErdosMatching5.shiftMember (ErdosMatching5/ShiftingAndReduction.lean:457-459) — Section 2 (Shifting: S_ij)

```
def shiftMember {n : ℕ} (i j : Fin n) (F : Finset (Finset (Fin n))) (A : Finset (Fin
  ↪ n)) :
    Finset (Fin n) :=
  if shiftSet i j A ∈ F then A else shiftSet i j A
```

ErdosMatching5.shift (ErdosMatching5/ShiftingAndReduction.lean:462-463) — Section 2 (Shifting: S_ij)

```
def shift {n : ℕ} (i j : Fin n) (F : Finset (Finset (Fin n))) : Finset (Finset (Fin
  ↪ n)) :=
  F.image (shiftMember i j F)
```

ErdosMatching5.IsShifted (ErdosMatching5/ShiftingAndReduction.lean:657-658) — Section 2 (left-shifted)

```
def IsShifted {n : ℕ} (F : Finset (Finset (Fin n))) : Prop :=
  ∀ i j : Fin n, i.val < j.val → shift i j F = F
```

ErdosMatching5.prefixCount (ErdosMatching5/AllRAndRefutation.lean:60-61) — Section 2 (Indexing: c_E(m))

```
def prefixCount {n : ℕ} (P : Finset (Fin n)) (m : ℕ) : ℕ :=
  (P.filter (fun x => x.val < m)).card
```

ErdosMatching5.HasDisjointAvoiders (ErdosMatching5/AllRAndRefutation.lean:892-893) — Section 2 (avoider invariant)

```
def HasDisjointAvoiders {n : ℕ} (j : ℕ) (F : Finset (Finset (Fin n))) : Prop :=
  ∃ B ∈ F, ∃ C ∈ F, Disjoint B C ∧ (∀ x ∈ B, j ≤ x.val) ∧ (∀ x ∈ C, j ≤ x.val)
```

ErdosMatching5.jInvariant (ErdosMatching5/AllRAndRefutation.lean:909-911) — Section 2 (avoider invariant J_r)

```
def jInvariant {n : ℕ} (r : ℕ) (F : Finset (Finset (Fin n))) : ℤ :=
  let S := (Finset.range r).filter (fun j => HasDisjointAvoiders j F)
  if hS : S.Nonempty then ((S.max' hS : ℕ) : ℤ) else -1
```

ErdosMatching5.hybridM1Down (ErdosMatching5/AllRAndRefutation.lean:937-939) — Section 2 (first part of H_{r,j})

```
def hybridM1Down (r j : ℕ) : Finset (Finset (Fin (3 * r + 1))) :=
  ((Finset.univ : Finset (Fin (3 * r + 1))).powersetCard r).filter
    (fun E => j ≤ prefixCount E (3 * j - 1) ∧ j + 1 ≤ prefixCount E (r + 2 * j + 1))
```

ErdosMatching5.hybridM2Down (ErdosMatching5/AllRAndRefutation.lean:943-945) — Section 2 (second part of H_{r,j})

```
def hybridM2Down (r j : ℕ) : Finset (Finset (Fin (3 * r + 1))) :=
  ((Finset.univ : Finset (Fin (3 * r + 1))).powersetCard r).filter
    (fun E => r ≤ prefixCount E (r + 2 * j + 1))
```

ErdosMatching5.hybridH (ErdosMatching5/AllRAndRefutation.lean:949-950) — Section 2 (the family H_{r,j})

```
def hybridH (r j : ℕ) : Finset (Finset (Fin (3 * r + 1))) :=
  hybridM1Down r j ∪ hybridM2Down r j
```

ErdosMatching5.hiltonMilnerTwo (ErdosMatching5/HiltonMilnerExact.lean:41-43) — Section 2 (Hilton-Milner type family)

```
def hiltonMilnerTwo (n r : ℕ) (p q : Fin n) (E₀ : Finset (Fin n)) : Finset (Finset
  ↪ (Fin n)) :=
  ((Finset.univ : Finset (Fin n)).powersetCard r).filter
    (fun E => p ∈ E ∨ (q ∈ E ∧ ¬ Disjoint E E₀) ∨ E = E₀)
```

ErdosMatching5.raneyZeroFamily (ErdosMatching5/LPBarriers.lean:52-54) — Definition (Raney sets Z_r)

```
def raneyZeroFamily (r : ℕ) : Finset (Finset (Fin (3 * r + 1))) :=
  ((Finset.univ : Finset (Fin (3 * r + 1))).powersetCard r).filter
    (fun E => ∀ m : Fin (3 * r + 2), 3 * prefixCount E m.val ≤ m.val)
```

ErdosMatching5.raneyStarWeight (ErdosMatching5/LPBarriers.lean:742-745) — Definition (weights: Raney-star weight)

```
def raneyStarWeight (r : ℕ) (E : Finset (Fin (3 * r + 1))) : ℚ :=
  if (⟨0, by omega⟩ : Fin (3 * r + 1)) ∈ E then 1
  else if E ∈ raneyZeroFamily r then 0
  else (1 : ℚ) / 2
```

ErdosMatching5.rawLinearRamp (ErdosMatching5/LPBarriers.lean:988-990) — Definition (weights: linear ramp before the Raney filter)

```
def rawLinearRamp (r : ℕ) (E : Finset (Fin (3 * r + 1))) : ℚ :=
  if prefixCount E (2 * r + 2) ≤ 2 then 0
  else ((prefixCount E (2 * r + 2) - 2 : ℕ) : ℚ) / (r - 2 : ℚ)
```

ErdosMatching5.linearRampWeight (ErdosMatching5/LPBarriers.lean:994-995) — Definition (weights: y*)

```
def linearRampWeight (r : ℕ) (E : Finset (Fin (3 * r + 1))) : ℚ :=
  if E ∈ raneyZeroFamily r then 0 else rawLinearRamp r E
```

ErdosMatching5.checkRUPProof (ErdosMatching5/SATCert5.lean:358-364) — Definition (RUP checker check)

```
def checkRUPProof {V : Type} [DecidableEq V] : CNF V → List (Clause V) → Bool
  | _, [] => false
  | db, c :: cs =>
    if isRUP db c then
      if c.isEmpty then true
      else checkRUPProof (c :: db) cs
    else false
```

ErdosMatching5.shiftedFamilyAtLeastCNF (ErdosMatching5/SATCert5.lean:487-489) — Definition (formula Phi(n,r,T))

```
def shiftedFamilyAtLeastCNF (n r T : ℕ) : CNF (EMCVar n) :=
  shiftClauses n r ++ noThreeDisjointClauses n r ++
    atLeastCardClauses (chooseLists (List.finRange n) r) T
```

ErdosMatching5.shiftedEMCCNF (ErdosMatching5/SATCert5.lean:492-493) — Definition (formula Psi_r)

```
def shiftedEMCCNF (r : ℕ) : CNF (EMCVar (3 * r + 1)) :=
  shiftedFamilyAtLeastCNF (3 * r + 1) r (Nat.choose (3 * r + 1) r - Nat.choose (3 * r
    ↪ - 1) r + 1)
```

ErdosMatching5.shiftedEMC4_CNF (ErdosMatching5/SATCert5.lean:496-496) — Definition (Psi_4)

```
abbrev shiftedEMC4_CNF : CNF (EMCVar 13) := shiftedEMCCNF 4
```

ErdosMatching5.shiftedEMC5_CNF (ErdosMatching5/SATCert5.lean:499-499) — Definition (Psi_5)

```
abbrev shiftedEMC5_CNF : CNF (EMCVar 16) := shiftedEMCCNF 5
```

ErdosMatching5.shiftedEMC5_forceAvoider_CNF (ErdosMatching5/SATCert5.lean:990-991) — Definition (Psi_5 star)

```
def shiftedEMC5_forceAvoider_CNF : CNF (EMCVar 16) :=
  shiftedFamilyAtLeastCNF 16 5 2366 ++ [[.pos (.mem firstAvoiderList16_5)]]
```

ErdosMatching5.firstAvoiderList16_5 (ErdosMatching5/SATCert5.lean:981-982) — Definition (the set {3,4,5,6,7})

```
def firstAvoiderList16_5 : List (Fin 16) :=
  [⟨2, by decide⟩, ⟨3, by decide⟩, ⟨4, by decide⟩, ⟨5, by decide⟩, ⟨6, by decide⟩]
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): ErdosMatching5.checkBoardDualCert_r2 (checkBoardDualCert 2 11 dualCertR2 = true); ErdosMatching5.rup_check_emc1 (checkRUPProof (shiftedEMCCNF 1) rupProofEMC1 = true).

Supporting Semantic Closure Index (non-Box-1 helpers): ErdosMatching5.Lit (ErdosMatching5/SATCert5.lean), ErdosMatching5.Lit.negate (ErdosMatching5/SATCert5.lean), ErdosMatching5.Clause (ErdosMatching5/SATCert5.lean), ErdosMatching5.CNF (ErdosMatching5/SATCert5.lean), ErdosMatching5.Eval (ErdosMatching5/SATCert5.lean), ErdosMatching5.evalLit (ErdosMatching5/SATCert5.lean), ErdosMatching5.ClauseStatus (ErdosMatching5/SATCert5.lean), ErdosMatching5.PassResult (ErdosMatching5/SATCert5.lean), ErdosMatching5.classifyClause (ErdosMatching5/SATCert5.lean), ErdosMatching5.hasContradiction (ErdosMatching5/SATCert5.lean), ErdosMatching5.rupPassGo (ErdosMatching5/SATCert5.lean), ErdosMatching5.rupLoop (ErdosMatching5/SATCert5.lean), ErdosMatching5.isRUP (ErdosMatching5/SATCert5.lean), ErdosMatching5.EMCVar (ErdosMatching5/SATCert5.lean), ErdosMatching5.chooseLists (ErdosMatching5/ShiftingAndReduction.lean), ErdosMatching5.shiftSortedList (ErdosMatching5/SATCert5.lean), ErdosMatching5.shiftClauses (ErdosMatching5/SATCert5.lean), ErdosMatching5.noThreeDisjointClauses (ErdosMatching5/SATCert5.lean), ErdosMatching5.atLeastCardClauses (ErdosMatching5/SATCert5.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

ErdosMatching5.erdos_1020_r5_s2_all_n (ErdosMatching5/SATCert5.lean:1214-1225) src:a48d453e873b4a

```
theorem erdos_1020_r5_s2_all_n
    (h_rup_4 : ∃ proof, checkRUPProof shiftedEMC4_CNF proof = true)
    (h_rup_5 : ∃ proof, checkRUPProof shiftedEMC5_CNF proof = true)
    {n : ℕ} (hn : 14 ≤ n) :
    (∀ F : Finset (Finset (Fin n)), IsUniform 5 F → NoThreeMatching F →
      F.card ≤ max 2002 (Nat.choose n 5 - Nat.choose (n - 2) 5)) ∧
    (∃ F : Finset (Finset (Fin n)), IsUniform 5 F ∧ NoThreeMatching F ∧
      F.card = max 2002 (Nat.choose n 5 - Nat.choose (n - 2) 5))
```

Assumptions: CONDITIONAL, hypotheses not discharged: Hypotheses h_rup_4 and h_rup_5 (existence of clause lists accepted by checkRUPProof); no such list is in this bundle. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: unknown_or_empty).

ErdosMatching5.uniqueness_unshifted_starTwo_ge_16_5_of_rup (ErdosMatching5/UnshiftedUniqueness.lean:1797-1821) src:485711a1846932

```
theorem uniqueness_unshifted_starTwo_ge_16_5_of_rup
    (h_rup_4 : ∃ proof, checkRUPProof shiftedEMC4_CNF proof = true)
    (h_rup_5 : ∃ proof, checkRUPProof shiftedEMC5_CNF proof = true)
    (h_rup_unique : ∃ proof, checkRUPProof shiftedEMC5_forceAvoider_CNF proof = true)
    {n : ℕ} (hn : 16 ≤ n)
    (F : Finset (Finset (Fin n))) (hU : IsUniform 5 F) (hN : NoThreeMatching F) :
    F.card = Nat.choose n 5 - Nat.choose (n - 2) 5 ↔ IsStarTwo n 5 F
```

Assumptions: CONDITIONAL, hypotheses not discharged: Hypotheses h_rup_4, h_rup_5 and h_rup_unique (clause lists accepted by checkRUPProof); no such list is in this bundle. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: unknown_or_empty).

ErdosMatching5.erdos_1020_r5_s2_n15 (ErdosMatching5/SATCert5.lean:1153-1159) src:22f07fa76fea16

```
theorem erdos_1020_r5_s2_n15 :
    (∀ F : Finset (Finset (Fin 15)), IsUniform 5 F → NoThreeMatching F → F.card ≤
      ↪ 2002) ∧
    (∃ F : Finset (Finset (Fin 15)), IsUniform 5 F ∧ NoThreeMatching F ∧ F.card =
      ↪ 2002)
```

Assumptions: None beyond Box 1 definitions. Applicability: Unconditional (closed statement; no outer premise witness needed).

ErdosMatching5.erdos_1020_s2_r_le_3 (ErdosMatching5/ShiftingAndReduction.lean:2420-2429) src:7a166bb4b96b88

```
theorem erdos_1020_s2_r_le_3 {r n : ℕ} (hr1 : 1 ≤ r) (hr3 : r ≤ 3) (hn : 3 * r - 1 ≤
  ↪ n)
    (F : Finset (Finset (Fin n))) (hU : IsUniform r F) (hN : NoThreeMatching F) :
    F.card ≤ max (Nat.choose (3 * r - 1) r) (Nat.choose n r - Nat.choose (n - 2) r)
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.uniqueness_unshifted_starTwo_ge_7_2 (ErdosMatching5/UnshiftedUniqueness.lean:1657-1666) src:184f8c9857f24e

```
theorem uniqueness_unshifted_starTwo_ge_7_2 {n : ℕ} (hn : 7 ≤ n)
    (F : Finset (Finset (Fin n))) (hU : IsUniform 2 F) (hN : NoThreeMatching F) :
    F.card = Nat.choose n 2 - Nat.choose (n - 2) 2 ↔ IsStarTwo n 2 F
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.uniqueness_unshifted_starTwo_ge_10_3 (ErdosMatching5/UnshiftedUniqueness.lean:1754-1764) src:73e72d65456f9c

```
theorem uniqueness_unshifted_starTwo_ge_10_3 {n : ℕ} (hn : 10 ≤ n)
    (F : Finset (Finset (Fin n))) (hU : IsUniform 3 F) (hN : NoThreeMatching F) :
    F.card = Nat.choose n 3 - Nat.choose (n - 2) 3 ↔ IsStarTwo n 3 F
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.theorem_K_unshifted (ErdosMatching5/AllRAndRefutation.lean:2675-2779) src:c1462691a006ef

```
theorem theorem_K_unshifted {r : ℕ} (hr : 2 ≤ r)
    (F : Finset (Finset (Fin (3 * r + 1)))) (hU : IsUniform r F) (hN :
      ↪ NoThreeMatching F)
    (p q : Fin (3 * r + 1)) (hpq : p ≠ q)
    (hpwi : ∀ E₁ ∈ F, ∀ E₂ ∈ F, p ∉ E₁ → q ∉ E₁ → p ∉ E₂ → q ∉ E₂ → ¬ Disjoint E₁ E₂)
      ↪ :
    F.card ≤ (starTwoFamily (3 * r + 1) r).card
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.as6_hilton_milner_gap (ErdosMatching5/AllRAndRefutation.lean:1977-2146) src:9f479281dc5123

```
theorem as6_hilton_milner_gap {r : ℕ} (hr : 2 ≤ r)
    (F : Finset (Finset (Fin (3 * r + 1)))) (hU : IsUniform r F) (hN :
      ↪ NoThreeMatching F)
    (p q : Fin (3 * r + 1)) (hpq : p ≠ q)
    (havoid : (F.filter (fun E => p ∉ E ∧ q ∉ E)).Nonempty) :
    F.card + Nat.choose (2 * r - 1) (r - 1) ≤
      (starTwoFamily (3 * r + 1) r).card + (F.filter (fun E => p ∉ E ∧ q ∉ E)).card
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.hiltonMilnerTwo_diag_gt_hybridH1 (ErdosMatching5/HiltonMilnerExact.lean:391-430) src:55893475b7a362

```
theorem hiltonMilnerTwo_diag_gt_hybridH1 {r : ℕ} (hr : 4 ≤ r)
    {p q : Fin (3 * r + 1)} (hpq : p ≠ q) {E₀ : Finset (Fin (3 * r + 1))}
    (hE₀ : E₀.card = r) (hp : p ∉ E₀) (hq : q ∉ E₀) :
    (hybridH r 1).card < (hiltonMilnerTwo (3 * r + 1) r p q E₀).card
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.refutation_conjecture_H_ge_8 (ErdosMatching5/AllRAndRefutation.lean:1713-1731) src:ae611206d23581

```
theorem refutation_conjecture_H_ge_8 {r : ℕ} (hr : 8 ≤ r) :
    IsShifted (hybridH r 2) ∧
    IsUniform r (hybridH r 2) ∧
    NoThreeMatching (hybridH r 2) ∧
    jInvariant r (hybridH r 2) = 2 ∧
    (frankIA (3 * r + 1) r 3).card < (hybridH r 2).card ∧
    (hybridH r 2).card < (frankIA (3 * r + 1) r 2).card ∧
    (hybridH r 2).card < (starTwoFamily (3 * r + 1) r).card
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.refutation_conjecture_H_j1_ge_6 (ErdosMatching5/AllRAndRefutation.lean:1736-1752) src:3723d122fa1b8b

```
theorem refutation_conjecture_H_j1_ge_6 {r : ℕ} (hr : 6 ≤ r) :
    IsShifted (hybridH r 1) ∧
    IsUniform r (hybridH r 1) ∧
    NoThreeMatching (hybridH r 1) ∧
    jInvariant r (hybridH r 1) = 1 ∧
    (franklA (3 * r + 1) r 2).card < (hybridH r 1).card ∧
    (hybridH r 1).card < (starTwoFamily (3 * r + 1) r).card
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.as6_j_ge_r_sub_2_lt_starTwo (ErdosMatching5/AllRAndRefutation.lean:3487-3547) src:99ebdba7422ce1

```
theorem as6_j_ge_r_sub_2_lt_starTwo {r : ℕ} (hr : 5 ≤ r)
    (F : Finset (Finset (Fin (3 * r + 1))))
    (hSh : IsShifted F) (hU : IsUniform r F) (hN : NoThreeMatching F)
    (hJ : HasDisjointAvoiders (r - 2) F) :
    F.card < (starTwoFamily (3 * r + 1) r).card
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.raneyStarWeight_gt_starTwo (ErdosMatching5/LPBarriers.lean:970-983) src:f62572621495e6

```
theorem raneyStarWeight_gt_starTwo {r : ℕ} (hr : 4 ≤ r) :
    ((starTwoFamily (3 * r + 1) r).card : ℚ) <
      ∑ E ∈ (Finset.univ : Finset (Fin (3 * r + 1))).powersetCard r, raneyStarWeight
        ↪ r E
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

ErdosMatching5.linearRampWeight_gt_starTwo (ErdosMatching5/LPBarriers.lean:1766-1771) src:6c9a7069f03b43

```
theorem linearRampWeight_gt_starTwo {r : ℕ} (hr : 7 ≤ r) :
    ((starTwoFamily (3 * r + 1) r).card : ℚ) <
      ∑ E ∈ (Finset.univ : Finset (Fin (3 * r + 1))).powersetCard r, linearRampWeight
        ↪ r E
```

Assumptions: None beyond Box 1 definitions; binders range over ordinary types. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
as6_hilton_milner_gap	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
as6_hilton_milner_gap	nat_sub	- 1	intended_convention	The statement assumes r >= 2, so 2r - 1 and r - 1 are exact.
as6_j_ge_r_sub_2_lt_starTwo	nat_sub	- 2	intended_convention	The statement assumes r >= 5, so the avoider index r - 2 is exact.
atLeastCardClauses	nat_sub	- 1	intended_convention	The counter clause with k - 1 is emitted only under the guard 2 <= k.
erdos_1020_r5_s2_all_n	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
erdos_1020_r5_s2_all_n	nat_sub	- Nat.choose	intended_convention	Monotone upper index gives C(b, r) <= C(a, r), so subtraction is exact.
erdos_1020_r5_s2_all_n	nat_sub	- 2	intended_convention	The statement assumes n >= 14, so n - 2 is exact.
erdos_1020_s2_r_le_3	nat_sub	- 1	intended_convention	3r - 1 is exact because the statement assumes r >= 1.
erdos_1020_s2_r_le_3	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
erdos_1020_s2_r_le_3	nat_sub	- Nat.choose	intended_convention	Monotone upper index gives C(b, r) <= C(a, r), so subtraction is exact.
erdos_1020_s2_r_le_3	nat_sub	- 2	intended_convention	Hypotheses r >= 1 and n >= 3r - 1 force n >= 2, so n - 2 is exact.
franklA	nat_sub	- 1	intended_convention	All paper uses have l >= 1, so 3l - 1 is exact (l = 0 never enters a main result).
hybridM1Down	nat_sub	- 1	intended_convention	The hybrid families are used with j >= 1 (Theorems on H_{r,1} and H_{r,2}), where 3j - 1 is exact.
jInvariant	finset_range	Finset.range	intended_convention	Finset.range r is the set of j with 0 <= j < r, matching the paper definition of J_r; the value -1 is the empty case.

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
raneyStarWeight	nat_div	/ 2	intended_convention	The weight (1 : Q) / 2 is rational division by the literal two, which is nonzero.
rawLinearRamp	nat_sub	- 2	intended_convention	The subtraction is in the else branch of the guard prefixCount <= 2, so it is exact.
rawLinearRamp	nat_div	/ (r - 2 : ℚ)	intended_convention	Rational division by (r : Q) - 2, nonzero for r >= 3; the theorem using it assumes r >= 7.
shiftedEMCCNF	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
shiftedEMCCNF	nat_sub	- Nat.choose	intended_convention	Monotone upper index gives C(b, r) <= C(a, r), so subtraction is exact.
shiftedEMCCNF	nat_sub	- 1	intended_convention	Exact for r >= 1 (used at r = 4, 5); truncates to 0 at r = 0 (Section 2).
uniqueness_unshifted_starTwo_ge_10_3	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
uniqueness_unshifted_starTwo_ge_10_3	nat_sub	- Nat.choose	intended_convention	Monotone upper index gives C(b, r) <= C(a, r), so subtraction is exact.
uniqueness_unshifted_starTwo_ge_10_3	nat_sub	- 2	intended_convention	The statement assumes n >= 10, so n - 2 is exact.
uniqueness_unshifted_starTwo_ge_16_5_of_rup	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
uniqueness_unshifted_starTwo_ge_16_5_of_rup	nat_sub	- Nat.choose	intended_convention	Monotone upper index gives C(b, r) <= C(a, r), so subtraction is exact.
uniqueness_unshifted_starTwo_ge_16_5_of_rup	nat_sub	- 2	intended_convention	The statement assumes n >= 16, so n - 2 is exact.
uniqueness_unshifted_starTwo_ge_7_2	classical_choose	Classical.choose	intended_convention	Source match on Nat.choose; no Classical.choose, Exists.choose, or Nat.find.
uniqueness_unshifted_starTwo_ge_7_2	nat_sub	- Nat.choose	intended_convention	Monotone upper index gives C(b, r) <= C(a, r), so subtraction is exact.
uniqueness_unshifted_starTwo_ge_7_2	nat_sub	- 2	intended_convention	The statement assumes n >= 7, so n - 2 is exact.
decidableHasDisjointAvoiders	local_instance_or_notation	decidableHasDisjointAvoiders	intended_convention	A Decidable instance obtained by unfolding the predicate and infer_instance; it changes no statement.

[a] Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: if checkRUPProof accepts clause lists for the shifted (13,4) and (16,5) CNFs, then m(n,5,2) = max{2002, C(n,5)-C(n-2,5)} for all n >= 14	Lean Kernel	ErdosMatching5.erdos_1020_r5_s2_all_n	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	CONDITIONAL, hypotheses not discharged: Hypotheses h_rup_4 and h_rup_5 (existence of clause lists accepted by check RUPProof); no such list is in this bundle.
Thm. 2: under h_rup_4, h_rup_5, h_rup_unique, for n >= 16 a 5-uniform family with no 3-matching has size C(n,5)-C(n-2,5) exactly when it is a complete 2-star	Lean Kernel	ErdosMatching5.uniqueness_unshifted_starTwo_ge_16_5_of_rup	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	CONDITIONAL, hypotheses not discharged: Hypotheses h_rup_4, h_rup_5 and h_rup_unique (clause lists accepted by check RUPProof); no such list is in this bundle.
Thm. 3: m(15,5,2) = 2002	Lean Kernel	ErdosMatching5.erdos_1020_r5_s2_n15	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions.
Thm. 4: for 1 <= r <= 3 and n >= 3r-1, m(n,r,2) <= max{C(3r-1,r), C(n,r)-C(n-2,r)}	Lean Kernel	ErdosMatching5.erdos_1020_s2_r_le_3	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 5: for n >= 7, a 2-uniform family with no 3-matching has size C(n,2)-C(n-2,2) exactly when it is a complete 2-star	Lean Kernel	ErdosMatching5.uniqueness_unshifted_starTwo_ge_7_2	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 6: for n >= 10, a 3-uniform family with no 3-matching has size C(n,3)-C(n-2,3) exactly when it is a complete 2-star	Lean Kernel	`ErdosMatching5.uniqueness_unshifted_starTwo_ge_10_3`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 7: r >= 2, F r-uniform on [3r+1] with no 3-matching, p != q, and pairwise intersecting {p,q}-avoiders imply |F| <= |A_1(3r+1,r)|	Lean Kernel	`ErdosMatching5.theorem_K_unshifted`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 8: r >= 2, F r-uniform on [3r+1] with no 3-matching, p != q, and F_0 = all members of F avoiding p and q nonempty imply |F| + C(2r-1,r-1) <= |A_1(3r+1,r)| + |F_0|	Lean Kernel	`ErdosMatching5.as6_hilton_milner_gap`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 9: for r >= 4, distinct p, q in [3r+1] and an r-set E_0 avoiding p and q, |H_{r,1}| < |H^(r)(3r+1,2;p,q,E_0)|	Lean Kernel	`ErdosMatching5.hiltonMilnerTwo_diag_gt_hybridH1`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 10: for r >= 8, H_{r,2} is shifted, r-uniform, has no 3-matching, J = 2, |A_3| < |H_{r,2}| < |A_2| and |H_{r,2}| < |A_1|	Lean Kernel	`ErdosMatching5.refutation_conjecture_H_ge_8`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 11: for r >= 6, H_{r,1} is shifted, r-uniform, has no 3-matching, J = 1 and |A_2| < |H_{r,1}| < |A_1|	Lean Kernel	`ErdosMatching5.refutation_conjecture_H_j1_ge_6`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 12: for r >= 5, a shifted r-uniform F on [3r+1] with no 3-matching and two disjoint [r-2]-avoiders has |F| < |A_1(3r+1,r)|	Lean Kernel	`ErdosMatching5.as6_j_ge_r_sub_2_lt_starTwo`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 13: for r >= 4, the Raney-star weight has total exceeding |A_1(3r+1,r)|	Lean Kernel	`ErdosMatching5.raneyStarWeight_gt_starTwo`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Thm. 14: for r >= 7, the linear-ramp weight has total exceeding |A_1(3r+1,r)|	Lean Kernel	`ErdosMatching5.linearRampWeight_gt_starTwo`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	None beyond Box 1 definitions; binders range over ordinary types.
Conj. 15: D(r) for every r >= 4 (open; r = 4 and 5 follow from the RUP hypotheses of Theorem 1)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Unsettled; not counted in any Lean denominator.

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: eb39d309eead | Lean mode: committed_receipt | Closure SHA-256: c8fd7b6121bd | Git anchor: 89f435d9ac11 | Axioms SHA-256: 9265f1659108 | Fidelity SHA-256: fb0546ee060e.

6. PRIOR WORK AND OPEN QUESTIONS

Matching bounds and thresholds. Erdős [6] introduced $f(n;r,k)$ (printed pages 93–95, with $f(n;r,3) = m(n,r,2)+1$), following Erdős–Gallai [7] ($r = 2$) and Erdős–Ko–Rado [5] ($s = 1$). Kleitman treated the nonuniform problem [20] [to check], while Section 3.1 gives the uniform $n = 3r$ layer directly. Progressive large-n thresholds [2, 18, 8] precede Frankl [9], proving (1) and 2-star uniqueness for $n \geq (2s+1)r - s$, that is, $n \geq 23$ at $(r,s) = (5,2)$; thus only the seven sizes $16 \leq n \leq 22$ lie below the threshold of [9]. Later improvements require s sufficiently large and therefore do not cover $s = 2$: Frankl and Kupavskii [12] [to check] proved $n \geq (5r-2)s/3$ for fixed r and large s, Cao, Liu and Zhang [3] [to check] improved the linear range to $n \geq (r+1)s$ for $s \geq s_0(r)$, and Chi, Sun and Wang [4] [to check] treat every admissible $n \geq r(s+1)$ for fixed $r \geq 3$ and sufficiently large s.

Diagonal structure and recent computations. Frankl's link induction and $r = 3$ proof [10] complement Łuczak–Mieczkowska [22], while Frankl and Kupavskii [11] establish the $n - r - 2$ Hilton–Milner count. Martin and Patkós [23]

(indexing by forbidden matching size 3) obtain a uniform diagonal bound $(\frac{2}{3}-\varepsilon)\binom{3r+1}{r}$. At $r=4$, Hou, Hu and Liu [17] resolve $s \geq 6{,}004$ (local threshold 3,481); Babanskyy [1] treats all $s \geq 1$, with a 13-point calculation (Proposition 3.6) giving a rational bound of floor $384 < 386$ on shifted families with a maximum matching avoiding vertex 1 (not replayed in Lean here). At $r=5$, Han, Liu and Wang [15] [to check] supply a degree-4 quadratic majorant whose hypergraph transfer lies outside their repository.

Open questions. (1) Prove $D(r)$ for $r \geq 6$; Theorems 7 and 12 reduce this to shifted families with $2 \leq J_r \leq r-3$. (2) Does every shifted r-uniform family on $[3r+1]$ with no three disjoint members and $J_r \geq 2$ have size at most $\max\{|A_3(3r+1,r)|, |\mathcal{H}_{r,2}|\}$? (3) Exhibit clause lists accepted by check for Ψ_4, Ψ_5, and $\Psi_5^\star$ and prove acceptance in Lean, discharging Theorems 1 and 2; alternatively, checking refutations of verified serializations outside Lean yields $D(4)$ and $D(5)$ through the CNF bridge as a certificate-dependent tier without supplying the RUP hypotheses. (4) Identify nonlinear constraints overcoming the barriers of Theorems 13 and 14.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

The Lean library `ErdosMatching5` (8 files) resides in the `mathdisc` repository at `findings/erdos-matching-conjecture-s2/lean` (`lake build ErdosMatching5` and `lake env lean ErdosMatching5/Verify.lean`). This manuscript, `certificate.py`, `recompute.py`, `DiagonalWindows.lean`, figure scripts, and ledgers live in `papers/erdos-matching-conjecture-s2/draft3`; run `python3 certificate.py`, `python3 recompute.py`, and `ourotrix paper check -p . -b -x -B`, and replay `DiagonalWindows.lean` via `lake env lean` from the library directory against `DiagonalWindows.out`.

AI research agents built and directed by Tyler B. Hughes found the results, wrote and checked the Lean proofs, and drafted this paper. No human mathematician has yet refereed the proof.

A. SUPPORTING UNIFORM AND ASYMPTOTIC PROOFS

Full $r=3$ complementary-slackness chase (Theorem 6). With $B_{49}=\{3,4,5\}$ and $x_{49}=1$, the tight triple $(B_{27},B_{43},B_{49}) = (\{1,8,9\},\{2,6,7\},\{3,4,5\})$ gives $x_{27}+x_{43}=1$. If $x_{27}=1$, then $x_{43}=0$; $B_{43} \preceq B_{47}=\{2,7,9\}$ and $B_{43} \preceq B_{69}=\{4,6,8\}$ force $x_{47}=x_{69}=0$, so the tight triple (B_8,B_{47},B_{69}) with $B_8=\{1,3,5\}$ forces $x_8=2$, impossible. If $x_{27}=0$, then $x_{43}=1$; $B_{27} \preceq B_{48}=\{2,8,9\}$ and $B_{18}=\{1,5,6\}, B_{39}=\{2,5,6\} \preceq B_{43}$ give $x_{48}=0$ and $x_{18}=x_{39}=1$. Tight shift rows $B_{43} \preceq B_{44}=\{2,6,8\} \preceq B_{46}=\{2,7,8\}$ give $x_{46}=1$, and the Gale comparison $B_{25}=\{1,7,8\} \preceq B_{46}$ gives $x_{25}=1$. The tight triple (B_{18},B_{48},B_{51}) $(B_{51}=\{3,4,7\})$ forces $x_{51}=1$; tight shift rows $B_{51} \preceq B_{52}=\{3,4,8\} \preceq B_{53}=\{3,4,9\}$ give $x_{53}=1$, making (B_{25},B_{39},B_{53}) sum to 3 instead of 2.

Gale–Hall packing (`gale_hall_packing_iff`). For $sr \leq n$ and r-subsets $P_1,\dots,P_s \subseteq [n]$, pairwise disjoint $A_1,\dots,A_s \subseteq [sr]$ with $A_t \preceq P_t$ exist exactly when $\sum_{t=1}^{s} c_{P_t}(m) \leq m$ for every $1 \leq m < sr$: necessity counts $[m]$, and sufficiency applies Hall's marriage theorem [14] to the bipartite graph joining slot (t,i) to $[\min(p_i^{(t)}, sr)]$. Setting $s=3$ and $P_1=P_2=P_3=E$ shows every shifted family with no three disjoint members avoids $\mathcal{Z}_r$.

Frankl's exact difference identity (`franklA_card_diff_identity`). For $1 \leq \ell$ and $q=r-\ell \geq 1$, split $[3r+1]$ into $L=[3\ell-1]$, $M=\{3\ell,3\ell+1,3\ell+2\}$, and the remaining $3q-1$ points R. Classifying E by $(|E\cap L|,|E\cap M|)$, membership in $A_\ell \setminus A_{\ell+1}$ occurs only at $(\ell,0)$ (count $\binom{3\ell-1}{\ell}\binom{3q-1}{q}$), while $A_{\ell+1}\setminus A_\ell$ splits into $(\ell-1,2)$, $(\ell-1,3)$, and $(\ell-2,3)$ (counts $3b_\ell b_q + b_\ell\binom{3q-1}{q-2} + \binom{3\ell-1}{\ell-2}b_q$, where $b_m=\binom{3m-1}{m-1}$; at $(r,\ell)=(4,2)$ this is $75+5+5=85$). Using $\binom{3m-1}{m}=2b_m$ and $(2m+1)\binom{3m-1}{m-2}=(m-1)b_m$ gives $(2\ell+1)(2q+1)(|A_\ell|-|A_{\ell+1}|)=3(r+1)b_\ell b_q>0$.

Raney's cycle lemma and LP feasibility (`raneyZeroFamily_card`, `LPBarriers.lean`). Adding 0 to $[3r+1]$ and reading $E \in \binom{\mathbb{Z}/(3r+2)}{r}$ as a cyclic walk with step -2 on E and $+1$ off E (net $+2$) gives exactly 2 cyclic starts keeping the walk strictly positive [25], so $2\binom{3r+2}{r}=(3r+2)|\mathcal{Z}_r|$, i.e. $(r+1)|\mathcal{Z}_r|=\binom{3r+1}{r}$. In `LPBarriers.lean` (lines 747–808, 997–1077, 1188–1190), both $\hat{y}$ (all $r \geq 1$) and y^* ($r \geq 3$) are verified to satisfy $[0,1]$-bounds, zeroing on $\mathcal{Z}_r$, Gale antitonicity, 3-matching cuts (weight sum at most 2 on any pairwise disjoint triple), and all $3r+1$ Kleitman vertex-deletion cuts (total avoiding any point at most $\binom{3r-1}{r}$), with $y^*_E=1$ on $\binom{[2r+2]}{r}$ ($r \geq 3$).

B. COMPANION DIAGONAL SIZE WINDOWS

Outside the audited `ErdosMatching5` library (and outside the G11 denominators), the companion file `DiagonalWindows.lean` derives four Lean 4 theorems (standard three axioms only) sharpening the open diagonal search. Applying the slice inequality $2r|\mathcal{F}| \leq (7r-2)\binom{3r-2}{r-1}$ at $r=4$ gives $8|\mathcal{F}| \leq 3{,}120$, so $|\mathcal{F}| \leq 390$ whenever disjoint [2]-avoiders exist; combined with Theorem 7, any failure of $D(4)$ has size in $\{386,\dots,390\}$. At $r=5$, two disjoint [3]-avoiders force $10|\mathcal{F}| \leq 23{,}595$, hence $|\mathcal{F}| \leq 2{,}359 < 2{,}366$. Trimming any oversized family to exact cardinality 386 ($r=4$) or 2,367 ($r=5$) before shifting preserves admissibility and recovers $J_4 \geq 2$ and $J_5=2$, respectively (`rank_four_window`, `rank_five_window`, `exact_obstruction_sizes`).

Excluding these two fixed-size shifted classes implies the all-n ($n \geq 14$) rank-5 maximum without specifying a proof format (`maximum_of_excluded_windows`), though neither exclusion nor uniqueness is proved here.

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