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Working Draft

A Conditional Fourier Reduction for Erdős's Ternary-Digit Problem

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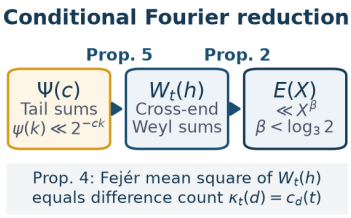
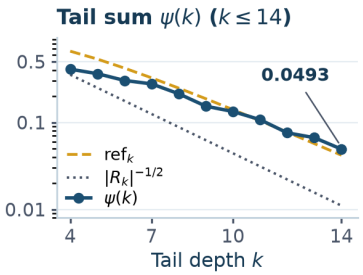
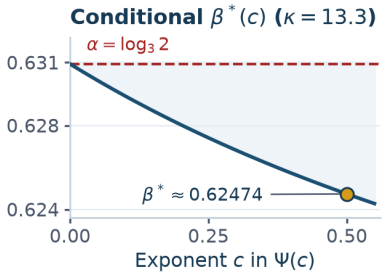
Ⓢ *Supporting Information:* **`recompute.py`** *and* **`recompute.out`** *(standalone Python verifier,* $t, k \le 14$*, 60-digit exponents)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.







ABSTRACT: Erdős conjectured that 2^n contains base-3 digit 2 for all $n > 8$. Let $E(X)$ count $0 \le n < X$ whose base-3 expansion omits 2 and set $\alpha = \log_3 2 \approx 0.63093$. Narkiewicz (1980) proved $E(X) \le 1.62\,X^\alpha$, sharpened to $1.3\,X^\alpha$ by Holdum, Klausen, and Rasmussen (2015); Lagarias (2009, §1.6) asked for $O(X^\beta)$ $(\beta < \alpha)$ and quantitative leading–trailing independence. For $u_t = 2 \cdot 3^{t-1}$, let $R_t \subset [0, u_t)$ be the 2^{t-1} exponents whose last t ternary digits of 2^r omit 2, with $S_k(j) = \sum_{r \in R_k} e(jr/u_k)$ and $\psi(k) = \max_{3 \nmid j} |S_k(j)|/|R_k|$. We prove three propositions reducing both questions to one unproved tail hypothesis $\Psi(c)$: $\psi(k) \le C \cdot 2^{-ck}$. Beurling–Selberg majorants bound $E(u_t)$ via $W_t(h) = \sum_{r \in R_t} e(hr\alpha)$; the Fejér mean square of $W_t(h)$ equals the difference count of R_t at $d\alpha$, whose cyclic wrap is $\kappa_t(d)$ of $C(1, 2^d)$; and expanding $e(hr\alpha)$ mod u_t with Lagarias's $\kappa = 13.3$ shows $\Psi(c)$ $(c > 0)$ gives $E(X) \ll_\beta X^\beta$ for all $\beta > \alpha - \alpha(1-\alpha)\,Lc/((\kappa+1)(L+c))$ $(L = \log_2 3)$. At $c = 1/2$, every $\beta > 0.624740407688\ldots$ follows (e.g. $X^{0.6248}$). Checks in `recompute.py` verify $t \le 14$, with $\psi(k)$ falling from 0.4087 $(k = 4)$ to 0.0493 $(k = 14)$. Hypothesis $\Psi(c)$ is unsettled; no Lean 4 proof is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 7 in Lean Kernel (0%) · 1 Cert Only · 6 written · 1 open (not counted)
Abstract claims: 0 of 6 in Lean Kernel (0%) · 1 Cert Only · 5 written

Full Trust Boundary: Section 5.

MSC 2020: 11A63, 11L07, 11J71 **Keywords:** Erdős ternary-digit conjecture, exponential sums, Beurling–Selberg majorant, Fejér kernel, irrationality measure, 3-adic Cantor sets

1. INTRODUCTION AND MAIN RESULTS

In base 3, the powers $2^0 = (1)_3$, $2^2 = (11)_3$, and $2^8 = 256 = (100111)_3$ contain only the digits 0 and 1, whereas $2^9 = 512 = (200222)_3$ has four digits 2 and $2^{10} = 1024 = (1101221)_3$ has trailing 2-free length 1 and leading 2-free length 4. Erdős [4] (`[to check]`) conjectured in 1979 that $n \in \{0, 2, 8\}$ are the only nonnegative integers for which the base-3 expansion of 2^n omits the digit 2. Computational searches have verified the conjecture through $n \le 2 \cdot 3^{45} \approx 5.9 \times 10^{21}$ (Saye [13]), yet proving finiteness remains completely open.

Define the exceptional counting function

$$E(X) := \#\{0 \le n < X : 2^n \text{ has no digit 2 in base 3}\},$$

and set $\alpha := \log_3 2 = 0.630929753571\ldots$ and $L := \log_2 3 = 1/\alpha = 1.584962500721\ldots$. Because 2 is a primitive root modulo 3^t for every $t \ge 1$, the last t ternary digits of 2^n are periodic of exact period $u_t := 2 \cdot 3^{t-1}$, and precisely 2^{t-1} residue classes modulo u_t omit the digit 2 in those t positions. Choosing t so that $u_t \asymp X$ immediately yields $E(X) \le 2X^\alpha$ (Lagarias [7], Theorem 1.4). Narkiewicz [11] (`[to check]`) refined the leading constant to $E(X) \le 1.62\,X^\alpha$, and Holdum, Klausen, and Rasmussen [5] (Theorem 2.13) sharpened the constant further to $E(X) \le 1.3\,X^\alpha$. No unconditional improvement on the exponent $\alpha = \log_3 2$ is known.

In §1.6 of his 2009 study of ternary expansions of powers of 2, Lagarias [7] asked two coupled questions: whether combining constraints on both ends of $(2^n)_3$ can yield an upper bound $E(X) = O(X^\beta)$ for some $\beta < \log_3 2$, and whether the high-order (leading) and low-order (trailing) ternary digits of 2^n can be shown to be "uncorrelated" in some quantifiable way. At fixed depths (t_1, t_2), standard equidistribution of $(n \bmod u_{t_1}, \{n\alpha\})$ implies asymptotic independence as $X \to \infty$, but a cell-by-cell discrepancy sum cannot improve on X^α when $t_1 + t_2$ grows beyond $\log_3 X$ because the number of cells already exceeds X^α. Any improvement on α requires cancellation *across* tail residue classes inside a single tail period $0 \le r < u_t$.

Tail sets, cross-end sums, and open Fourier hypotheses. Write $e(x) := \exp(2\pi i x)$ and $\|x\| := \min_{m\in\mathbb{Z}} |x-m|$. For each integer $t \geq 1$, let $u_t := 2\cdot 3^{t-1}$ and define the *tail set*

$$R_t := \{0 \leq r < u_t : \text{the last } t \text{ ternary digits of } 2^r \text{ lie in } \{0,1\}\},$$

which satisfies $|R_t| = 2^{t-1}$ (Lemma 6). For integers $h \geq 1$ and $j \bmod u_t$, introduce two exponential sums over R_t:

$$W_t(h) := \sum_{r\in R_t} e(hr\alpha) \qquad \text{(cross-end Weyl sum)},$$

$$S_t(j) := \sum_{r\in R_t} e\!\left(\frac{jr}{u_t}\right) \qquad \text{(tail-set character sum)},$$

and define the normalized primitive character-sum maximum at depth $k \geq 1$ by

$$\psi(k) := \max_{\substack{0\leq j<u_k \\ 3\nmid j}} \frac{|S_k(j)|}{|R_k|}.$$

We formulate three asymptotic cancellation hypotheses (all unproved and explicitly unsettled): each asserts that there exists a constant $C > 0$ such that for all $t \geq 1$ (respectively $k \geq 1$),

- **Hypothesis** $W(\delta)$**:** $|W_t(h)| \leq C|R_t|2^{-\delta t}$ for all $1 \leq h \leq 2^{\delta t}$;
- **Hypothesis** $W_2(\delta)$**:** $\sum_{h=1}^{H} |W_t(h)|^2 \leq C|R_t|^2 2^{-2\delta t}$ with $H = \lfloor 2^{\delta t}\rfloor$;
- **Hypothesis** $\Psi(c)$**:** $\psi(k) \leq C2^{-ck}$ for all $k \geq 1$.

Note that $\Psi(c)$ involves the trailing ternary digits *alone*: neither $\alpha = \log_3 2$ nor the leading digits appear in its statement.

Theorem 1 (Conditional Reduction to Trailing-Digit Character Sums). *Let $\kappa > 0$ and $c_0 > 0$ be constants such that $\|q\alpha\| \geq c_0 q^{-\kappa}$ for every integer $q \geq 1$ (by Lagarias [7], Lemma 2.2(1), one may take $\kappa = 13.3$ and $c_0 = 1/1200$). Suppose Hypothesis $\Psi(c)$ holds for some $c > 0$. Then:*

1. *Hypothesis $W(\delta)$ holds for every exponent δ satisfying*
$$0 < \delta < \delta^*(c,\kappa) := \frac{Lc}{(\kappa+1)(L+c)}.$$
2. *Consequently, the exceptional counting function satisfies $E(X) \ll_\beta X^\beta$ for every real exponent*
$$\beta > \beta^*(c,\kappa) := \alpha - \alpha(1-\alpha)\,\delta^*(c,\kappa),$$
strictly below $\alpha = \log_3 2 \approx 0.63093$, and the leading and trailing ternary digits obey a quantitative power-saving independence bound over one tail period.

In particular, with $\kappa = 13.3$:

- *$\Psi(1/2)$ implies $W(\delta)$ for every $\delta < 0.026579983686\ldots$ and $E(X) \ll_\beta X^\beta$ for every $\beta > 0.624740407688\ldots$ (for example, $\delta = 0.0265$ gives $E(X) \ll X^{0.6248}$);*
- *$\Psi(0.35)$ implies $W(\delta)$ for every $\delta < 0.020048341228\ldots$ and $E(X) \ll_\beta X^\beta$ for every $\beta > 0.626261348966\ldots$ (for example, $\delta = 0.0200$ gives $E(X) \ll X^{0.6263}$).*

If in addition α had irrationality exponent 2 ($\kappa = 1+\varepsilon$ with $\varepsilon > 0$ arbitrarily small, unproved), then $\Psi(1/2)$ would imply $W(\delta)$ for every $\delta < 0.190046883357\ldots$ and $E(X) \ll_\beta X^\beta$ for every $\beta > 0.586675930504\ldots$ (for example, $E(X) \ll X^{0.5867}$).

Notice the rounding discipline in Theorem 1: every displayed threshold δ is rounded *down* below $\delta^*(c,\kappa)$ and every upper-bound exponent β is rounded *up* strictly above $\alpha - \alpha(1-\alpha)\delta$, so the strict exponent slack absorbs the $\log X$ factor from $W(\delta)$. In particular, at $(\kappa, c) = (13.3, 1/2)$, one has $0.6247 < \beta^*(1/2, 13.3) = 0.624740407688\ldots < 0.6248$, so $X^{0.6248}$ is established conditionally whereas $X^{0.6247}$ would overstate the bound.

Theorem 1 is assembled from three self-contained written propositions and one corollary (Propositions 2, 4, and 5, and Corollary 3), stated in full below. Let $I_{t_2} \subset [0,1)$ denote the head set of fractional parts $\{n\alpha\}$ for which the leading $t_2 \geq 2$ ternary digits of 2^n lie in $\{0,1\}$ (with leading digit 1), let $q(t_2) := |I_{t_2}|$ be its Lebesgue measure, and set $N_t(t_2) := \#\{r \in R_t : \{r\alpha\} \in I_{t_2}\}$.

Proposition 2 (Cross-End Majorant Reduction). *For every integer $t \geq 1$, $t_2 \geq 2$, and $H \geq 1$, the head set I_{t_2} is a disjoint union of $J = 2^{t_2-2}$ half-open arcs of total measure $q(t_2) \leq 0.92\cdot(2/3)^{t_2-1}$. With $n_0(t_2) := \lceil (t_2-1)/\alpha \rceil$, one has*

$$E(u_t) \leq n_0(t_2) + N_t(t_2)$$

and

$$N_t(t_2) \leq |R_t|\,q(t_2) + J\left[\frac{|R_t|}{H+1} + 2\sum_{h=1}^{H}\left(\frac{1}{H+1} + \frac{1}{\pi h}\right)|W_t(h)|\right].$$

Moreover, the same error term bounds $|N_t(t_2) - |R_t|\,q(t_2)|$ from both sides.

Corollary 3 (Power Saving from Cross-End Cancellation). *If Hypothesis $W_2(\delta)$ holds for some $\delta > 0$, then necessarily $\delta < 1$ and*

$$E(X) \ll X^{\alpha-\alpha(1-\alpha)\delta},$$

where $\alpha(1-\alpha) = 0.232857399629\ldots$. Under the pointwise Hypothesis $W(\delta)$, one has $E(X) \ll X^{\alpha-\alpha(1-\alpha)\delta}\log X$. In either case, for $t_2 = \max(2, \lceil \delta t/L \rceil)$ one obtains a power saving over X^α and a quantitative leading–trailing independence estimate on $N_t(t_2) - |R_t|\,q(t_2)$.

The second proposition links the Fejér mean square of $W_t(h)$ to the difference counts $\nu_t(d) := \#\{(r,r') \in R_t^2 : r - r' = d\}$ and to the level-t admissible counts $\kappa_t(d) := \#\{x \in F_t : 2^d x \bmod 3^t \in F_t\}$ of the 3-adic Cantor intersections $C(1, 2^d)$ studied by Abram and Lagarias [1], where F_t is the set of 2-free units modulo 3^t. Let $F_H(x) := \sum_{|h|\leq H}\left(1 - \frac{|h|}{H+1}\right)e(hx)$ be the Fejér kernel of degree H.

Proposition 4 (Fejér Mean-Square and Autocorrelation Bridge). *For all integers $t \geq 1$ and $H \geq 1$:*

1. $\displaystyle\sum_{|h|\leq H}\left(1 - \frac{|h|}{H+1}\right)|W_t(h)|^2 = \sum_{|d|<u_t} \nu_t(d)\,F_H(d\alpha);$
2. *$\kappa_t(0) = \nu_t(0) = |R_t|$, $\kappa_t(d) = 0$ for all odd d, and for every $0 < d < u_t$, $\kappa_t(d) = \nu_t(d) + \nu_t(u_t - d)$;*
3. *for every $j \bmod u_t$,*
$$|S_t(j)|^2 = \sum_{d \bmod u_t} \kappa_t(d)\,e\!\left(-\frac{jd}{u_t}\right).$$

By part (3) of Proposition 4, Hypothesis $\Psi(c)$ is equivalent to a power-saving bound on the discrete Fourier transform of the level-k Cantor-intersection counts $d \mapsto \kappa_k(d)$ over

all d mod u_k at primitive frequencies $3 \nmid j$:

$$\left| \sum_{d \bmod u_k} \kappa_k(d)\, e\!\left(-\frac{jd}{u_k}\right) \right| \le C^2\, |R_k|^2\, 2^{-2ck}.$$

Proposition 5 (Cross-End Sums from Primitive Tail Character Sums). *For all integers $t \ge 1$ and $h \ge 1$,*

$$\frac{|W_t(h)|}{|R_t|} \le \min\!\left(1, \frac{1}{2u_t\|h\alpha\|}\right) + \sum_{k=1}^{t} \psi(k) \left[\min\!\left(1, \frac{1}{2 \cdot 3^{t-k}\|u_k h\alpha\|}\right) + 3^{k-t}(2 + \ln u_k)\right].$$

Alongside the written proofs, our standalone deterministic verifier `recompute.py` checks every identity, inequality, planted-fault sensitivity control, 60-digit conditional exponent, and empirical tail-sum profile through $t, k \le 14$ (Proposition 7, Section 4). Figure 1 maps the proof flow and explicit verification boundary.

Concept in Action: Safe Conditional Exponent & R_3 Check

```
import math
a = math.log(2) / math.log(3); L = 1.0 / a
d_star = L * 0.5 / (14.3 * (L + 0.5))
b_star = a - a * (1.0 - a) * d_star
assert 0.0265 < d_star < 0.0266
assert 0.6247 < b_star < 0.6248  # safe upper bound: 0.6248
R3 = [r for r in range(18) if all(
    (pow(2, r, 27) // 3**j) % 3 != 2 for j in range(3))]
assert R3 == [0, 2, 6, 8]        # |R_3| = 2^(3-1) = 4
```

Roadmap. Section 2 fixes notation, proves Lemma 6 (periodicity, uniform 2-of-3 lifting, and the character-sum lift recursion), and illustrates the tail-head split on small powers of 2. Section 3 proves Propositions 2, 4, and 5, Corollary 3, and Theorem 1, and explains why the plain additive-character Riesz-product bound fails. Section 4 states Proposition 7 and reports the verification tables, planted-fault controls, and empirical decay of $\psi(k)$ and $W_t(h)$ through $t, k \le 14$. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 reviews prior work, unread sources (`[to check]`), and open questions (Conjecture 8), and Section 7 documents reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND TAIL-SET LIFTING

For a nonnegative integer x, let $d_m(x) := \lfloor x/3^{m-1} \rfloor \bmod 3 \in \{0,1,2\}$ denote its m-th ternary digit from the right ($m \ge 1$), so $d_1(x) = x \bmod 3$ is the units digit. For each integer $t \ge 1$, set

$$u_t := 2 \cdot 3^{t-1} = \varphi(3^t),$$

let $F_t := \{x \in \{1, \dots, 3^t - 1\} : d_1(x) = 1 \text{ and } d_m(x) \in \{0,1\} \text{ for } 2 \le m \le t\}$ be the set of 2-free units modulo 3^t, and define the tail set

$$R_t := \{0 \le r < u_t : 2^r \bmod 3^t \in F_t\}.$$

Notice that any integer $n \ge 0$ whose full base-3 expansion omits 2 satisfies $2^n \bmod 3^t \in F_t$ for every $t \ge 1$ (even when $2^n < 3^{t-1}$, since leading zero digits are $0 \in \{0,1\}$), and therefore $n \bmod u_t \in R_t$.

Worked examples. The three known 2-free powers are $2^0 = 1 = (1)_3$, $2^2 = 4 = (11)_3$, and $2^8 = 256 = 243 + 9 + 3 + 1 = (100111)_3$; hence $0, 2, 8 \in R_t$ whenever $u_t > n$. The next power $2^9 = 512 = 2 \cdot 243 + 2 \cdot 9 + 2 \cdot 3 + 2 = (200222)_3$ has four digits 2, while $2^{10} = 1024 = (1101221)_3$ has one 2-free trailing digit and four 2-free leading digits. At small depths, brute-force enumeration gives $R_1 = \{0\}$ ($u_1 = 2$), $R_2 = \{0,2\}$ ($u_2 = 6$), $R_3 = \{0,2,6,8\}$ ($u_3 = 18$), and $R_4 = \{0,2,8,18,20,24,26,42\}$ ($u_4 = 54$), all consisting solely of even exponents.

The following elementary lemma collects the algebraic lifting properties of R_t (see Lagarias [7], Theorem 1.4, and Saye [13], Lemma 3.1) together with the resulting character-sum recursion for $S_{k+1}(j)$.

Lemma 6 (Primitive-Root Periodicity, Uniform Lifting, and Character-Sum Recursion). *For every integer $k \ge 1$:*

1. *$2^{u_k} = 1 + 3^k a_k$ for an integer $a_k \equiv 1 \pmod 3$. Consequently, the multiplicative order of 2 modulo 3^k is u_k, the map $r \mapsto 2^r \bmod 3^k$ is a bijection from $\{0, 1, \dots, u_k - 1\}$ onto $(\mathbb{Z}/3^k\mathbb{Z})^\times$ sending R_k bijectively onto F_k, $|R_k| = |F_k| = 2^{k-1}$, and every $r \in R_k$ is even.*
2. *For every $r \in R_k$ and $i \in \{0,1,2\}$, the last k ternary digits of 2^{r+iu_k} agree with those of 2^r, and the $(k+1)$-th digit satisfies*
$$d_{k+1}(2^{r+iu_k}) \equiv d_{k+1}(2^r) + i \pmod 3.$$
Hence exactly one lift $i_{\mathrm{bad}}(r) \equiv 2 - d_{k+1}(2^r) \pmod 3$ is excluded from R_{k+1}, and for every $t \ge k$, each $r' \in R_k$ has exactly 2^{t-k} lifts in R_t, all congruent to $r' \pmod{u_k}$.
3. *For every integer j with $3 \nmid j$, the primitive character sum at level $k+1$ obeys the exact recursion*
$$S_{k+1}(j) = -e\!\left(\frac{2j}{3}\right) \sum_{r \in R_k} e\!\left(\frac{jr}{u_{k+1}} - \frac{j\, d_{k+1}(2^r)}{3}\right).$$

Proof. For (1), we induct on $k \ge 1$. At $k = 1$, $u_1 = 2$ and $2^2 = 4 = 1 + 3 \cdot 1$, so $a_1 = 1$. If $2^{u_k} = 1 + 3^k a_k$ with $a_k \equiv 1 \pmod 3$, cubing gives

$$2^{u_{k+1}} = (1 + 3^k a_k)^3 = 1 + 3^{k+1}(a_k + 3^k a_k^2 + 3^{2k-1} a_k^3),$$

and since $k \ge 1$, both 3^k and 3^{2k-1} are multiples of 3, giving $a_{k+1} \equiv a_k \equiv 1 \pmod 3$. The order of 2 modulo 3^k divides $\varphi(3^k) = u_k = 2 \cdot 3^{k-1}$ and is even because $2^1 \equiv 2 \not\equiv 1 \pmod 3$. For $k \ge 2$, any proper even divisor of $2 \cdot 3^{k-1}$ divides $u_{k-1} = 2 \cdot 3^{k-2}$, whereas $2^{u_{k-1}} = 1 + 3^{k-1} a_{k-1} \not\equiv 1 \pmod{3^k}$. Hence $\mathrm{ord}_{3^k}(2) = u_k$, making $r \mapsto 2^r \bmod 3^k$ a bijection onto $(\mathbb{Z}/3^k\mathbb{Z})^\times$ mapping R_k onto F_k. Every unit modulo 3^k has $d_1(x) \in \{1,2\}$, so omitting 2 forces $d_1(x) = 1$ (which forces r even since $2^r \equiv (-1)^r \pmod 3$) and leaves 2^{k-1} free choices for $(d_2, \dots, d_k) \in \{0,1\}^{k-1}$.

For (2), since $3^{2k} \equiv 0 \pmod{3^{k+1}}$ for $k \ge 1$ and $a_k \equiv 1 \pmod 3$, the binomial expansion yields $2^{iu_k} = (1 + 3^k a_k)^i \equiv 1 + i \cdot 3^k \pmod{3^{k+1}}$. Multiplying by 2^r and using $2^r \equiv d_1(2^r) = 1 \pmod 3$ for $r \in R_k$ gives

$$2^{r+iu_k} \equiv 2^r + i \cdot 3^k \pmod{3^{k+1}}.$$

Adding $i \cdot 3^k$ modulo 3^{k+1} leaves the lowest k ternary digits unchanged and shifts d_{k+1} by $i \bmod 3$ (discarding any carry above 3^k). Thus as i ranges over $\{0,1,2\}$, $d_{k+1}(2^{r+iu_k})$ takes each value $0, 1, 2$ once; the unique excluded lift satisfies $d_{k+1}(2^r) + i_{\mathrm{bad}}(r) \equiv 2 \pmod 3$, leaving two lifts in R_{k+1}.

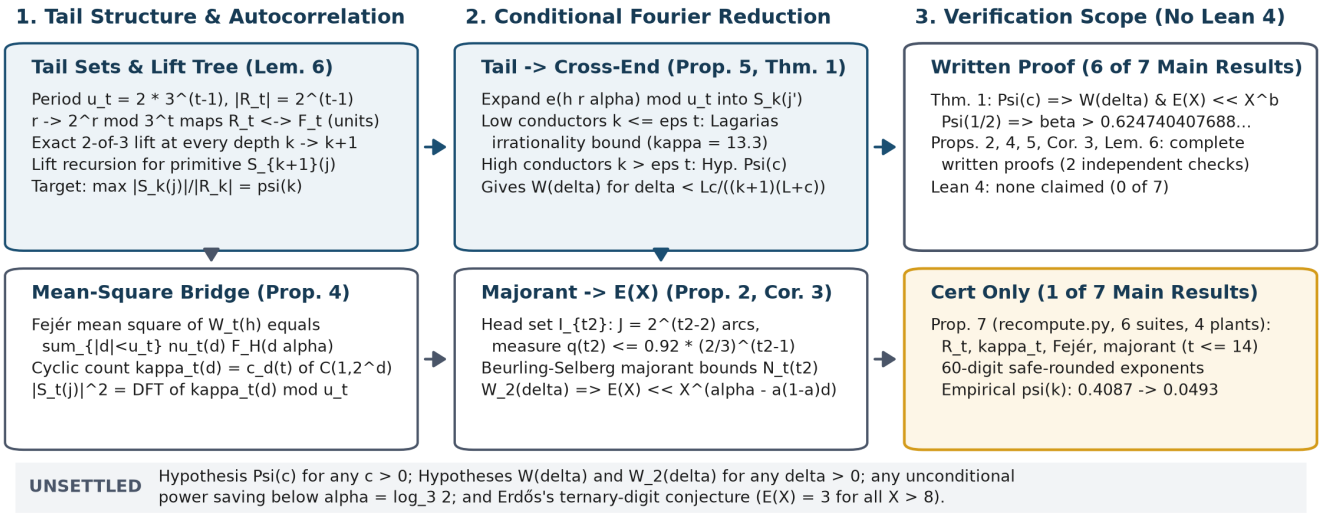


Figure 1. Proof and verification architecture for Theorem 1, Propositions 2, 4, 5, and 7, Corollary 3, and Lemma 6. No Lean 4 formalization is claimed (0 of 7 proved main results in Lean 4; 6 of 7 by self-contained written proofs checked by two independent adversarial review passes; 1 of 7 by deterministic Python verification in `recompute.py` with 4 planted-fault controls). Hypothesis $\Psi(c)$ for any $c > 0$, any unconditional power saving below $\alpha = \log_3 2$, and Erdős's ternary-digit conjecture remain unsettled.

Induction on $t - k \geq 0$ gives 2^{t-k} lifts in R_t congruent to $r' \pmod{u_k}$.

For (3), write each element of R_{k+1} as $r + iu_k$ with $r \in R_k$ and $i \in \{0,1,2\}$, $i \neq i_{\mathrm{bad}}(r)$. Because $u_{k+1} = 3u_k$ and $3 \nmid j$, we have $\sum_{i=0}^{2} e(ij/3) = 0$, so

$$\begin{aligned} S_{k+1}(j) &= \sum_{r \in R_k} e\left(\frac{jr}{u_{k+1}}\right) \sum_{\substack{i \in \{0,1,2\} \\ i \neq i_{\mathrm{bad}}(r)}} e\left(\frac{ij}{3}\right) \\ &= -\sum_{r \in R_k} e\left(\frac{jr}{u_{k+1}} + \frac{j\, i_{\mathrm{bad}}(r)}{3}\right) \\ &= -e\left(\frac{2j}{3}\right) \sum_{r \in R_k} e\left(\frac{jr}{u_{k+1}} - \frac{j\, d_{k+1}(2^r)}{3}\right). \end{aligned}$$ □

Lemma 6(3) gives a direct spectral interpretation of Hypothesis $\Psi(c)$: over $r \in R_k$, the next ternary digit $d_{k+1}(2^r)$ must fail to correlate with any primitive phase $e(jr/u_{k+1})$ $(3 \nmid j)$ with a power saving in 2^k.

3. PROOFS OF THE MAIN RESULTS

We now prove Propositions 2, 4, and 5, Corollary 3, and Theorem 1 in sequence.

Cross-End Majorant (Proposition 2 and Corollary 3).

Proof of Proposition 2. Fix integers $t \geq 1$, $t_2 \geq 2$, and $H \geq 1$. For any $n \geq n_0(t_2) = \lceil (t_2 - 1)/\alpha \rceil$, we have $n\alpha \geq t_2 - 1$ and hence $2^n \geq 3^{t_2-1}$, so the base-3 expansion of 2^n has at least t_2 digits. Its leading t_2 digits form an integer $d \in [3^{t_2-1}, 3^{t_2}-1]$ if and only if $d \cdot 3^m \leq 2^n < (d+1) \cdot 3^m$ with $m = \lfloor n\alpha \rfloor - (t_2 - 1) \geq 0$. Taking $\log_3$, this condition is equivalent to

$$\{n\alpha\} \in [\log_3 d - (t_2 - 1),\ \log_3(d+1) - (t_2 - 1)).$$

If 2^n is 2-free, its leading digit is 1 and the next $t_2 - 1$ digits lie in $\{0,1\}$, yielding 2^{t_2-1} admissible integers $d \in [3^{t_2-1}, 2 \cdot 3^{t_2-1})$. Since $t_2 \geq 2$, these integers pair into $J = 2^{t_2-2}$ consecutive pairs $\{d, d+1\}$ with $d \equiv 0 \pmod 3$ (while $d + 2 \equiv 2 \pmod 3$ is excluded). Each pair's two adjacent half-open intervals merge into a single arc $[\log_3 d - (t_2 - 1), \log_3(d+2) - (t_2 - 1))$, so the head set I_{t_2} is a disjoint union of $J = 2^{t_2-2}$ arcs in $[0,1)$.

For each admissible $d \geq 3^{t_2-1}$, the inequality $\ln(1 + 1/d) < 1/d$ gives

$$\log_3(d+1) - \log_3 d = \frac{\ln(1 + 1/d)}{\ln 3} < \frac{1}{d \ln 3} \leq \frac{1}{3^{t_2-1} \ln 3}.$$

Summing over all 2^{t_2-1} admissible d yields

$$q(t_2) = |I_{t_2}| < \frac{1}{\ln 3} \left(\frac{2}{3}\right)^{t_2-1} \leq 0.92 \cdot \left(\frac{2}{3}\right)^{t_2-1},$$

since $1/\ln 3 = 0.910239\ldots < 0.92$. Every 2-free $n < u_t$ with $n \geq n_0(t_2)$ belongs to R_t (as its last t ternary digits lie in $\{0,1\}$, padding with leading zeros if $2^n < 3^{t-1}$) and satisfies $\{n\alpha\} \in I_{t_2}$. This establishes $E(u_t) \leq n_0(t_2) + N_t(t_2)$.

Now let $I \subset \mathbb{R}/\mathbb{Z}$ be any one of the J disjoint arcs of I_{t_2}. By the Beurling–Selberg extremal trigonometric approximation (Vaaler [14] `[to check]`; Montgomery [10] `[to check]`, Chapter 1), there exist trigonometric polynomials $S_I^+(x) = \sum_{|h| \leq H} \hat{S}_I^+(h)\, e(hx)$ and $S_I^-(x) = \sum_{|h| \leq H} \hat{S}_I^-(h)\, e(hx)$ of degree H such that $S_I^-(x) \leq \mathbf{1}_I(x) \leq S_I^+(x)$ for all $x \in \mathbb{R}/\mathbb{Z}$, with constant terms $\hat{S}_I^{\pm}(0) = |I| \pm \frac{1}{H+1}$ and nonzero Fourier coefficients bounded by

$$|\hat{S}_I^{\pm}(h)| \leq \frac{1}{H+1} + \min\left(|I|, \frac{1}{\pi|h|}\right) \leq \frac{1}{H+1} + \frac{1}{\pi|h|}$$

for $1 \leq |h| \leq H$. Summing over $r \in R_t$ gives

$$\#\{r \in R_t : \{r\alpha\} \in I\} \leq \sum_{r \in R_t} S_I^+(r\alpha) = \sum_{|h| \leq H} \hat{S}_I^+(h)\, W_t(h).$$

Using $W_t(0) = |R_t|$, $W_t(-h) = \overline{W_t(h)}$, and pairing h with $-h$

yields

$$\#\{r \in R_t : \{r\alpha\} \in I\} - |R_t|\,|I| \le \frac{|R_t|}{H+1} + 2\sum_{h=1}^{H}\left(\frac{1}{H+1}+\frac{1}{\pi h}\right)|W_t(h)|.$$

The identical calculation with S_I^- yields the matching lower bound on $\#\{r \in R_t : \{r\alpha\} \in I\} - |R_t|\,|I|$. Summing over the J disjoint arcs of I_{t_2} (whose lengths sum to $q(t_2)$) completes the proof. □

Proof of Corollary 3. First observe that any nonvacuous Hypothesis $W_2(\delta)$ or $W(\delta)$ necessarily has $\delta < 1$. Indeed, suppose $\delta \ge 1$ and set $H = \lfloor 2^{\delta t} \rfloor$; for every $t \ge 1$, $H \ge 2^t = 2|R_t|$. By Proposition 4(1) (whose proof below is independent of Corollary 3), since $F_H(x) \ge 0$ for all x and $\nu_t(0) = |R_t|$ with $F_H(0) = H+1$,

$$|R_t|^2 + 2\sum_{h=1}^{H}\left(1-\frac{h}{H+1}\right)|W_t(h)|^2 \ge |R_t|\,(H+1) > 2|R_t|^2,$$

forcing $\sum_{h=1}^{H}|W_t(h)|^2 > \frac{1}{2}|R_t|^2$. This contradicts $W_2(\delta)$ (and also $W(\delta)$, which implies $\sum_{h=1}^{H}|W_t(h)|^2 \le C^2 H|R_t|^2 2^{-2\delta t} \le C^2|R_t|^2 2^{-\delta t}$) as $t \to \infty$. Hence $\delta < 1$, so $1-\delta(1-\alpha) > \alpha > 0$.

Assume $W_2(\delta)$ holds with constant $C > 0$, and let t be large enough that $H = \lfloor 2^{\delta t} \rfloor \ge 2$ and $\lceil \delta t/L \rceil \ge 2$. By Cauchy–Schwarz and $(a+b)^2 \le 2a^2 + 2b^2$,

$$\sum_{h=1}^{H}\left(\frac{1}{H+1}+\frac{1}{\pi h}\right)^2 \le \frac{2H}{(H+1)^2} + \frac{2}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2} < \frac{2}{H+1} + \frac{1}{3} \le 1$$

for all $H \ge 2$. Hence

$$\sum_{h=1}^{H}\left(\frac{1}{H+1}+\frac{1}{\pi h}\right)|W_t(h)| \le \left(\sum_{h=1}^{H}|W_t(h)|^2\right)^{1/2} \le \sqrt{C}\,|R_t|\,2^{-\delta t}.$$

With $t_2 = \max(2, \lceil \delta t/L \rceil)$ and $1/(H+1) \le 2^{-\delta t}$, Proposition 2 gives

$$N_t(t_2) \le |R_t|\left[0.92\cdot\left(\frac{2}{3}\right)^{t_2-1} + 2^{t_2-2}(1+2\sqrt{C})\,2^{-\delta t}\right].$$

For $t > L/\delta$, the choice $t_2 = \lceil \delta t/L \rceil$ satisfies $2^{\delta t} \le 3^{t_2} \le 3\cdot 2^{\delta t}$, so $2^{t_2}2^{-\delta t} \le 3\cdot(2/3)^{t_2}$. Both terms inside the brackets are therefore $O((2/3)^{t_2})$, and

$$\left(\frac{2}{3}\right)^{t_2} = 3^{-t_2(1-\alpha)} \le 2^{-\delta(1-\alpha)t}$$

because $3^{t_2} \ge 2^{\delta t}$ and $3^\alpha = 2$. Since $n_0(t_2) = O(t)$ is exponentially smaller than $|R_t|\,2^{-\delta(1-\alpha)t} = \frac{1}{2}2^{(1-\delta(1-\alpha))t}$ (as $1-\delta(1-\alpha) > 0$), we obtain

$$E(u_t) \ll 2^{(1-\delta(1-\alpha))t} = (3^t)^{\alpha-\alpha(1-\alpha)\delta}.$$

For any real $X \ge 2$, choose the least integer $t \ge 1$ with $u_t = 2\cdot 3^{t-1} \ge X$; then $u_t < 3X$, so $3^t < \frac{9}{2}X$ and $E(X) \le E(u_t) \ll X^{\alpha-\alpha(1-\alpha)\delta}$.

Under Hypothesis $W(\delta)$, the harmonic sum adds a factor

$$\sum_{h=1}^{H}\left(\frac{1}{H+1}+\frac{1}{\pi h}\right) \le 1 + \frac{1+\ln H}{\pi} = O(t) = O(\log X),$$

giving $E(X) \ll X^{\alpha-\alpha(1-\alpha)\delta}\log X$. Finally, the two-sided inequality of Proposition 2 directly bounds $|N_t(t_2) - |R_t|\,q(t_2)|$ by the same power-saving error term. □

Autocorrelation Bridge (Proposition 4).

Proof of Proposition 4. For (1), expand the squared modulus:

$$|W_t(h)|^2 = \sum_{r,r' \in R_t} e(h(r-r')\alpha) = \sum_{|d|<u_t}\nu_t(d)\,e(hd\alpha),$$

where $\nu_t(d) = \#\{(r,r') \in R_t^2 : r - r' = d\}$ satisfies $\nu_t(-d) = \nu_t(d)$. Multiplying by $(1-|h|/(H+1))$ and summing over $|h| \le H$ exchanges the two finite sums to yield

$$\sum_{|h|\le H}\left(1-\frac{|h|}{H+1}\right)|W_t(h)|^2 = \sum_{|d|<u_t}\nu_t(d)\,F_H(d\alpha).$$

For (2), by Lemma 6(1) the map $r \mapsto 2^r \bmod 3^t$ sends $\{0,\dots,u_t-1\}$ bijectively onto $(\mathbb{Z}/3^t\mathbb{Z})^\times$ and R_t bijectively onto F_t. Hence for every integer d,

$$\begin{aligned}\kappa_t(d) &= \#\{x \in F_t : 2^d x \bmod 3^t \in F_t\}\\ &= \#\{r' \in R_t : (r'+d) \bmod u_t \in R_t\}.\end{aligned}$$

At $d = 0$, this equals $|R_t| = \nu_t(0)$. Because every $r \in R_t$ is even and $u_t = 2\cdot 3^{t-1}$ is even, $r'+d \bmod u_t$ is odd whenever d is odd, forcing $\kappa_t(d) = 0$ for all odd d. For $0 < d < u_t$, a pair $(r',r) \in R_t^2$ satisfies $r \equiv r'+d \pmod{u_t}$ either without wraparound ($r - r' = d$, counted by $\nu_t(d)$) or with wraparound ($r' - r = u_t - d$, counted by $\nu_t(u_t-d)$), and the two cases are disjoint. Therefore $\kappa_t(d) = \nu_t(d) + \nu_t(u_t-d)$.

For (3), expanding $|S_t(j)|^2$ over $r,r' \in R_t$ and grouping by the cyclic difference $d \equiv r - r' \pmod{u_t}$ gives

$$|S_t(j)|^2 = \sum_{r,r'\in R_t} e\left(\frac{j(r-r')}{u_t}\right) = \sum_{d \bmod u_t}\kappa_t(d)\,e\left(\frac{jd}{u_t}\right).$$

Since $\kappa_t(-d) = \kappa_t(d)$ for every $d \bmod u_t$, replacing d by $-d$ yields $|S_t(j)|^2 = \sum_{d \bmod u_t}\kappa_t(d)\,e(-jd/u_t)$. □

Cross-End Sums from Tail Character Sums (Proposition 5 and Theorem 1).

Proof of Proposition 5. Fix integers $t \ge 1$ and $h \ge 1$. Expand the function $r \mapsto e(hr\alpha)$ on $\{0,1,\dots,u_t-1\}$ in the discrete Fourier basis modulo u_t:

$$e(hr\alpha) = \sum_{j=0}^{u_t-1} c_j\,e\left(\frac{jr}{u_t}\right), \qquad c_j := \frac{1}{u_t}\sum_{r=0}^{u_t-1} e\left(r\left(h\alpha - \frac{j}{u_t}\right)\right).$$

For every real $\theta \notin \mathbb{Z}$, the geometric-series bound gives

$$\left|\sum_{r=0}^{u_t-1} e(r\theta)\right| \le \min\left(u_t, \frac{1}{|\sin(\pi\theta)|}\right) \le \min\left(u_t, \frac{1}{2\|\theta\|}\right).$$

Since α is irrational, $h\alpha - j/u_t \notin \mathbb{Z}$, so the coefficients satisfy

$$|c_j| \le \min\left(1, \frac{1}{2u_t\|h\alpha - j/u_t\|}\right).$$

Summing $e(hr\alpha)$ over $r \in R_t$ gives $W_t(h) = \sum_{j=0}^{u_t-1} c_j S_t(j)$.

The $j = 0$ term has $S_t(0) = |R_t|$, contributing at most $|R_t| \min(1, 1/(2u_t\|h\alpha\|))$. Every nonzero $j \in \{1, \dots, u_t - 1\}$ has 3-adic valuation $v_3(j) \in \{0, 1, \dots, t-1\}$ because $u_t = 2 \cdot 3^{t-1}$. Setting $k := t - v_3(j) \in \{1, \dots, t\}$ expresses j uniquely as $j = 3^{t-k}j'$ with $1 \le j' < u_k$ and $3 \nmid j'$. Then $j/u_t = j'/u_k$, and by Lemma 6(2), each $r' \in R_k$ has exactly 2^{t-k} lifts in R_t all congruent to $r' \pmod{u_k}$. Consequently,

$$S_t(j) = \sum_{r \in R_t} e\left(\frac{j'r}{u_k}\right) = 2^{t-k} S_k(j'),$$

so $|S_t(j)| \le 2^{t-k}|R_k|\,\psi(k) = |R_t|\,\psi(k)$.

For each fixed conductor $k \in \{1, \dots, t\}$, enlarge the primitive index set $\{j' \in [1, u_k) : 3 \nmid j'\}$ to all u_k residues $j' \bmod u_k$, which only increases the upper bound. The points $j'/u_k \bmod 1$ form a uniform grid of spacing $1/u_k$. The grid point j'_0/u_k nearest to $h\alpha \bmod 1$ lies at distance θ_0/u_k where $\theta_0 = \|u_k h\alpha\| \in (0, 1/2]$, contributing

$$\min\left(1, \frac{1}{2u_t(\theta_0/u_k)}\right) = \min\left(1, \frac{1}{2 \cdot 3^{t-k}\|u_k h\alpha\|}\right),$$

since $u_t/u_k = 3^{t-k}$. Among the remaining $u_k - 1$ grid points on $\mathbb{R}/\mathbb{Z}$, the m-th point moving away from $h\alpha$ on the side opposite j'_0/u_k ($1 \le m \le u_k/2$) lies at distance $(m - \theta_0)/u_k \ge (m - 1/2)/u_k$, while the m-th point beyond j'_0/u_k on the same side ($1 \le m \le u_k/2 - 1$) lies at distance $(m + \theta_0)/u_k \ge m/u_k$. Setting $M := u_k/2 \ge 1$, the elementary integral comparisons

$$\sum_{m=1}^{M} \frac{1}{m - 1/2} \le 2 + \int_1^M \frac{dx}{x - 1/2} = 2 + \ln(2M - 1) < 2 + \ln u_k,$$

$$\sum_{m=1}^{M-1} \frac{1}{m} \le 1 + \ln M < 1 + \ln u_k$$

(with the second sum empty when $M = 1$) give

$$\sum_{\substack{j' \bmod u_k \\ j' \ne j'_0}} |c_{3^{t-k}j'}| \le \frac{u_k}{2u_t}(3 + 2\ln u_k) = 3^{k-t}\left(\frac{3}{2} + \ln u_k\right) < 3^{k-t}(2 + \ln u_k).$$

Multiplying by $|R_t|\,\psi(k)$, summing over $k = 1, \dots, t$, and adding the $j = 0$ bound yields Proposition 5. □

Proof of Theorem 1. Assume $\Psi(c)$ holds with constant $C > 0$ ($\psi(k) \le C\,2^{-ck}$ for all $k \ge 1$) and $\|q\alpha\| \ge c_0\, q^{-\kappa}$ for all integers $q \ge 1$. Fix any δ with $0 < \delta < \delta^*(c, \kappa) = \frac{Lc}{(\kappa+1)(L+c)}$. Then

$$\frac{\delta}{c} < \frac{1}{\kappa + 1} - \frac{\delta}{L},$$

so we may choose a parameter ε strictly inside the open interval $(\delta/c,\ \frac{1}{\kappa+1} - \frac{\delta}{L}) \subset (0, 1)$. By choice of ε, both

$$\eta_1 := c\varepsilon - \delta > 0,$$
$$\eta_2 := -\delta - (L((\kappa + 1)\varepsilon - 1) + \kappa\delta) > 0.$$

Let $t \ge 1$ and $1 \le h \le 2^{\delta t}$. Apply Proposition 5 and split the sum over $k = 1, \dots, t$ at εt:

- *Zero frequency* ($j = 0$). Since $\|h\alpha\| \ge c_0 h^{-\kappa} \ge c_0 2^{-\kappa\delta t}$ and $u_t = \frac{2}{3}3^t = \frac{2}{3}2^{Lt}$,
$$\frac{1}{2u_t\|h\alpha\|} \le \frac{3}{4c_0} 2^{(\kappa\delta - L)t} \ll 2^{(-\delta-\eta_2)t}$$
because $\kappa\delta - L < L((\kappa + 1)\varepsilon - 1) + \kappa\delta = -\delta - \eta_2$.
- *Low conductors* ($1 \le k \le \varepsilon t$). Bound $\psi(k) \le 1$. Since $u_k = 2 \cdot 3^{k-1} < 3^k$, the irrationality bound gives $\|u_k h\alpha\| \ge c_0 (u_k h)^{-\kappa} > c_0 (3^k 2^{\delta t})^{-\kappa}$, so
$$\frac{1}{2 \cdot 3^{t-k}\|u_k h\alpha\|} \le \frac{1}{2c_0} 3^{(\kappa+1)k - t} 2^{\kappa\delta t} \le \frac{1}{2c_0} 2^{(-\delta-\eta_2)t}.$$
The accompanying harmonic term is bounded by $2^{L(\varepsilon-1)t}$ times $(2 + t\ln 3)$, which decays faster because $L(\varepsilon - 1) < L((\kappa + 1)\varepsilon - 1) + \kappa\delta$. Summing over at most εt indices gives $O(t^2 2^{(-\delta-\eta_2)t}) = O(2^{-\delta t})$.
- *High conductors* ($\varepsilon t < k \le t$). Bound the minimum term by 1 and invoke $\psi(k) \le C\,2^{-ck}$. Each term is at most $C\,2^{-ck}(3 + t\ln 3)$, and summing the geometric series over $k > \varepsilon t$ yields $O(t\,2^{-c\varepsilon t}) = O(t\,2^{(-\delta-\eta_1)t}) = O(2^{-\delta t})$.

Combining all three contributions proves $|W_t(h)|/|R_t| \ll_{\delta,c,\kappa} 2^{-\delta t}$ uniformly for $1 \le h \le 2^{\delta t}$, establishing Hypothesis $W(\delta)$.

For part (2), fix any $\beta > \beta^*(c, \kappa) = \alpha - \alpha(1 - \alpha)\delta^*(c, \kappa)$. Choose $\delta \in (0, \delta^*(c, \kappa))$ close enough to $\delta^*(c, \kappa)$ that $\beta_1 := \alpha - \alpha(1 - \alpha)\delta < \beta$. Part (1) gives $W(\delta)$, and Corollary 3 gives $E(X) \ll X^{\beta_1} \log X \ll_\beta X^\beta$ because $\beta - \beta_1 > 0$ absorbs $\log X$. Finally, substituting Lagarias's explicit irrationality measure $\|q\alpha\| \ge q^{-13.3}/1200$ [7] (Lemma 2.2(1), multiplying $|\alpha - p/q| \ge q^{-14.3}/1200$ by q) into $\delta^*(c, \kappa)$ and $\beta^*(c, \kappa)$ yields the displayed numerical bounds, certified to 60 decimal digits in Section 4. □

Why the Plain Additive-Character Riesz-Product Bound Fails. It is natural to ask whether $\Psi(c)$ or $W(\delta)$ follows directly by expanding $\mathbf{1}_{F_t}$ in additive characters modulo 3^t. For a unit $\xi \bmod 3^k$, the additive Fourier transform of $\mathbf{1}_{F_t}$ at $3^{t-k}\xi$ is a digit Riesz product of modulus

$$3^{-t}\,2^{t-k} \prod_{m=1}^{k-1} |1 + e(\xi/3^m)| = 3^{-t}\,2^{t-1} \prod_{m=1}^{k-1} |\cos(\pi\xi/3^m)|.$$

In the expansion of $W_t(h)$ or $S_t(j)$, this Riesz product multiplies the twisted exponential sum $K_k(\xi, h) := \sum_{r=0}^{u_k - 1} e(\xi \cdot 2^r/3^k + rh\alpha)$. By the primitive Gauss-sum modulus $|\tau_j| = 3^{k/2}$ (for $3 \nmid j$) and the geometric-series bound, $|K_k(\xi, h)| \le 3^{k/2}(3 + \ln u_k)$, so the total triangle-inequality contribution of conductor-k frequencies is proportional to

$$\frac{L_1(k-1)}{(2\sqrt{3})^k}\,|R_t|\,(3 + \ln u_k),$$

$$L_1(m) := \sum_{\substack{\xi \bmod 3^m \\ 3 \nmid \xi}} \prod_{j=1}^{m} |1 + e(\xi/3^j)|.$$

While Parseval gives $\sum_{\xi \bmod 3^m} \prod_{j=1}^{m} |1 + e(\xi/3^j)|^2 = 6^m$, direct computation in Section 4 shows that $L_1(m)$ grows by a step factor of approximately 3.8899 per digit on $6 \le m \le 14$, strictly above $2\sqrt{3} \approx 3.4641$. Although $m \le 14$ is finite data

rather than an asymptotic lower bound, it demonstrates why the plain triangle-inequality Riesz-product bound blows up rather than decaying at moderate depths: proving $\Psi(c)$ requires cancellation across frequencies or deeper arithmetic input on $2^r \bmod 3^k$ (for instance via a transfer operator on Lemma 6(3)'s lift recursion, or a large-values estimate analogous to Maynard's restricted-digit method [9]).

Note also that Parseval already gives the *mean-square* analogue of $\Psi(1/2)$ unconditionally:

$$\frac{1}{u_k}\sum_{j \bmod u_k} |S_k(j)|^2 = |R_k|,$$

so the typical size of $|S_k(j)|/|R_k|$ is $|R_k|^{-1/2} = 2^{-(k-1)/2}$. What remains open is controlling the *maximum* over primitive j (or establishing a large-values bound showing that large $|S_k(j)|$ avoid the rotated points $u_k\{h\alpha\}$, $h \le H$, where the coefficients c_j of Proposition 5 concentrate).

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Every finite numerical claim, table entry, and 60-digit threshold in this draft is independently computed by the deterministic script `recompute.py` (output saved verbatim in `recompute.out`, runtime roughly 5 seconds).

Proposition 7 (Finite Verification and Empirical Tail-Sum Profile $(t, k \le 14)$). *Deterministic computation in* `recompute.py` *verifies the following six finite suites (with 4 planted-fault controls caught and 0 failures):*

1. *Exact base-3 expansions of* $2^0 = (1)_3$, $2^2 = (11)_3$, $2^8 = (100111)_3$, $2^9 = (200222)_3$, *and* $2^{10} = (1101221)_3$, *and confirmation that* $\{0, 2, 8\}$ *are the sole 2-free exponents in* $[0, 1000]$.
2. *Exact lift-tree construction of* R_t *for* $1 \le t \le 14$ *with* $|R_t| = 2^{t-1}$ ($|R_{14}| = 8{,}192$), *all* $r \in R_t$ *even,* $2^{u_k} \equiv 1 + 3^k \pmod{3^{k+1}}$, *brute-force equality for* $t \le 6$, *and bijection* $r \mapsto 2^r \bmod 3^t$ *onto* F_t (*planted shift* $42 \to 44$ *in* R_4 *fails bijection*).
3. *Proposition 4: Fejér mean-square identity holds to relative error* $\le 3.94 \times 10^{-12}$ ($6 \le t \le 12$, $H \in \{8, 64\}$; *planted removal of Fejér weights fails with gap* $\ge 3.02 \times 10^{-3}$); $\kappa_t(d) = \nu_t(d) + \nu_t(u_t - d)$ *matches direct counting for all* $d < u_t$ ($t \le 8$) *and selected* d ($t \le 14$), *with* $\kappa_t(2) = F_t$ (*Fibonacci through* $F_{14} = 377$) *and* $\kappa_t(8) = c_8(t)$ *matching an independent carry automaton through* $t = 14$; *cyclic Fourier identity holds to* $4.44 \times 10^{-16}|R_t|^2$ ($t \le 12$).
4. *Proposition 2: exact arc counts* $J = 2^{t_2-2}$ *and measures* $q(t_2) \le 0.92 \cdot (2/3)^{t_2-1}$ ($2 \le t_2 \le 5$), 0 *violations across* 100 *tested* (t, t_2, H) *cases for both* R_t *and its adversarial head-restricted subset* (*planted removal of* W_t *violates* 90/90 *cases with* $q + J/(H+1) < 1$; *planted rational* $\alpha = 11/18$ *locks* $|W_t(1)|/|R_t| = 2\cos(2\pi/9)/4 = 0.383022$ *for all* $3 \le t \le 14$).
5. *Lemma 6(3) lift recursion verified to* $9.55 \times 10^{-16}|R_k|$ ($1 \le k \le 11$, *all* $3 \nmid j$), *Proposition 5 lift scaling verified to* $6.21 \times 10^{-17}|R_4|$, *empirical tables of* $\psi(k)$ ($1 \le k \le 14$), M_t *and* MS_t ($6 \le t \le 14$), *and Riesz-product* $L_1(k)$ *with Parseval check* 6^k ($1 \le k \le 14$).
6. *60-digit decimal certification of* α, L, $\alpha(1-\alpha)$, *and all safe-rounded thresholds and exponents of Theorem 1* (*including* $0.6247 < \beta^*(1/2, 13.3) = 0.624740407688\ldots < 0.6248$).

Table 1 and Figure 2 summarize the empirical behavior of $\psi(k)$ and $W_t(h)$ through $k, t \le 14$ (at $k = 1, 2, 3$, one has $\psi(1) = 1.0000$, $\psi(2) = 0.5000$, and $\psi(3) = 0.4698$). Four empirical observations stand out, each strictly finite data and not a proof of any asymptotic hypothesis:

1. *Decay of* $\psi(k)$. Over $7 \le k \le 14$, $\psi(k)$ falls from 0.2772 to 0.0493, with mean per-level ratio $(\psi(14)/\psi(7))^{1/7}$ equal to 0.7813 (against $2^{-1/2} \approx 0.7071$). On $7 \le k \le 14$, $\psi(k)/\mathrm{ref}_k$ sits between 0.821 and 1.173. Because every $r \in R_k$ is even, $S_k(j + u_k/2) = S_k(j)$ and $S_k(-j) = \overline{S_k(j)}$, so ref_k serves only as a heuristic scale rather than a calibrated null law.
2. *Convergent resonance in* $W_t(h)$. For $6 \le t \le 10$, the maximum of $|W_t(h)|/|R_t|$ on $1 \le h \le 1024$ sits at $h = 527$: since $1054 = 2 \cdot 527$ is a convergent denominator of α whose distance $\|1054\alpha\|$ equals 3.9736×10^{-5} and every $r \in R_t$ is even, the phases $527r\alpha$ cluster near 0 mod 1 while $u_t \ll 10^5$. For $t \ge 11$ ($u_{11} = 118{,}098$), $h = 527$ breaks resonance and M_t falls from 0.1135 ($t = 11$) to 0.0524 ($t = 14$).
3. *Sub-random mean square and head counts.* For $11 \le t \le 14$, the normalized Fejér mean square $MS_t(1024)$ lies in $[0.848, 0.872]$, slightly below the independent-phase value 1. Across all 20 pairs (t, t_2) with even $6 \le t \le 14$ and $2 \le t_2 \le 5$, the count $N_t(t_2)$ stays within 1.14 Poisson units $\sqrt{q(t_2)|R_t|}$ of $q(t_2)|R_t|$. At $t = 14$, Proposition 2 improves on $|R_{14}|$, giving $0.42\,|R_{14}|$ at $t_2 = 3, 4$.
4. *Asymptotic character of Proposition 5.* At $t = 14$, the harmonic tail sum

$$\sum_{k=4}^{14} \psi(k)\, 3^{k-14}(2 + \ln u_k) = 1.402 > 1,$$

so Proposition 5 is not numerically discriminating at $t \le 14$; it becomes nontrivial only at large t where 2^{-ck} beats $(2 + \ln u_k)$.

Table 2 records the 60-digit exact decimal evaluations from `recompute.py` illustrating Theorem 1 across three parameter pairs (κ, c), together with the explicit safe rounding directions.

Table 1. Empirical primitive tail character-sum maxima $\psi(k)$ $(k = 1, \dots, 14)$ and cross-end Weyl-sum maxima $M_t = \max_{1 \le h \le 1024} |W_t(h)|/|R_t|$ $(t = 6, \dots, 14)$, computed by `recompute.py`. Here $P_k = 4 \cdot 3^{k-2}$ is the count of primitive $j \bmod u_k$, $\mathrm{ref}_k = \sqrt{(\ln(P_k/2) + \gamma)/|R_k|}$ is a heuristic independent-phase scale (not a proved null distribution), and $MS_t(H) = \frac{2}{H|R_t|} \sum_{h=1}^{H} (1 - h/(H+1))|W_t(h)|^2$.

Primitive Tail Character Sums $\psi(k)$							Cross-End Weyl Sums $W_t(h)$ $(h \le 1024)$				
k	$\lvert R_k\rvert$	u_k	$\psi(k)$	$\lvert R_k\rvert^{-1/2}$	ref_k	ψ/ref	t	M_t (at h)	M_t/ref_{1024}	$MS_t(64)$	$MS_t(1024)$
4	8	54	0.4087	0.3536	0.6584	0.621	—	—	—	—	—
5	16	162	0.3632	0.2500	0.5342	0.680	—	—	—	—	—
6	32	486	0.3034	0.1768	0.4207	0.721	6	0.9998 (527)	2.064	1.082	0.971
7	64	1,458	0.2772	0.1250	0.3251	0.853	7	0.9986 (527)	2.915	1.102	0.972
8	128	4,374	0.2134	0.0884	0.2478	0.861	8	0.9873 (527)	4.076	1.008	0.987
9	256	13,122	0.1536	0.0625	0.1871	0.821	9	0.8870 (527)	5.179	1.494	1.111
10	512	39,366	0.1334	0.0442	0.1402	0.952	10	0.2646 (527)	2.185	1.384	0.956
11	1,024	118,098	0.1076	0.0312	0.1044	1.031	11	0.1135 (182)	1.325	0.873	0.867
12	2,048	354,294	0.0759	0.0221	0.0774	0.981	12	0.0959 (996)	1.583	0.827	0.872
13	4,096	1,062,882	0.0668	0.0156	0.0571	1.170	13	0.0696 (332)	1.625	0.701	0.855
14	8,192	3,188,646	0.0493	0.0110	0.0420	1.173	14	0.0524 (797)	1.732	0.687	0.848

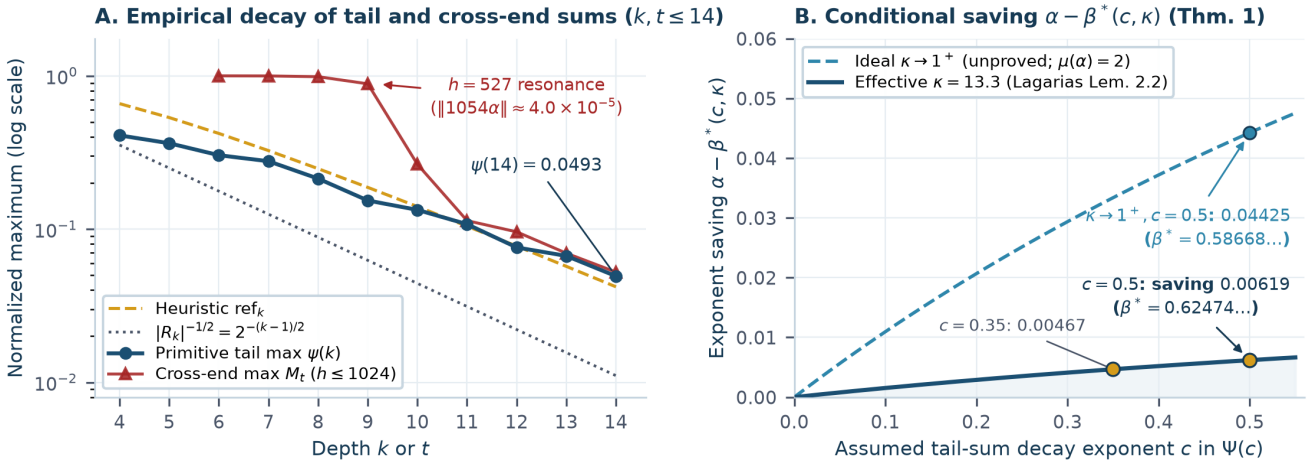


Figure 2. Empirical cancellation through depth 14 and conditional power-saving exponents. Panel A: log-scale decay of the primitive tail character-sum maximum $\psi(k)$ $(k = 4, \dots, 14$, dropping to 0.0493 at $k = 14)$ alongside $|R_k|^{-1/2}$, heuristic scale ref_k, and cross-end Weyl maxima M_t $(h \le 1024)$, showing the convergent resonance at $h = 527$ $(\|1054\alpha\| \approx 3.97 \times 10^{-5})$ breaking above $t = 10$. Panel B: conditional exponent saving $\alpha - \beta^*(c, \kappa)$ from Theorem 1 under Lagarias's effective $\kappa = 13.3$ (solid navy; saving 0.00619 at $c = 1/2$, $\beta^* = 0.624740407688\ldots$) versus the unproved optimal irrationality-measure benchmark $\kappa \to 1^+$ (dashed teal; saving 0.04425 at $c = 1/2$, $\beta^* = 0.586675930504\ldots$).

Table 2. Conditional thresholds $\delta^*(c, \kappa)$ and limiting exponents $\beta^*(c, \kappa)$ in Theorem 1 against $\alpha = \log_3 2 = 0.630929753571\ldots$, certified to 60 decimal digits in `recompute.py`. Safe δ is rounded down; safe β is rounded up above $\beta(\delta)$ to absorb $\log X$.

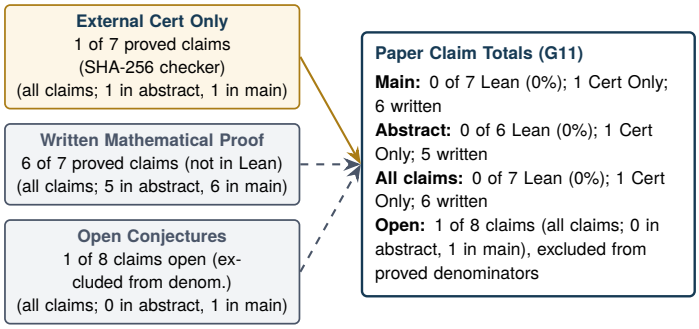
κ	c	$\delta^*(c, \kappa)$	Safe δ	$\beta^*(c, \kappa)$	Safe β
13.3	0.35	0.020048341228…	0.0200	0.626261348966…	0.6263
13.3	0.50	0.026579983686…	0.0265	0.624740407688…	0.6248
1^+	0.50	0.190046883357…	0.1900	0.586675930504…	0.5867

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Under Hypothesis Psi(c) (c > 0) and ‖q alpha‖ >= c_0 q^(-kappa), W(delta) holds for all 0 < delta < Lc/((kappa+1)(L+c)) and E(X) « X^beta for all beta > alpha - alpha(1-alpha)*delta* (with kappa = 13.3, Psi(1/2) gives beta > 0.624740407688..., e. g. X^0.6248)	Written	Written proof	Lean: none; fidelity: n/a	Conditional on unproved Hypothesis Psi(c); invokes Lagarias (2009) Lemma 2.2(1) (kappa = 13.3, c_0 = 1/1200) and Beurling-Selberg majorant (Vaaler 1985 / Montgomery 1994 [to check]). No Lean 4 formalization; 60-digit decimal arithmetic checked in recompute.py.
Prop. 2: E(u_t) <= n_0(t_2) + N_t(t_2) and two-sided Beurling-Selberg bound on N_t(t_2) - |R_t| q(t_2) in terms of cross-end Weyl sums W_t(h) (t >= 1, t_2 >= 2, H >= 1)	Written	Written proof	Lean: none; fidelity: n/a	Written proof using Beurling-Selberg extremal trigonometric polynomials (Vaaler 1985, Montgomery 1994 [to check]) and elementary head-arc pairing J = 2^(t_2 - 2), q(t_2) <= 0.92*(2/3)^(t_2 - 1). No Lean 4 formalization.
Cor. 3: Hypothesis W_2(delta) (delta > 0) implies delta < 1 and E(X) « X^(alpha - alpha(1-alpha)*delta); Hypothesis W(delta) gives the same with factor log X and quantitative leading-trailing independence	Written	Written proof	Lean: none; fidelity: n/a	Conditional on unproved Hypothesis W_2(delta) or W(delta); follows from Proposition 2 via Cauchy-Schwarz and Proposition 4(1) for delta < 1. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 4: Fejer mean square of W_t(h) equals sum_{|d| < u_t} nu_t(d) F_H(d alpha); cyclic wrap kappa_ t(d) = nu_t(d) + nu_t(u_t - d) equals level-t count of C(1, 2^d); |S_t(j)|^2 is the DFT of kappa_t(d) mod u_t	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof using Lemma 6 bijection r -> 2^r mod 3^t and finite Fourier identities (Section 3). No Lean 4 formalization.
Prop. 5: Pointwise upper bound on |W_t(h)|/|R_t| in terms of primitive tail character-sum maxima psi(k) and Diophantine distances ||u_k h alpha|| (all t >= 1, h >= 1)	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via discrete Fourier expansion mod u_t, Lemma 6 uniform lifting S_ t(3^(t-k) j') = 2^(t-k) S_k(j'), and harmonic grid bounds (Section 3). No Lean 4 formalization.
Lem. 6: Primitive-root periodicity u_k = 2*3^(k-1), bijection R_k <-> F_k with |R_k| = 2^(k-1), uniform 2-of-3 lifting, and exact primitive character-sum recursion for S_{k+1}(j)	Written	Written proof	Lean: none; fidelity: n/a	Elementary written induction on 2^(u_k) == 1 + 3^k (mod 3^(k+1)) (Section 2; periodicity and lifting follow Lagarias 2009 and Saye 2022). No Lean 4 formalization.
Prop. 7: Deterministic finite verification through t, k <= 14 (small base-3 expansions, |R_t|, bridge & majorant checks, 4 planted-fault controls, empirical psi(k) falling to 0.0493, and 60-digit conditional exponents)	Cert Only	recompute.py (607ccc03dfdc) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Verified by standalone deterministic Python script recompute.py (exact integer arithmetic for R_t/kappa_t, 60-digit Decimal for exponents, IEEE-754 double FFT/Weyl sums for t, k <= 14). Finite data proves no asymptotic hypothesis.
Conj. 8: Open Hypothesis Psi(c) for any c > 0, unconditional power saving E(X) « X^beta below log_3 2, and Erdos's ternary-digit conjecture (E(X) = 3 for X > 8)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open problems (excluded from G11 denominators).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: cb01fce0d59e | Fidelity SHA-256: d4a9427940ee.

6. PRIOR WORK AND OPEN QUESTIONS

Erdős [4] (`[to check]`) posed the ternary-digit problem for 2^n in 1979. Narkiewicz [11] (`[to check]`) established the upper bound $E(X) \le 1.62\, X^{\log_3 2}$ by counting admissible low-order residue classes; Lagarias [7] (Theorem 1.4) gave a six-line proof of $2X^{\log_3 2}$ for every nonzero 3-adic multiple $\lambda \cdot 2^n$, and Holdum, Klausen, and Rasmussen [5] (Theorem 2.13) improved the leading constant to $1.3\, X^{\log_3 2}$ at the same exponent $\alpha = \log_3 2$. On the 3-adic Hausdorff-dimension side, Lagarias [7] proved $\dim E(\mathbb{Z}_3) \le 1/2$, Abram and Lagarias [1] introduced the finite-automaton intersections $C(1, M)$ whose level-t counts $\kappa_t(d)$ appear in Proposition 4, and Abram, Bolshakov, and Lagarias [2] sharpened the dimension bound to $\dim E(\mathbb{Z}_3) \le \log_3 \varphi \approx 0.438018$. Complementary bounds restrict sparse expansions: Dimitrov and Howe [3] (`[to check]`) showed that no $n \notin \{0, 2, 8\}$ expresses 2^n as a sum of at most 25 distinct powers of 3.

Recent computational and structural work includes Saye [13], who verified Erdős's conjecture up to $n \le 2 \cdot 3^{45}$ via the binary lifting rule used in Lemma 6; Roettger and Ren [12], who analyzed leading-digit Benford statistics up to $n = 10^6$; Li and Zhao [8] (`[to check]`), who studied non-Wieferich generalizations and note that the exponent $\log_3 2$ remains unimproved; and Idowu [6], who reformulated the trailing-digit lifting tree in base-4 repunit coordinates without establishing a cross-end Fourier reduction. The analytic ingredients used in Propositions 2, 4, and 5—Beurling–Selberg majorants (Vaaler [14] `[to check]`; Montgomery [10] `[to check]`), Fejér mean squares, discrete Fourier inversion, and Lagarias's effective irrationality measure for $\log_3 2$ [7] (Lemma 2.2(1))—are standard tools, and priority is not claimed for the individual devices.

Prior-work searches logged. Direct arXiv and web literature searches together with a 5-query Google Scholar sweep on 2026-11-22 (41 returned records inspected, plus seven open-access full-text and abstract checks) surfaced Holdum, Klausen,

and Rasmussen [5] (correcting the background constant from 1.62 to 1.3 at exponent $\log_3 2$) and Idowu [6], while finding no source in the searches logged that states the joint leading–trailing law with an error term or derives a power saving below $X^{\log_3 2}$ from a trailing-digit character-sum bound $\Psi(c)$.

Unread and unsearched items (`[to check]`). The following sources remain unread in full or unsearched and are flagged `[to check]` in `references-checked.md`: (1) direct full-text inspection of Narkiewicz [11] (read only through secondary citations in [7, 5, 8]) and Erdős [4]; (2) original publisher PDFs of Vaaler [14] and Montgomery [10]; (3) full-text reading beyond abstract and §1 of Dimitrov and Howe [3] and Li and Zhao [8]; (4) paywalled publisher items surfaced in the 2026-11-22 Scholar sweep (Moore and Schulman, 2014, ACM Digital Library; Loxton, 1989, Springer), whose snippets indicate distinct problems (base-3/2 tree codes and Mahler automata); and (5) direct MathSciNet and zbMATH full-text citation-index sweeps.

Conjecture 8 (Open Hypotheses and Unconditional Power Saving). *The following problems remain completely unsettled:*

1. (Hypothesis $\Psi(c)$.) *Prove that there exist constants $c > 0$ and $C > 0$ such that $\psi(k) = \max_{3\nmid j} |S_k(j)|/|R_k| \le C \cdot 2^{-ck}$ for all $k \ge 1$ (or establish a large-values estimate for $S_k(j)$ sufficient to imply $W_2(\delta)$ for some $\delta > 0$).*
2. (Unconditional power saving.) *Prove unconditionally that $E(X) \ll X^{\beta}$ for some exponent $\beta < \log_3 2$.*
3. (Erdős's ternary-digit conjecture.) *Prove that $E(X) = 3$ for all $X > 8$ (equivalently, that 2^n contains the ternary digit 2 for every $n > 8$).*

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, and the standalone deterministic verifier reside in `papers/erdos-ternary-reduction/draft1/`. Running

```
python3 papers/erdos-ternary-reduction/draft1/recompute.py
```

executes in roughly 5 seconds using only the Python standard library and `numpy`, deterministically reproducing `recompute.out` (all six verification suites of Proposition 7, four planted-fault controls, Tables 1 and 2, and the Section 1 Python snippet). No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes derived the reduction, carried out two independent adversarial verification passes on the written proofs and rounding directions, wrote and rechecked the verification and plotting code, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 24, 2026

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