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Working Draft

Unique Fibonacci-Sum Permutations: Proof of Hajnal's Conjecture and Reshetnikov's Floor Formula

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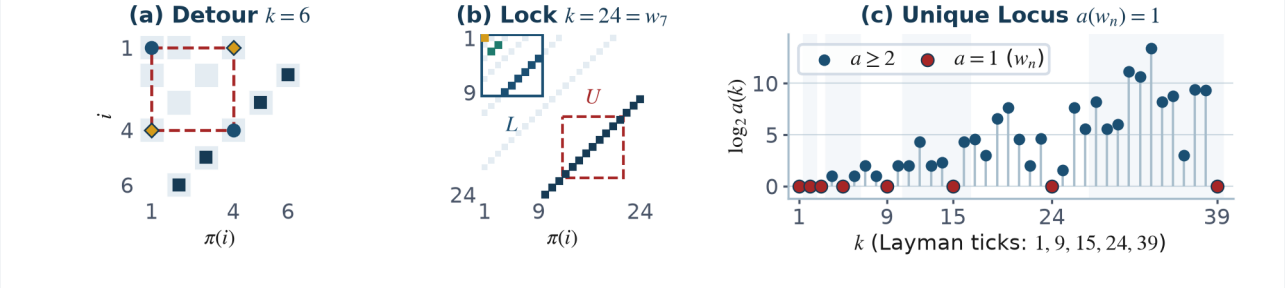
Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For $k \geq 1$, let $a(k)$ (OEIS A097082) count permutations π of $\{1,\dots,k\}$ with every $i+\pi(i)$ a positive Fibonacci number. Layman (2004, OEIS A097083) conjectured $a(k)=1$ precisely on $w_1=1$, $w_2=2$, $w_{n+2}=F_{n+4}-w_n-1$ $(1,2,3,5,9,15,24,\dots)$, and Reshetnikov (2015) conjectured $w_n=\lfloor F_{n+3}/\sqrt{5}\rfloor$. Hajnal (2022) proved "only if" in $M_4(\ell)=F_\ell+F_{\ell-4}+\cdots$ notation and posed the converse as Conjecture 1. We prove $a(k)=1 \iff \exists n\geq 1, k=w_n \iff \exists \ell \geq 2, k=M_4(\ell)$, the identity $M_4(n+1)=w_n$, and Reshetnikov's floor formula $w_n=\lfloor F_{n+3}/\sqrt{5}\rfloor$. A Step Lemma propagates uniqueness and absence of sum F_{n+1} along Layman's recurrence, proving the strengthened invariant on $[w_n]$; Cassini's identity and $5w_n+1\leq L_{n+3}\leq 5w_n+4$ give $5w_n^2<F_{n+3}^2<5(w_n+1)^2$. All seven proved results are verified in Lean 4 (46/46 pure-kernel declarations, standard axioms only). Growth-rate records of $a(k)^{1/k}$ at $k=F_j-1$ $(j\geq 5)$ remain open; no prior proof of "if" was found in logged searches (MathSciNet and print problem sections unsearched).

Verification Scope (G11): WARN: Human referee: pending
Main results: 7 of 7 in Lean Kernel (100%) · 1 open (not counted)
Abstract claims: 6 of 6 in Lean Kernel (100%) · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 05A05, 11B39, 05C70, 68V20 **Keywords:** Fibonacci numbers, Lucas numbers, Layman permutations, unique perfect matchings, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Can the integers $1,2,\dots,k$ be paired with a rearrangement $\pi(1),\dots,\pi(k)$ of themselves so that every sum $i+\pi(i)$ is a positive Fibonacci number? Sun [7] proved that such a permutation exists for every $k\geq 1$, and for most k there are many. Yet along the sparse sequence $k=1,2,3,5,9,15,24,39,64,\dots$, there is *exactly one* valid permutation.

Writing $a(k)$ for the count of permutations $\pi\in S_k$ with $i+\pi(i)$ a positive Fibonacci number for all $1\leq i\leq k$ (OEIS A097082 [5]), Layman [6] observed in 2004 that the integers $k\geq 1$ with $a(k)=1$ satisfy

$$w_1=1,\quad w_2=2,\quad w_{n+2}=F_{n+4}-w_n-1\ \ (n\geq 1),\qquad (1)$$

where $F_0=0$, $F_1=1$, $F_{m+2}=F_{m+1}+F_m$ are the standard Fibonacci numbers $(1,1,2,3,5,8,13,21,34,55,\dots)$. In 2015, Reshetnikov [6] conjectured the closed form $w_n=\lfloor F_{n+3}/\sqrt{5}\rfloor$. In 2022, Hajnal [3] named such permutations *Layman permutations*, defined the partial-sum sequence

$$M_4(0)=M_4(1)=0,\qquad M_4(2)=F_2=1,$$
$$M_4(3)=F_3=2,\qquad M_4(\ell+4)=F_{\ell+4}+M_4(\ell),\qquad (2)$$

and proved the "only if" direction: if $k\geq 1$ is not of the form $M_4(\ell)$, then $a(k)\geq 2$ [3, Theorems 1–2]. Hajnal isolated the converse as his Conjecture 1, remarking that "the hardness of Layman's conjecture (Conjecture 1) is to prove that for $n=M_4(\ell)$ we have a unique Layman permutation" [3, p. 237].

Theorem 1 (Characterization of Unique Fibonacci-Sum Permutations; Hajnal's Conjecture 1). *For every integer $k\geq 1$, $a(k)=1$ if and only if there exists an integer $n\geq 1$ such that $k=w_n$.*

Corollary 2 (Hajnal's M_4 Formulation). *For every integer $k\geq 1$, $a(k)=1$ if and only if there exists an integer $\ell\geq 2$ such that $k=M_4(\ell)$.*

The "only if" implication of Theorem 1 is Hajnal's the-

orem [3]; we include a self-contained proof in Section 3 matching our formalization. The converse "if" direction follows from a self-contained Step Lemma (Theorem 5) that propagates both uniqueness and absence of a forbidden Fibonacci sum along Layman's recurrence, yielding the strengthened invariant on $[w_n]$ (Theorem 6). We also prove $M_4(n+1) = w_n$ (Proposition 3) and Reshetnikov's floor formula $w_n = \lfloor F_{n+3}/\sqrt{5} \rfloor$ via exact Cassini–Lucas algebra (Theorem 7).

Every one of our seven proved results (Theorems 1, 5, 6, 7, Corollary 2, and Propositions 3, 4) is verified unconditionally inside the Lean 4 pure kernel (`lean/FibSumPerm/`, 46/46 declarations, zero `sorry`, zero `native_decide`, standard axioms only; Section 5) and registered in the Palomar Registry (`PALOMAR-2026-11-22-000029`, version 1, providing mechanical Lean checking plus automated editorial review). Only the empirical growth-rate records law (Conjecture 8) remains open. The standalone 10-line Python check below reproduces both uniqueness on $1 \le k \le 16$ and the integer square sandwich for $1 \le n \le 6$ verbatim (executed by `recompute.py`).

Concept in Action (10 lines, verified verbatim by recompute.py)

```
F, W = [0, 1], [0, 1, 2]
for _ in range(12): F.append(F[-1] + F[-2])
S = set(F[1:])
for n in range(3, 7): W.append(F[n+2] - W[n-2] - 1)
def cnt(k, i=1, u=0):
    return 1 if i > k else sum(cnt(k, i+1, u | 1<<j)
        for j in range(1, k+1) if i+j in S and not u & 1<<j)
assert [k for k in range(1, 17) if cnt(k) == 1] == W[1:]
for n in range(1, 7):
    assert 5*W[n]**2 < F[n+3]**2 < 5*(W[n]+1)**2
```

2. DEFINITIONS, SCALES, AND LAYMAN'S SEQUENCE

Write $\mathbb{N} = \{0, 1, 2, \dots\}$ and $[k] := \{1, 2, \dots, k\}$ for $k \in \mathbb{N}$ (with $[0] = \emptyset$). Matching `lean/FibSumPerm/Defs.lean` verbatim, define $(F_m)_{m\ge0}$ by $F_0 = 0, F_1 = 1, F_{m+2} = F_{m+1} + F_m$, and define the Lucas sequence $(L_m)_{m\ge0}$ by $L_0 = 2, L_1 = 1$, $L_{m+2} = L_{m+1} + L_m$ $(2, 1, 3, 4, 7, 11, 18, 29, 47, \dots)$. A natural number m satisfies IsFib(m) if $m > 0$ and $F_j = m$ for some $j \in \mathbb{N}$; note that $F_0 = 0$ is excluded while $F_1 = F_2 = 1$. Since $j - 1 \le F_j$ for all $j \ge 0$, any witness index satisfies $j < m + 2$, making IsFib(m) constructively decidable.

A permutation $\pi \in S_k$ of $[k]$ is a *valid Fibonacci-sum permutation* (or *Layman permutation*) if IsFib$(i + \pi(i))$ holds for every $i \in [k]$. Set

$$a(k) := \#\{\pi \in S_k : \pi \text{ is a valid Fibonacci-sum permutation}\},$$

with $a(0) = 1$ (the empty permutation is vacuously valid). If π is valid, then so is π^{-1}, since $i + \pi^{-1}(i) = \pi^{-1}(i) + \pi(\pi^{-1}(i))$. Equivalently, $a(k)$ is the permanent of the symmetric $k \times k$ binary matrix $(A_k)_{i,j} = \mathbf{1}_{\text{IsFib}(i+j)}$.

Fibonacci scale properties. For $n \ge 2$, $F_{n+1} = F_n + F_{n-1} > F_n$ since $F_{n-1} \ge 1$, so $(F_m)_{m\ge2}$ is strictly increasing. Consequently, no positive Fibonacci number lies strictly between F_a and F_{a+1}, and any positive Fibonacci number m satisfying $F_a < m < F_{a+2}$ must equal F_{a+1}. Moreover, for every integer $k \ge 1$, since $2k \ge 2 = F_3$ and $2k < F_{2k+2}$, taking the least r with $2k < F_r$ and setting $j := r - 1 \ge 3$ gives the unique *Fibonacci scale* satisfying $F_j \le 2k < F_{j+1}$.

Extend Layman's sequence (1) by $w_0 := 0$, so $w_1 = 1$, $w_2 = 2$, and $w_{n+3} = F_{n+5} - w_{n+1} - 1$ for $n \ge 0$ (leading terms $1, 2, 3, 5, 9, 15, 24, 39, 64, 104, 168, 272, 441, \dots$).

Proposition 3 (Equivalence of M_4 and Layman's Sequence). *For every integer $n \ge 1$, $M_4(n + 1) = w_n$.*

Proof. We establish the universal scale bounds

$$F_{n+1} \le w_n, \qquad w_n + 1 \le F_{n+2} \le 2w_n < F_{n+3} \tag{3}$$

for all $n \ge 1$ by strong induction. For $n = 1$: $F_2 = 1 \le 1$, $2 \le F_3 = 2 \le 2 < F_4 = 3$. For $n = 2$: $F_3 = 2 \le 2$, $3 \le F_4 = 3 \le 4 < F_5 = 5$. For $n = m + 3$ $(m \ge 0)$, the hypothesis at $m + 1$ gives $F_{m+2} \le w_{m+1}$ and $w_{m+1} + 1 \le F_{m+3} \le 2w_{m+1} < F_{m+4}$. Since $w_{m+1} + 1 \le F_{m+3} < F_{m+5}$, the subtraction $w_{m+3} = F_{m+5} - w_{m+1} - 1$ never underflows in $\mathbb{N}$, giving $w_{m+1} + w_{m+3} + 1 = F_{m+5}$ and hence:

$$\begin{aligned} w_{m+3} &= F_{m+5} - (w_{m+1} + 1) \ge F_{m+5} - F_{m+3} = F_{m+4}, \\ w_{m+3} + 1 &= F_{m+5} - w_{m+1} \le F_{m+5} - F_{m+2} \le F_{m+5}, \\ 2w_{m+3} &\ge 2F_{m+5} - (F_{m+4} - 1) - 2 \\ &= F_{m+5} + F_{m+3} - 1 \ge F_{m+5}, \\ 2w_{m+3} &\le 2F_{m+5} - F_{m+3} - 2 = F_{m+6} - 2 < F_{m+6}. \end{aligned}$$

The last line simultaneously gives the sharper upper bound

$$2w_n + 2 \le F_{n+3} \qquad (n \ge 3), \tag{4}$$

with equality at $n = 3$ $(2w_3 + 2 = 8 = F_6)$. Since $w_0 + w_2 + 1 = 3 = F_4$, the exact identity $w_n + w_{n+2} + 1 = F_{n+4}$ holds for all $n \ge 0$. Subtracting it at n from $n + 2$ yields the period-4 additive step

$$w_{n+4} = F_{n+5} + w_n \qquad (n \ge 0). \tag{5}$$

Finally, strong induction on $n \ge 1$ proves $M_4(n+1) = w_n$: direct check gives $M_4(2) = 1 = w_1$, $M_4(3) = 2 = w_2$, $M_4(4) = F_4 + M_4(0) = 3 = w_3$, $M_4(5) = F_5 + M_4(1) = 5 = w_4$; for $n = m+5$, (2) and (5) yield $M_4(m+6) = F_{m+6} + M_4(m+2) = F_{m+6} + w_{m+1} = w_{m+5}$. □

3. BLOCK REVERSAL, STEP LEMMA, AND HAJNAL'S CONJECTURE

Following Wasserman [6] and Hajnal [3], whenever $k < s \le 2k$ and $t := s - k - 1 \in [0, k - 1]$, any permutation $q \in S_t$ extends to $\text{ext}_{k,s}(q) \in S_k$ by setting $\text{ext}_{k,s}(q)(i) := q(i)$ for $1 \le i \le t$ and $\text{ext}_{k,s}(q)(i) := s - i$ for $t + 1 \le i \le k$. Because $i \mapsto s - i$ reverses $[t + 1, k]$ onto itself, $\text{ext}_{k,s}(q)$ is a bijection with inverse $\text{ext}_{k,s}(q^{-1})$.

Proposition 4 (Block-Reversal Extension Bound). *Whenever s is a positive Fibonacci number satisfying $k < s \le 2k$, we have $a(s - k - 1) \le a(k)$.*

Proof. For any valid $q \in S_t$ $(t = s - k - 1)$, $\text{ext}_{k,s}(q)$ is valid: on $i \le t$ the sum is $i + q(i)$, while on $t + 1 \le i \le k$ the sum is $i + (s - i) = s$. Since q is recovered as the restriction of $\text{ext}_{k,s}(q)$ to $[t]$, the map $q \mapsto \text{ext}_{k,s}(q)$ is injective, giving $a(t) \le a(k)$. □

Two consequences follow immediately (formalized as `fibPermCount_pos` and `two_le_fibPermCount_of_two_fibs` in `StepLemma.lean`). First, $a(k) \ge 1$ for every $k \ge 0$ (Sun [7], Hajnal [3]): $a(0) = 1$, and for $k \ge 1$ the unique scale $F_j \le$

$2k < F_{j+1}$ ($j \geq 3$) satisfies $F_{j-1} < F_j$, so $2k < F_j + F_{j-1} < 2F_j$, hence $k < F_j \leq 2k$ and $0 \leq F_j - k - 1 < k$, closing the strong induction via Proposition 4. Second, if two distinct positive Fibonacci numbers satisfy $k < s_1 < s_2 \leq 2k$, then picking valid $q_r \in S_{s_r-k-1}$ ($r = 1, 2$) produces valid permutations $\pi_r := \mathrm{ext}_{k,s_r}(q_r) \in S_k$ with $\pi_1(k) = s_1 - k \neq s_2 - k = \pi_2(k)$, forcing $a(k) \geq 2$.

Why uniqueness alone is not inductive. At $k = 6$ (Figure 1(a)), we have $F = 8, G = 5, H = 3, I = 2$, and $t = 8 - 6 - 1 = 1$, where $a(1) = 1$. Yet $a(6) = 2$: the move $1 \mapsto 1$ (sum $I = 2$) pairs with $4 \mapsto 4$ (sum $F = 8$) to form the detour $1 \mapsto 4$, $4 \mapsto 1$ of sum $G = 5$. Forbidding sum I on $[t]$ blocks every detour and propagates cleanly up the Fibonacci ladder.

Theorem 5 (The Step Lemma). *Let $j \geq 3$ and set $I := F_j$, $H := F_{j+1}$, $G := F_{j+2}$, and $F := F_{j+3}$. Suppose integers $k, t \geq 1$ satisfy*

$$G \leq k, \quad 2k < F + G, \quad t = F - 1 - k, \quad H \leq 2t < G.$$

If $a(t) = 1$ and every valid permutation of $[t]$ uses no sum equal to I, then $a(k) = 1$ and every valid permutation of $[k]$ uses no sum equal to G.

Proof. We have $G = H + I$, $F = G + H$, $F_{j+4} = F + G$, $2 \leq I < H < G < F$, $k < F \leq 2k$, and $t < H < G \leq k$. Let $p_0 \in S_t$ be the unique valid permutation of $[t]$, and set $\pi_{\mathrm{ext}} := \mathrm{ext}_{k,F}(p_0) \in S_k$, which is valid by Proposition 4. We show every valid $\pi \in S_k$ equals π_{ext} in four steps. (The hypothesis $H \leq 2t$ records the scale of t matching `FibSumPerm.step_lemma`; the deduction below uses only $2t < G$ and $t \geq 1$.)

Step 1 (Forced strips). Partition $[k]$ into $L := [1, t]$, $B_{\mathrm{lo}} := (t, H)$, $U := [H, G]$, and $B_{\mathrm{hi}} := (G, k]$. For any valid $r \in S_k$ and $x \in B_{\mathrm{hi}}$, we have $F_{j+2} = G < x + r(x) \leq 2k < F + G = F_{j+4}$, so the gap property forces $x + r(x) = F$. For $y \in B_{\mathrm{lo}}$, $m := F - y \in (G, k] = B_{\mathrm{hi}}$; applying the high-strip equation to r^{-1} at m gives $r^{-1}(m) = y$, i.e., $r(y) = F - y$. Thus every valid r swaps $B_{\mathrm{hi}} \leftrightarrow B_{\mathrm{lo}}$ via $x \mapsto F - x$ and preserves $R := L \cup U$.

Step 2 (Forced sums on $U \times U$ and $L \times U$). Let $r \in S_k$ be valid. If $x, r(x) \in U$, then $F_{j+2} = G < 2H \leq x + r(x) \leq 2G < F_{j+4}$, forcing $x + r(x) = F$. If $x \in L$ and $r(x) \in U$, then $F_{j+1} = H < 1 + H \leq x + r(x) \leq t + G < F = F_{j+3}$, forcing $x + r(x) = G$.

Step 3 (Detour pairing). Suppose a valid $r \in S_k$ sends some $y \in L$ to $r(y) \in U$. By Step 2, $y + r(y) = G = H + I$, so $r(y) \geq H$ forces $y \leq I$. Put $u := H + y \in U \subset R$, so $r(u) \in R = L \cup U$. If $r(u) \in U$, Step 2 would give $r(u) = F - u = (G + H) - (H + y) = G - y = r(y)$, contradicting injectivity of r since $u > t \geq y$. Thus $r(u) \in L$; applying Step 2 to r^{-1} at $r(u)$ gives $r(u) + u = G$, whence

$$r(H + y) = G - (H + y) = I - y \in L.$$

Hence $1 \leq I - y \leq t$ (so $y < I$) and $r^{-1}(I - y) = H + y \in U$.

Step 4 (Closing detours on $[t]$). For any valid $\pi \in S_k$, define $\pi', \rho' : L \to L$ by $\pi'(y) := I - y$ if $\pi(y) \in U$ and $\pi(y)$ otherwise, and $\rho'(z) := I - z$ if $\pi^{-1}(z) \in U$ and $\pi^{-1}(z)$ otherwise. By Steps 1 and 3 (for π and π^{-1}), both map L into L. If $\pi(y) \in U$, then $\pi^{-1}(I - y) = H + y \in U$, so $\rho'(\pi'(y)) = I - (I - y) = y$; if $\pi(y) \in L$, then $\pi^{-1}(\pi(y)) = y \in L$, so $\rho'(\pi'(y)) = y$. Symmetrically $\pi'(\rho'(z)) = z$, so $\pi' \in S_t$ is a permutation, and it is valid because detour pairs sum to $I = F_j$ while non-detour pairs keep π's sums.

Since every valid permutation of $[t]$ avoids sum I, no $y \in L$ can satisfy $\pi(y) \in U$. Symmetrically (via Step 3 for π^{-1}), no $z \in L$ satisfies $\pi^{-1}(z) \in U$. Thus π preserves L and U separately: $\pi|_L = \pi' = p_0$ by uniqueness, Step 2 forces $\pi(x) = F - x$ on U, and Step 1 forces $\pi(x) = F - x$ on $B_{\mathrm{lo}} \cup B_{\mathrm{hi}}$. Therefore $\pi = \pi_{\mathrm{ext}}$, proving $a(k) = 1$. Finally, π_{ext} has $i + \pi_{\mathrm{ext}}(i) \leq 2t < G$ on L and $i + \pi_{\mathrm{ext}}(i) = F > G$ on $[t + 1, k]$, so it avoids sum G. □

Theorem 6 (Strengthened Uniqueness Invariant; "If" Direction). *For every integer $n \geq 1$, $a(w_n) = 1$ and every valid permutation $\pi \in S_{w_n}$ satisfies $i + \pi(i) \neq F_{n+1}$ for all $i \in [w_n]$.*

Proof. We use strong induction on $n \geq 1$.

- $n = 1$ ($w_1 = 1, F_2 = 1$): the sole permutation $1 \mapsto 1$ has sum $2 \neq 1$.
- $n = 2$ ($w_2 = 2, F_3 = 2$): $2 + \pi(2) \in \{3, 4\}$ forces $\pi(2) = 1$, hence $\pi = (2, 1)$ is unique with sum $3 \neq 2$.
- $n = 3$ ($w_3 = 3, F_4 = 3$): $3 + \pi(3) \in \{4, 5, 6\}$ forces $\pi(3) = 2$, and applying this to π^{-1} gives $\pi(2) = 3$, leaving $\pi = (1, 3, 2)$ with sums $2, 5, 5$, avoiding 3.
- $n = m + 4 \geq 4$: set $j := m + 3 \geq 3$, $k := w_{m+4}$, $t := w_{m+2}$, so $I = F_{m+3}, H = F_{m+4}, G = F_{m+5}, F = F_{m+6}$, and $F + G = F_{m+7}$. By (3) and $w_{m+2} + w_{m+4} + 1 = F_{m+6}$, we have $G \leq k$, $2k < F + G$, $t = F - 1 - k \geq 1$, and $H \leq 2t < G$. By induction at $m + 2$, $a(t) = 1$ and no valid permutation of $[t]$ uses sum I. Theorem 5 yields $a(w_{m+4}) = 1$ and absence of sum $G = F_{(m+4)+1}$. □

Small non-uniqueness witnesses and proof of Theorem 1 and Corollary 2. Before closing the "only if" direction, check two small witness bounds (matching `fibPermCount_six` and `two_le_fibPermCount_ten` in `Main.lean`). At $k = 6$, any valid r satisfies $5 < 6 + r(6) \leq 12 < 13$ and $5 < 5 + r(5) \leq 11 < 13$, forcing $r(6) = 2, r(5) = 3$ and (via r^{-1}) $r(2) = 6, r(3) = 5$; the two valid choices on $\{1, 4\}$ give $\pi_{6,A} = (1, 6, 5, 4, 3, 2)$ and $\pi_{6,B} = (4, 6, 5, 1, 3, 2)$, so $a(6) = 2$. At $k = 10$, $\pi_{10,A} = (2, 1, 10, 9, 8, 7, 6, 5, 4, 3)$ (sums $3, 13$) and $\pi_{10,B} = (7, 6, 10, 9, 8, 2, 1, 5, 4, 3)$ (sums $8, 13$) are distinct and valid, so $a(10) \geq 2$ (in fact $a(10) = 4$). Likewise $a(4) = 2$, witnessed by $(1, 3, 2, 4)$ and $(4, 3, 2, 1)$.

Proof of Theorem 1 and Corollary 2. The "if" direction of Theorem 1 is the first clause of Theorem 6. For "only if", we argue by strong induction on $k \geq 1$ assuming $a(k) = 1$. Let $j \geq 3$ be the unique scale with $F_j \leq 2k < F_{j+1}$. If $j = 3$, then $2 \leq 2k < 3$ gives $k = 1 = w_1$; if $j = 4$, then $3 \leq 2k < 5$ gives $k = 2 = w_2$. Suppose $j \geq 5$, so $k < F_j \leq 2k$. If $k < F_{j-1}$, then $k < F_{j-1} < F_j \leq 2k$ would force $a(k) \geq 2$ by the two-Fibonacci bound, contradicting $a(k) = 1$; thus $F_{j-1} \leq k$.

Set $t := F_j - k - 1$. Since $2k \leq F_{j+1} - 1$ and $F_{j-2} \geq 2$ (as $j \geq 5$), we get $2t = 2F_j - 2k - 2 \geq F_j - F_{j-1} - 1 = F_{j-2} - 1 \geq 1$, so $1 \leq t < k$. By Proposition 4 and existence, $1 \leq a(t) \leq a(k) = 1$, hence $a(t) = 1$. By induction, $t = w_m$ for some $m \geq 1$. Now locate m:

1. Since $F_{j-1} \leq k$, we have $F_{m+1} \leq w_m = t \leq F_j - F_{j-1} - 1 < F_{j-2}$ by (3), so $m \leq j - 4$.

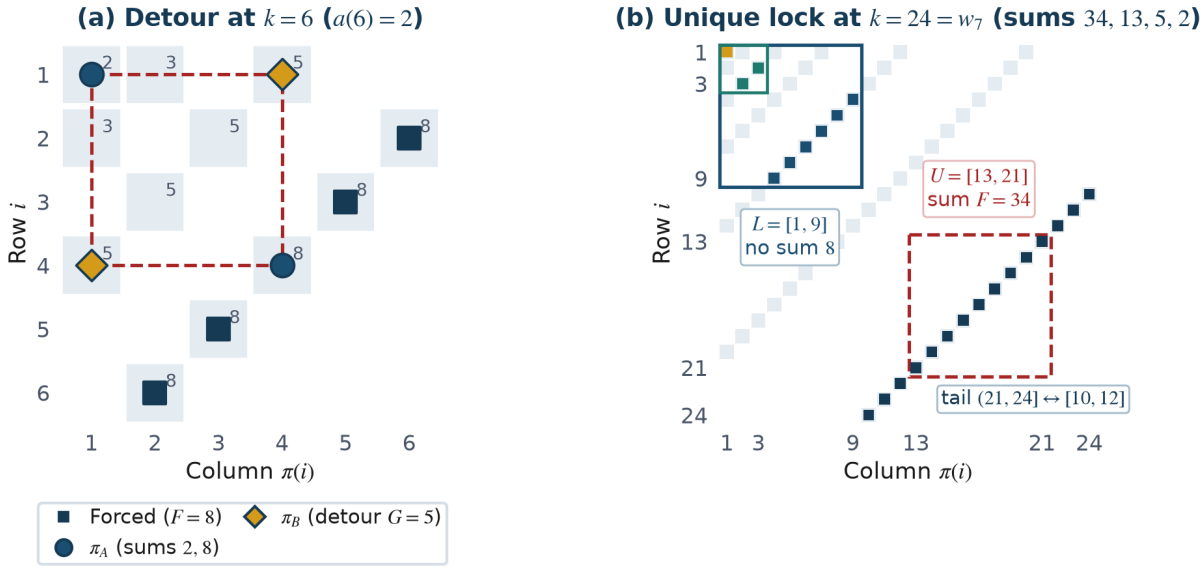


Figure 1. Visual anatomy of the Step Lemma. (a) At $k=6$ ($F=8, G=5, H=3, I=2, t=1$), the move $1\mapsto 1$ of sum $I=2$ inside $[1]$ pairs with $4\mapsto 4$ (sum $F=8$) to create a detour cycle $1\mapsto 4\mapsto 1$ of sum $G=5$, yielding $a(6)=2$. (b) At $k=24=w_7$ ($F=34, G=21, H=13, I=8, t=9=w_5$), the tail $(21,24]$ is forced onto $[10,12]$ (sum 34), the middle block $U=[13,21]$ reverses onto itself (sum 34), and the leftover $L=[1,9]$ avoids sum $I=8$, propagating uniqueness down $24\to 9\to 3\to 1$ with sums in $\{34,13,5,2\}$ only.

2. Since $F_{j-2}-1\le 2w_m<F_{m+3}$, having $m\le j-6$ would yield $F_{m+3}\le F_{j-3}=F_{j-2}-F_{j-4}\le F_{j-2}-1$, impossible. Thus $m\in\{j-5, j-4\}$.
3. If $m=j-5$ with $m\ge 3$, then (4) gives $2w_m+2\le F_{m+3}=F_{j-2}$, contradicting $F_{j-2}-1\le 2w_m$. If $m=1$ ($j=6$), $k=8-1-1=6$, contradicting $a(6)=2$; if $m=2$ ($j=7$), $k=13-2-1=10$, contradicting $a(10)\ge 2$.

Hence $m=j-4$. From $w_{j-4}+w_{j-2}+1=F_j=k+w_{j-4}+1$ we obtain $k=w_{j-2}$ with $j-2\ge 3$, proving Theorem 1. Combining Theorem 1 with Proposition 3 ($M_4(n+1)=w_n$) immediately gives Corollary 2. □

4. FLOOR FORMULA, COMPUTATIONS, AND OPEN RECORDS

Rather than invoking Binet's formula or asymptotic error bounds, we establish Reshetnikov's [6] closed form by pure integer algebra via Cassini's identity and a period-4 Lucas sandwich.

Theorem 7 (Reshetnikov's Floor Formula). *For every integer $n\ge 1$,*

$$w_n=\left\lfloor \frac{F_{n+3}}{\sqrt{5}}\right\rfloor.$$

Proof. We derive four exact integer identities (formalized in `Defs.lean` and `Main.lean`). First, $L_{m+1}=F_m+F_{m+2}$ holds for $m=0,1$ and satisfies the shared Fibonacci recurrence, so it holds for all $m\ge 0$; expanding $L_{m+4}=3L_{m+1}+2L_m$ then yields $L_{m+4}=L_m+5F_{m+2}$ for all $m\ge 0$ (checked directly at $m=0$: $7=2+5\cdot 1$). Second, Cassini's identity $F_mF_{m+2}-F_{m+1}^2=(-1)^{m+1}$ holds at $m=0$ ($-1=-1$) and negates under $m\mapsto m+1$ since $F_{m+1}F_{m+3}-F_{m+2}^2=-(F_mF_{m+2}-F_{m+1}^2)$. Third, writing $L_{k+1}=F_{k+1}+2F_k$ gives

$$L_{k+1}^2-5F_{k+1}^2=4(F_k^2+F_kF_{k+1}-F_{k+1}^2)=4(-1)^{k+1},$$

so $L_m^2-5F_m^2=4(-1)^m$ for all $m\ge 0$ (with $m=0$: $4=4$).

Fourth, direct evaluation gives $L_4-5w_1=2$, $L_5-5w_2=1$, $L_6-5w_3=3$, and $L_7-5w_4=4$. Combining $L_{n+7}=L_{n+3}+5F_{n+5}$ with (5) ($w_{n+4}=w_n+F_{n+5}$) shows $L_{(n+4)+3}-5w_{n+4}=L_{n+3}-5w_n$, so $r_n:=L_{n+3}-5w_n$ cycles strictly as $(2,1,3,4)$ for $n\equiv 1,2,3,0 \pmod 4$. In particular, $5w_n+1\le L_{n+3}\le 5w_n+4$ for every $n\ge 1$.

Now let $n\ge 1$, so $w_n\ge 1$. Using $5F_{n+3}^2=L_{n+3}^2-4(-1)^{n+3}$,

$$\begin{aligned}5F_{n+3}^2&\ge (5w_n+1)^2-4=25w_n^2+10w_n-3>25w_n^2,\\ 5F_{n+3}^2&\le (5w_n+4)^2+4\\ &=25w_n^2+40w_n+20<25(w_n+1)^2,\end{aligned}$$

since $10w_n-3\ge 7>0$ and $10w_n+5>0$. Dividing by 5 yields the exact integer square sandwich

$$5w_n^2<F_{n+3}^2<5(w_n+1)^2. \tag{6}$$

Taking positive square roots gives $w_n<F_{n+3}/\sqrt{5}<w_n+1$, whence $\lfloor F_{n+3}/\sqrt{5}\rfloor=w_n$. □

Independent exact-arithmetic checks and tables. Although all seven proved results sit inside the pure Lean 4 kernel without computational certificates, `recompute.py` performs an independent standard-library integer/rational verification (0 floating-point values, 0 complex numbers, 0 `ast.Div` operators): (i) executes the Section 1 Python box verbatim; (ii) computes $a(k)$ by bitmask dynamic programming ($1\le k\le 20$) and exact rational block reduction ($1\le k\le 40$), agreeing identically on $1\le k\le 20$ and finding $\{k\le 40 : a(k)=1\}=\{1,2,3,5,9,15,24,39\}$;

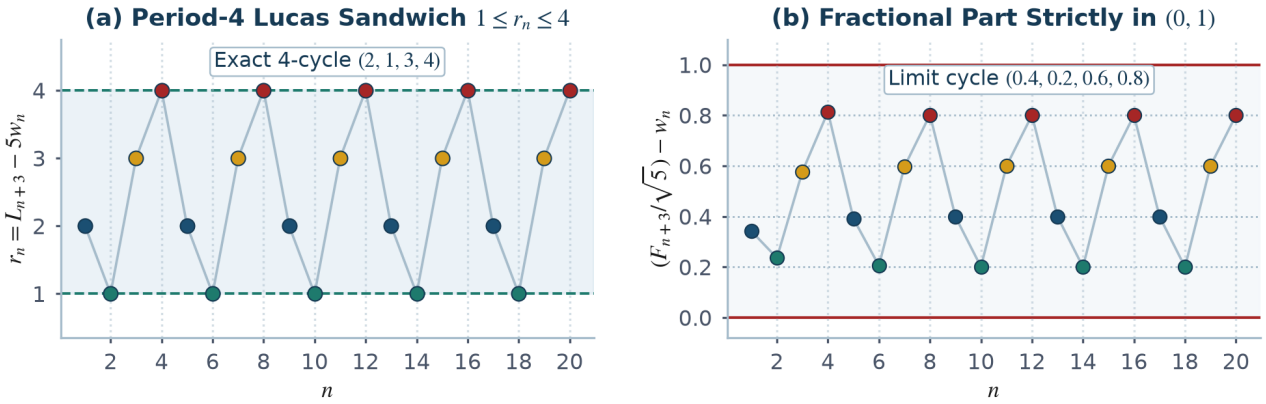


Figure 2. Algebraic certification of Reshetnikov's floor formula (Theorem 7). (a) The exact integer residue $r_n = L_{n+3} - 5w_n$ cycles with period 4 through $(2, 1, 3, 4)$, remaining strictly inside $[1, 4]$ for all $n \ge 1$. (b) The fractional part $(F_{n+3}/\sqrt{5}) - w_n$ lies strictly inside $(0, 1)$ (between 0.200016 and 0.813777 for $1 \le n \le 20$) and converges to the 4-cycle $(0.4, 0.2, 0.6, 0.8)$.

(iii) checks all algebraic identities and (6) through $n = 100$; (iv) parses and checks every row of Tables 1 and 2 directly from the LaTeX source; and (v) catches 6/6 negative-control mutations (`mut_w_recurrence` at $n = 3$, `mut_step_drop_noI` at $k = 6$, `mut_lucas_shift` at $n = 4$, `mut_floor_ceil` at $n = 1$, `mut_k6_unique`, and `mut_two_fib_bound` at $k = 4$).

Table 1. Exact counts $a(k)$ (OEIS A097082) for $1 \le k \le 16$, scale $F_j \le 2k < F_{j+1}$, extension lower bound $E(k) := \sum_{s \in (k, 2k]} a(s - k - 1)$, and classification.

k	$a(k)$	F_j	$E(k)$	Classification / unique π
1	1	2	1	w_1: (1)
2	1	3	1	w_2: (2, 1)
3	1	5	1	w_3: (1, 3, 2)
4	2	8	2	two Fibs {5, 8}
5	1	8	1	w_4: (2, 1, 5, 4, 3)
6	2	8	1	witness $k = 6$ (detour)
7	4	13	2	two Fibs {8, 13}
8	2	13	2	inherits $a(4) = 2$
9	1	13	1	w_5: (1, 3, 2, 9, 8, 7, 6, 5, 4)
10	4	13	1	witness $k = 10$ (detour)
11	4	21	2	two Fibs {13, 21}
12	20	21	3	two Fibs {13, 21}
13	4	21	4	inherits $a(7) = 4$
14	5	21	2	inherits $a(6) = 2$
15	1	21	1	w_6: unique ($t = 5 = w_4$)
16	20	21	2	inherits $a(4) = 2$

Table 2. Layman's sequence $w_n = M_4(n + 1)$, F_{n+3}, L_{n+3}, period-4 residue $r_n = L_{n+3} - 5w_n$, and positive integer margins $\Delta_- := F_{n+3}^2 - 5w_n^2$ and $\Delta_+ := 5(w_n + 1)^2 - F_{n+3}^2$ for $1 \le n \le 12$.

n	w_n	F_{n+3}	L_{n+3}	r_n	Δ_-	Δ_+
1	1	3	7	2	4	11
2	2	5	11	1	5	20
3	3	8	18	3	19	16
4	5	13	29	4	44	11
5	9	21	47	2	36	59
6	15	34	76	1	31	124
7	24	55	123	3	145	100
8	39	89	199	4	316	79
9	64	144	322	2	256	389
10	104	233	521	1	209	836
11	168	377	843	3	1009	676
12	272	610	1364	4	2180	545

Empirical landscape and open records conjecture. Figure 3 plots $\log_2 a(k)$ and $a(k)^{1/k}$ across $1 \le k \le 40$. Each Fibonacci scale band $F_j \le 2k < F_{j+1}$ contains a single unique index w_n (matching (3)), whereas interior counts reach $a(20) = 200$ and $a(33) = 10{,}800$. For $2 \le k \le 40$, every strict record of $a(k)^{1/k}$ occurs at $k = F_j - 1 \in \{4, 7, 12, 20, 33\}$; the closest non-record challenger is $k = 19$ ($a(19) = 96$ versus $a(12) = 20$, where $96^{12} < 20^{19}$ holds strictly). We fence this observation explicitly as an open conjecture outside our proved claim denominator:

Conjecture 8 (Open Growth-Rate Records Law). *For integers $k \ge 2$, the per-element growth rate $a(k)^{1/k}$ attains a strict record ($a(k)^{1/k} > a(m)^{1/m}$ for all $1 \le m < k$) if and only if $k = F_j - 1$ for some integer $j \ge 5$.*

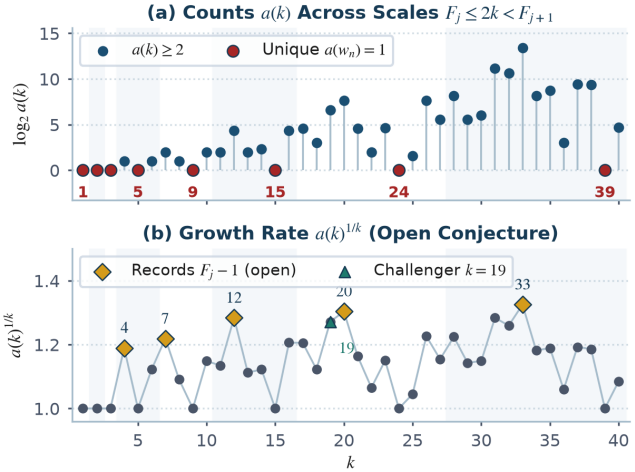


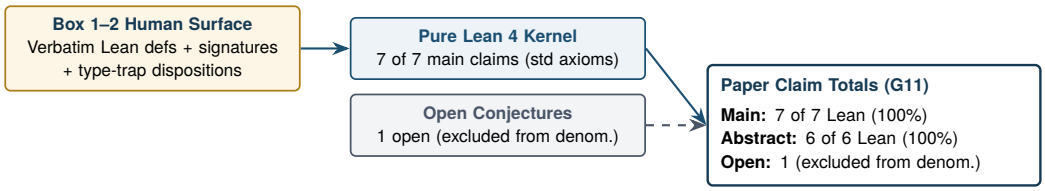
Figure 3. Exact counts and empirical growth rates for $1 \le k \le 40$. (a) $\log_2 a(k)$ over shaded Fibonacci scale bands $F_j \le 2k < F_{j+1}$; each band contains a unique Layman index $w_n \in \{1, 2, 3, 5, 9, 15, 24, 39\}$ with $a(w_n) = 1$. (b) Per-element growth rate $a(k)^{1/k}$, whose observed records for $k \ge 2$ occur at $k = F_j - 1 \in \{4, 7, 12, 20, 33\}$ (Conjecture 8, explicitly open).

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 87adcd555097 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Artifacts & Remaining Claims:** No external certificate artifacts are required; any claims marked Written or Open rely on standard written proofs or remain explicitly unproved.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 46/46 public theorems (source-level approx) | WARN: Fidelity: 7/7 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

FibSumPerm.IsFib (FibSumPerm/Defs.lean:25-26) — Section 2 (positive Fibonacci predicate m > 0 and exists j, F_j = m)

```
def IsFib (m : ℕ) : Prop :=
  0 < m ∧ ∃ j : ℕ, Nat.fib j = m
```

FibSumPerm.IsValidFibPerm (FibSumPerm/Defs.lean:73-74) — Section 2 (valid Fibonacci-sum permutation on {1, ..., k})

```
def IsValidFibPerm (k : ℕ) (p : Equiv.Perm (Fin k)) : Prop :=
  ∀ i : Fin k, IsFib ((i.1 + 1) + ((p i).1 + 1))
```

FibSumPerm.fibPermCount (FibSumPerm/Defs.lean:81-82) — Section 2 (exact count a(k) via Fintype.card, OEIS A097082)

```
def fibPermCount (k : ℕ) : ℕ :=
  Fintype.card { p : Equiv.Perm (Fin k) // IsValidFibPerm k p }
```

FibSumPerm.w (FibSumPerm/Defs.lean:169-173) — Section 1 (Layman recurrence w_1 = 1, w_2 = 2, w_{n+2} = F_{n+4} - w_n - 1)

```
def w : ℕ → ℕ
  | 0 => 0
  | 1 => 1
  | 2 => 2
  | n + 3 => Nat.fib (n + 5) - w (n + 1) - 1
```

FibSumPerm.M4 (FibSumPerm/Defs.lean:176-181) — Section 1 (Hajnal partial-sum sequence M_4(l) = F_l + F_{l-4} + ...)

```
def M4 : ℕ → ℕ
  | 0 => 0
  | 1 => 0
  | 2 => Nat.fib 2
  | 3 => Nat.fib 3
  | k + 4 => Nat.fib (k + 4) + M4 k
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): FibSumPerm.fibPermCount_four (fibPermCount 4 = 2); FibSumPerm.fibPermCount_five (fibPermCount 5 = 1); FibSumPerm.fibPermCount_six (fibPermCount 6 = 2).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

FibSumPerm.hajnal_conjecture_1 (FibSumPerm/Main.lean:550-555) `src:6bfe0cbece0880`

```
theorem hajnal_conjecture_1 (k : ℕ) (hk : 1 ≤ k) :
    fibPermCount k = 1 ↔ ∃ n : ℕ, 1 ≤ n ∧ k = w n
```

Assumptions: Positive integer k >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

FibSumPerm.hajnal_conjecture_1_M4 (FibSumPerm/Main.lean:559-569) src:91e2fc679c3b30

```
theorem hajnal_conjecture_1_M4 (k : ℕ) (hk : 1 ≤ k) :
    fibPermCount k = 1 ↔ ∃ l : ℕ, 2 ≤ l ∧ k = M4 l
```

Assumptions: Positive integer k >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

FibSumPerm.M4_eq_w (FibSumPerm/Defs.lean:240-255) src:99bbfff7983770

```
theorem M4_eq_w (n : ℕ) (hn : 1 ≤ n) : M4 (n + 1) = w n
```

Assumptions: Positive integer n >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

FibSumPerm.extension_count_ge (FibSumPerm/StepLemma.lean:103-110) src:b84e8c3d692308

```
theorem extension_count_ge {k s : ℕ} (hs : IsFib s) (hks : k < s) (hs2k : s ≤ 2 * k) :
    fibPermCount (s - k - 1) ≤ fibPermCount k
```

Assumptions: Positive Fibonacci number s with k < s <= 2k. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

FibSumPerm.step_lemma (FibSumPerm/StepLemma.lean:166-453) src:e84758c5ccb017

```
theorem step_lemma {j k t : ℕ} (hj : 3 ≤ j)
    (hGk : Nat.fib (j + 2) ≤ k)
    (hkFG : 2 * k < Nat.fib (j + 3) + Nat.fib (j + 2))
    (ht : t = Nat.fib (j + 3) - 1 - k)
    (ht1 : 1 ≤ t)
    (_hHt : Nat.fib (j + 1) ≤ 2 * t)
    (htG : 2 * t < Nat.fib (j + 2))
    (huniq : fibPermCount t = 1)
    (hnoI : ∀ q : Equiv.Perm (Fin t), IsValidFibPerm t q →
      ∀ i : Fin t, (i.1 + 1) + ((q i).1 + 1) ≠ Nat.fib j) :
    fibPermCount k = 1 ∧
      ∀ p : Equiv.Perm (Fin k), IsValidFibPerm k p →
        ∀ i : Fin k, (i.1 + 1) + ((p i).1 + 1) ≠ Nat.fib (j + 2)
```

Assumptions: j >= 3, k, t >= 1, F_{j+2} <= k, 2k < F_{j+3}+F_{j+2}, t = F_{j+3}-1-k, F_{j+1} <= 2t < F_{j+2}, a(t)=1, no sum F_j. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

FibSumPerm.layman_unique_strong (FibSumPerm/Main.lean:384-429) src:547a8ae8e3369f

```
theorem layman_unique_strong (n : ℕ) (hn : 1 ≤ n) :
    fibPermCount (w n) = 1 ∧
      ∀ p : Equiv.Perm (Fin (w n)), IsValidFibPerm (w n) p →
        ∀ i : Fin (w n), (i.1 + 1) + ((p i).1 + 1) ≠ Nat.fib (n + 1)
```

Assumptions: Positive integer n >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

FibSumPerm.w_eq_floor_fib_div_sqrt5 (FibSumPerm/Main.lean:599-618) src:41a39915f44d35

```
theorem w_eq_floor_fib_div_sqrt5 (n : ℕ) (hn : 1 ≤ n) :
    (w n : ℤ) = ⌊(Nat.fib (n + 3) : ℝ) / Real.sqrt 5⌋
```

Assumptions: Positive integer n >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 3. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
extension_count_ge	nat_sub	- k	intended_convention	Hypothesis hks : k < s gives s - k >= 1.
extension_count_ge	nat_sub	- 1	intended_convention	Hypothesis hks : k < s gives s - k - 1 >= 0.
fibPermCount	nat_div	/ /	intended_convention	Lean subtype syntax // (not division).
fibPermCount	nat_div	/ IsValidFibPerm	intended_convention	Subtype predicate separator // (not division).
step_lemma	nat_sub	- 1	intended_convention	hj : 3 <= j gives F_{j+3} >= 8 > 1.
step_lemma	nat_sub	- k	intended_convention	ht1 : 1 <= t forces k <= F_{j+3} - 2.
w	nat_sub	- w	proved_guard (w_bounds)	Proved by w_bounds (w n + 1 <= F_{n+2}).
w	nat_sub	- 1	proved_guard (w_bounds)	Proved by w_bounds (no Nat underflow).
w_eq_floor_fib_div_sqrt5	real_div	Real.sqrt	intended_convention	Real division by constant Real.sqrt 5 > 0.
w_eq_floor_fib_div_sqrt5	real_log_rpow	Real.sqrt	intended_convention	Positive constant 5 > 0 for Real.sqrt 5.
decidableIsFib	local_instance_or_notation	decidableIsFib	intended_convention	Standard Decidable via isFib_iff_bounded.
decidableIsValidFibPerm	local_instance_or_notation	decidableIsValidFibPerm	intended_convention	Standard Mathlib Decidable on Fin k.

[a]Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Unique Fib-sum perm a(k) = 1 if and only if k = w_n for some n >= 1	Lean Kernel	`FibSumPerm.hajnal_conjecture_1`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (for all k >= 1).
Cor. 2: Unique Fib-sum perm a(k) = 1 if and only if k = M_4(l) for some l >= 2	Lean Kernel	`FibSumPerm.hajnal_conjecture_1_M4`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (for all k >= 1).
Prop. 3: Equivalence M_4(n + 1) = w_n for every integer n >= 1	Lean Kernel	`FibSumPerm.M4_eq_w`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (for all n >= 1).
Prop. 4: Extension bound a(s - k - 1) <= a(k) for Fib s in (k, 2k]	Lean Kernel	`FibSumPerm.extension_count_ge`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (positive Fibonacci s, k < s <= 2k).
Thm. 5: Step Lemma propagating uniqueness and no sum F_{j+2} from [t] to [k]	Lean Kernel	`FibSumPerm.step_lemma`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (j >= 3, scale hypotheses, a(t) = 1, no sum F_j).
Thm. 6: Strengthened invariant a(w_n) = 1 and no sum F_{n+1} for all n >= 1	Lean Kernel	`FibSumPerm.layman_unique_strong`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (for all n >= 1).
Thm. 7: Reshetnikov floor formula w_n = floor(F_{n+3} / sqrt(5)) for n >= 1	Lean Kernel	`FibSumPerm.w_eq_floor_fib_div_sqrt5`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Mathlib Real.sqrt/Int.floor (for all n >= 1).
Conj. 8: Strict records of a(k)^(1/k) (k >= 2) occur if and only if k = F_j - 1 (j >= 5)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open conjecture (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 729b9e6a4f6b | Lean mode: committed_receipt | Closure SHA-256: b0656b6391b8 | Git anchor: 87adcd555097 | Axioms SHA-256: bbe1c3fded90 | Fidelity SHA-256: d58a555e7a6f. External validation receipts: Pinned commit | Palomar record.

6. PRIOR WORK AND OPEN QUESTIONS

J. W. Layman (2004) introduced $a(k)$ and its uniqueness locus as OEIS A097082 [5] and A097083 [6] (both entries read in full), conjecturing (1); D. Wasserman (2007) noted the block-reversal lower bound, and V. Reshetnikov (2015) conjectured the floor form $w_n = \lfloor F_{n+3}/\sqrt{5} \rfloor$ (still unproved at A097083 rev. 40, 2026-09-22). Z.-W. Sun [7] (preprint `arXiv:1811.10503` read in full; publisher copy not fetched) proved existence of a valid involution for every $k \geq 1$ for both Fibonacci and Lucas sums (Corollary 1.1), tabulated $a(1..20)$, and noted the unique permutation $(2,3)(4,9)(5,8)(6,7)$ in S_9. P. Hajnal [3] (all 8 pages read in full) proved the "only if" direction in $M_4(\ell)$ notation (Theorems 1–2) and posed the converse "if" direction as Conjecture 1. Nearby works touching the graph or sequence A097083 without addressing permutation uniqueness include Arman, Gunderson, and Li [1] [to check] (simple loop-free Fibonacci-sum graphs), Hinz and Stockmeyer [4] [to check] (Sierpiński-type graphs), and Chalebgwa and Szalay [2] [to check] (Tribonacci representations).

Systematic literature searches were logged across OEIS, arXiv, zbMATH, Crossref, Semantic Scholar citation lookups, MathOverflow, Math Stack Exchange, Zenodo, DataCite, CORE (plus Hajnal's 2026 doctoral thesis), and a companion 2026-11-22 Google Scholar sweep (5 queries) recorded in the published Palomar entry. No prior proof of the "if" direction ($a(w_n) = 1$) was found in the searches logged. Explicitly *not* searched were MathSciNet (paywalled), full zbMATH review texts beyond metadata, books in print (such as Guy's *Unsolved Problems in Number Theory*), print problem sections of *The Fibonacci Quarterly*, and unpublished Hungarian-language theses outside checked repositories; sources marked [to check] must be read in full before submission. Three concrete open questions remain: (1) Conjecture 8 on strict records of $a(k)^{1/k}$; (2) characterizing uniqueness for Lucas-sum permutations (including $L_0 = 2$); and (3) determining which constant-recursive target sets $S \subset \mathbb{N}_{>0}$ yield a periodic partial-sum uniqueness locus analogous to $M_4(\ell)$.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All Draft 2 sources live in `papers/fibonacci-sum-permutations/draft2/` and the Lean 4 library (5 files, 1,490 lines, 46/46 audited declarations under Lean `v4.35.0` + Mathlib `v4.35.0`) in `lean/FibSumPerm/`. From the repository root, run `python3 papers/fibonacci-sum-permutations/draft2/recompute.py` (verifying the Section 1 box, dual counts through $k = 40$, algebraic identities through $n = 100$, Tables 1–2, and 6/6 mutations), regenerate figures via `figures/make_hero.py` and `figures/make_figures.py`, regenerate Section 5 via `tools/trust_boundary.py -p papers/fibonacci-sum-permutations/draft2 -g`, and run the fail-closed gate `tools/check_paper_top20.py -p papers/fibonacci-sum-permutations/draft2 -b`. The formalization is also registered in the Palomar Registry as `PALOMAR-2026-11-22-000029` (version 1; mechanical Lean checking plus automated editorial review).

AI research agents built and directed by Tyler B. Hughes generated and mechanically verified the proofs, scripts, figures, and Lean 4 formalization. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 23, 2026

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(1) Arman, A.; Gunderson, D. S.; Li, P. C. Properties of the Fibonacci-sum graph. Preprint, `arXiv:1710.10303`, **2017**.

(2) Chalebgwa, T. P.; Szalay, L. Representations with k-generalized Fibonacci numbers. Preprint, `arXiv:2608.23193`, **2026**.

(3) Hajnal, P. A short note on Layman permutations. *Acta Univ. Sapientiae Math.* **2022**, *14* (2), 231–238. DOI: 10.2478/ausm-2022-0015.

(4) Hinz, A. M.; Stockmeyer, P. K. Precious metal sequences and Sierpiński-type graphs. *J. Integer Seq.* **2022**, *25*, Article 22.4.8.

(5) OEIS Foundation Inc. Entry A097082 (J. W. Layman, 2004; rev. 25, 2026). `https://oeis.org/A097082`.

(6) OEIS Foundation Inc. Entry A097083 (J. W. Layman, 2004; comments by D. Wasserman, 2007, and V. Reshetnikov, 2015; rev. 40, 2026). `https://oeis.org/A097083`.

(7) Sun, Z.-W. On permutations of $\{1, \dots, n\}$ and related topics. *J. Algebraic Combin.* **2021**. DOI: 10.1007/s10801-021-01028-8; preprint `arXiv:1811.10503`.