MATHDISCOVERIES
Mathematical Preprint

Working Draft

Free $(2,3,6)$ Affine Button Machines: Conjugacy Reduction, Stick Tilings, and Height-100 Census

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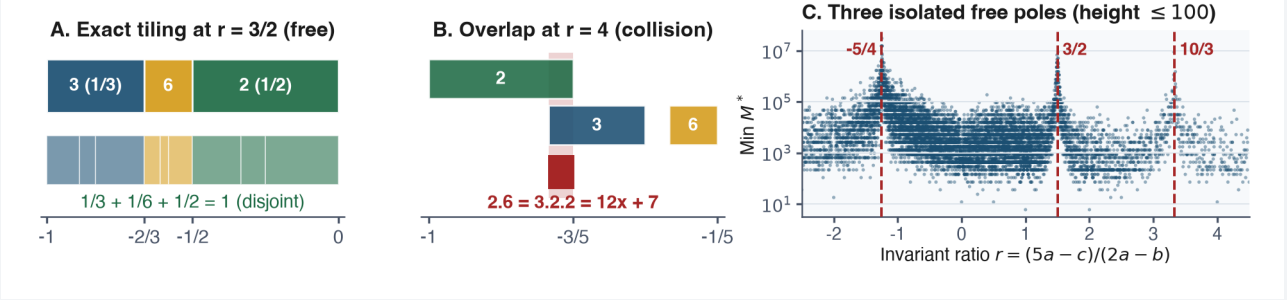
Ⓢ *Supporting Information:* `recompute.py` *(standalone exact-rational Python verifier, standard library only)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For rational or real (a,b,c), consider $S(a,b,c) = \langle 2x+a,\ 3x+b,\ 6x+c\rangle$. We prove three results: (1) outside $2a=b$, $5a=c$, $r=(5a-c)/(2a-b)\in\mathbb{P}^1$ is a complete affine-conjugacy invariant deciding freeness, and irrational r is free; (2) the six partitions of $[0,1]$ into lengths $1/2$, $1/3$, $1/6$ pair under reflection to $r\in\{-5/4,\ 3/2,\ 10/3\}$, matching Klarner's six free triples (1982); and (3) length-≤ 2 words kill $r\in\{0,1,2,5/2,\infty\}$, while minimal $M^*(r)$ collisions have distinct ends. Exact replay (`recompute.py`) verifies $2.6 = 3.2.2 = 12x+7$ $(r=4)$, $1296x+663$ $(-3/2)$, nine witnesses to $M^* = 725{,}594{,}112$, six tilings, counts (4,405 at $H\le 60$; 12,173 at $H\le 100$), and $H\le 20$ collisions. Sweeps find collisions at all 12,173 non-free $H\le 100$ ratios (18 letters at $-99/79$; max $M^* = 30{,}233{,}088$), all 1,460 near-free ($q\in[101,300]$, max 2,176,782,336 at $-321/257$), and 400 random ($q\le 1{,}000$), with 0 collisions in 96,959,662 words ($M\le 10^8$) per Klarner ratio. We conjecture every rational $r\notin\{-5/4,\ 3/2,\ 10/3\}$ collides (unsettled). No Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)
Abstract claims: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 20M05, 11B83, 28A80, 37E05 **Keywords:** affine semigroups, freeness problem, Klarner triples, self-similar tilings, ping-pong lemma, exact overlaps

1. INTRODUCTION AND MAIN RESULTS

Choose three rational or real numbers a,b,c and form three affine "buttons" on the number line:

$$f_2(x) = 2x+a, \qquad f_3(x) = 3x+b, \qquad f_6(x) = 6x+c.$$

Pressing buttons sequentially from left to right (a *word*, written with dots such as 2.6 for "apply f_2, then f_6") produces an affine map $f_w(x) = M(w)x + T(w;a,b,c)$ with multiplier $M(w) = 2^i3^j$. Two distinct words $u\neq v$ that build the exact same affine map form a *collision* (a nontrivial semigroup relation); the triple (a,b,c) is *free* if the generated semigroup $S(a,b,c) = \langle f_2,f_3,f_6\rangle$ has no collision at all. Because $2\cdot 3 = 6$, distinct words frequently share the same multiplier—for example, at $(a,b,c) = (1,1,1)$, both 2.6 and 3.2.2 yield $12x+7$—and only their translations can separate them.

The slopes $(2,3,6)$ sit precisely at the critical balance $1/2+1/3+1/6 = 1$. Klarner [4] (`[to check]`, read via Lagarias [7], entry 97) observed that $\sum 1/m_i \le 1$ is necessary for freeness of any affine semigroup $\langle m_ix+b_i\rangle$ and proved a sufficient interval-ordering criterion yielding six free integer triples for $(2,3,6)$:

$$(0,2,3),\ (1,0,2),\ (0,3,10),\ (2,1,0),\ (2,0,15),\ (1,6,0).$$

Geometrically, running the three buttons backwards cuts a closed interval into pieces of lengths $1/2$, $1/3$, and $1/6$ with pairwise disjoint interiors, so a ping-pong argument forces every word to occupy a unique subinterval $[4,5]$ (`[to check]`). Recent work has settled the analogous exact-tiling characterization for equal prime slopes [6] (`[to check]`) and established zero asymptotic density for the orbit sets of Klarner's six triples [2] (`[to check]`), while freeness for general integer affine functions remains open [6, 5, 9] (`[to check]`).

In this draft we separate cleanly what is proved algebraically, what is credited to Klarner [4], what is verified by exact computation, and what remains unsettled. We prove three algebraic structural results (Theorem 1 and Propositions 2 and 3), present two computational verification propositions (Propositions 4 and 5), and formulate the converse classification as Conjecture 6 in Section 6.

Theorem 1 (Complete Affine-Conjugacy Invariant and Irrational Freeness). *Let $K \in \{\mathbb{Q}, \mathbb{R}\}$ and $(a,b,c) \in K^3$. Set $\Delta_1 = 2a - b$ and $\Delta_2 = 5a - c$.*

1. (Degenerate locus.) *If $\Delta_1 = \Delta_2 = 0$ (equivalently $b = 2a$ and $c = 5a$), then (a,b,c) is conjugate in $\mathrm{Aff}(K)$ to $(0,0,0)$, all three generators commute pairwise, and $S(a,b,c)$ is not free.*
2. (Complete conjugacy invariant.) *Two non-degenerate triples $(a,b,c), (a',b',c') \in K^3$ are conjugate under an invertible affine change of coordinates $\phi(x) = \lambda x + \mu$ ($\lambda \in K^\times$, $\mu \in K$) if and only if their projective ratios*
$$r(a,b,c) := \frac{5a - c}{2a - b} \in \mathbb{P}^1(K) = K \cup \{\infty\}$$
coincide (with $r = \infty$ when $\Delta_1 = 0$ and $\Delta_2 \neq 0$). Consequently, freeness of $S(a,b,c)$ depends solely on $r(a,b,c)$. Every finite-r class is represented by $(0,1,r)$ (and for reduced rational $r = p/q$, $q \geq 1$, by the integer triple $(0,q,p)$); the class $r = \infty$ is represented by $(0,0,1)$, where $2.3 = 3.2$.
3. (Irrational ratios.) *Every distinct equal-multiplier pair $u \neq v$ imposes a nonzero integer linear relation $A_{u,v}\Delta_1 + B_{u,v}\Delta_2 = 0$, vanishing at a single $r \in \mathbb{P}^1(\mathbb{Q})$. Hence every irrational $r \in \mathbb{R}\setminus\mathbb{Q}$ defines a free semigroup $S(a,b,c)$.*

Proposition 2 (Equivalence of Interval Tilings, Klarner's Six Triples, and Three Ratios). *The six permutations of partitioning $[0,1]$ end-to-end into closed subintervals of lengths $1/2$, $1/3$, and $1/6$ yield six inverse-contraction triples that pair under the reflection $x \mapsto -x$ into exactly three affine-conjugacy classes:*
$$r \in \left\{-\frac{5}{4}, \frac{3}{2}, \frac{10}{3}\right\},$$
represented by Klarner's six integer triples $(0,2,3)$ and $(1,0,2)$ ($r = 3/2$), $(0,3,10)$ and $(2,1,0)$ ($r = 10/3$), and $(2,0,15)$ and $(1,6,0)$ ($r = -5/4$). Crediting Klarner [4] (`[to check]`; see also [5, 7]) for the "if" direction, every triple with $r \in \{-5/4, 3/2, 10/3\}$ generates a free semigroup via interval ping-pong.

Proposition 3 (Short-Word Collision Loci and Minimality of M^*). *The following two structural properties hold for collisions in $S(a,b,c)$:*

1. (Length $\leq$ 2 loci.) *Among all pairs of distinct equal-multiplier words of length at most 2, there are exactly five pairs, each killing one ratio: $2.6 = 6.2$ ($r = 0$, $M = 12$); $6 = 2.3$ ($r = 1$, $M = 6$); $6 = 3.2$ ($r = 2$, $M = 6$); $3.6 = 6.3$ ($r = 5/2$, $M = 18$); and $2.3 = 3.2$ ($r = \infty$, $M = 6$).*
2. (Minimality of M^*.) *Whenever $S(a,b,c)$ is not free, let $M^*(r)$ be the minimum multiplier $M(u) = M(v)$ over all collisions $u \neq v$. Any collision achieving $M^*(r)$ has distinct initial letters and distinct terminal letters. Consequently, the multiplier-class recurrence on $(0,q,p)$,*
$$T(N) = T(N/2) \cup (T(N/3) + (N/3)q) \cup (T(N/6) + (N/6)p),$$
detects $M^*(p/q)$ exactly whenever run up to any bound $B \geq M^*(p/q)$.

Proposition 4 (Exact Live Replay of Witnesses, Stick Tilings, Counts, and Height-$\leq$ 20 Census). *Standalone standard-library exact-rational evaluation in `recompute.py` (zero floating-point decisions) verifies:*

1. (Worked and extremal collisions.) *At $(1,1,1)$ ($r = 4$), $M^* = 12$ with $2.6 = 3.2.2 = 12x + 7$ (taking $x = 1$ to 19, after scanning 9 smaller-multiplier words); at $(1,0,8)$ ($r = -3/2$), $M^* = 1{,}296$ with $3.6.2.6.2.3 = 2.3.3.3.2.3.2.2 = 1296x + 663$ (lengths 6 and 8, taking $x = 1$ to 1,959, after 1,146 smaller words). Seven further extremal witnesses up to $M^* = 725{,}594{,}112$ and word length 23 (Table 2) hold identically across three affine conjugates each.*
2. (Stick tilings and overlap.) *At $(0,2,3)$ ($r = 3/2$), the inverse maps partition $[-1,0]$ into $[-1,-2/3]$, $[-2/3,-1/2]$, and $[-1/2,0]$, yielding 3, 9, and 27 disjoint-interior pieces of exact total length 1 at depths 1, 2, 3. All six permutations of $(1/2,1/3,1/6)$ tile $[0,1]$ and pair into $\{-5/4, 3/2, 10/3\}$. At $(1,1,1)$ ($r = 4$) on $[-1,-1/5]$, pieces 2 and 3 overlap on $[-2/3,-3/5]$ (length $1/15$) while leaving gap $[-2/5,-1/3]$, and 2.6 and 3.2.2 share the exact subinterval $[-2/3,-3/5]$.*
3. (Exact counts and word recurrence.) *There are 1,112, 4,408, and 12,176 distinct reduced ratios p/q ($\gcd(p,q) = 1$, ∞ once) of height $\max(|p|,q) \leq 30$, 60, and 100 (1,109, 4,405, and 12,173 excluding the three free ratios, correcting the duplicate $\pm 1/0$ height-$\leq$ 60 count 4,406). Among all 2,197 integer triples $0 \leq a,b,c \leq 12$, exactly 3 are degenerate, 88 have $r = \infty$, 37 are free (Klarner's 6 plus 31 conjugates), and all 2,069 remaining ratios have height ≤ 60. The exact word count $f(B) = |\{w \in \Sigma^* : M(w) \leq B\}|$ equals 1,003,900 ($B = 10^6$), 9,430,494 (10^7), 18,582,130 (2×10^7), and 96,959,662 (10^8).*
4. (Live height-$\leq$ 20 census and table audit.) *A pure-Python search over all 512 distinct ratios of height $\leq$ 20 finds collisions at all 509 non-free ratios (matching the height-$\leq$ 100 table on 509/509 entries) and zero collisions for $M \leq 10^5$ on $\{-5/4, 3/2, 10/3\}$. Direct audit confirms 12,173 collisions in `falsifier-allM.txt` (with $M^* \leq 30{,}233{,}088$, attained at $-86/69$), 1,460 near-free collisions in `falsifier-nearfree.tsv` ($M^* \leq 2{,}176{,}782{,}336$ at $-321/257$; log-log fit $1.771\ln(1/d) + 7.008$), and 1,109 collisions in `census.tsv` (0 disagreements).*

Proposition 5 (Exhaustive Height-$\leq$ 100 Census, Near-Free Probes, and Klarner Controls). *Combining the archived tables audited in Proposition 4 with the stratified C and Python search runs in Table 3:*

1. *Every one of the 12,173 non-free reduced ratios of height $\leq$ 100 (12,172 finite plus ∞) admits a collision verified in exact rational arithmetic; the longest colliding word has 18 letters ($r = -99/79$, $M^* = 15{,}116{,}544$) and the largest minimal multiplier is $M^* = 30{,}233{,}088$ ($r = -86/69$). Independent code agrees on 0 discrepancies across all 4,406 height-$\leq$ 60 rows, all 1,109 height-$\leq$ 30 pure-Python rows, and a 300-row random C recheck.*
2. *Every one of the 1,460 boundary-crowding ratios (the 4 reduced fractions nearest each free pole for every denominator $101 \leq q \leq 300$) admits a collision, with largest multiplier $M^* = 2{,}176{,}782{,}336$ at $r = -321/257$ (with 15 rows up to $M^* \leq 2 \times 10^7$ independently confirmed by a second C pro-*

gram).

3. *Every one of* 400 *random reduced ratios with* $101 \le q \le 1{,}000$ *in* $[-4, 6]$ *collides* (*hardest* $r = 1353/904$, $M^* = 214{,}990{,}848$, *lengths* 23 *and* 20; 76 *witnesses checked on non-normalized conjugates*).
4. *On each Klarner ratio* $r \in \{-5/4, 3/2, 10/3\}$, *no collision exists among all* 96,959,662 *words with* $M(w) \le 10^8$ (*and independently confirmed across* 9,430,494 *words up to* 10^7 *in pure Python and* 1,003,900 *words up to* 10^6 *in a second C program*).

Figure 1 maps the algebraic results, computational census, and explicit verification boundary.

Instant Witness Check: Exact Collisions at $r = 4$, $r = -3/2$, and $r = -99/79$

```
from fractions import Fraction
def run(word, a, b, c):
    m, t = 1, Fraction(0)
    add = {"2": a, "3": b, "6": c}
    for g in word: m, t = int(g) * m, int(g) * t + add[g]
    return m, t
assert run("26", 1, 1, 1) == run("322", 1, 1, 1) == (12, 7)
assert run("362623", 1, 0, 8) == run("23332322", 1, 0, 8) == (1296, 663)
u, v = "23333332232332222", "366662332663"
assert run(u, 0, 79, -99) == run(v, 0, 79, -99)
```

Roadmap. Section 2 introduces notation, works out the $M = 12$ words at $(1, 1, 1)$ and the 6- versus 8-press collision at $(1, 0, 8)$, and lists the six interval-tiling permutations. Section 3 proves Theorem 1 and Propositions 2 and 3. Section 4 documents the exact witness table, backward stick-tiling geometry, boundary scaling law, stratified sweep provenance, and five negative-mutation controls. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 surveys prior work, lists unread sources (`[to check]`), and states Conjecture 6 (with the two geometric obstructions and scoped orbit-density corollary), and Section 7 documents reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND WORKED EXAMPLES

Let $\Sigma = \{2, 3, 6\}$, and let Σ^* denote the free monoid of finite words over Σ (with empty word ε). For a field $K \in \{\mathbb{Q}, \mathbb{R}\}$ and parameters $(a, b, c) \in K^3$, write $(t_2, t_3, t_6) = (a, b, c)$ and define the three invertible affine maps $f_m(x) = mx + t_m$ ($m \in \Sigma$). A word $w = w_1 w_2 \cdots w_k \in \Sigma^*$ acts *left-to-right* by composition:

$$f_w := f_{w_k} \circ \cdots \circ f_{w_2} \circ f_{w_1}, \qquad f_\varepsilon(x) = x.$$

Inductively, $f_w(x) = M(w)x + T(w; a, b, c)$, where the multiplier is $M(w) = \prod_{j=1}^{k} w_j = 2^{i(w)} 3^{j(w)}$ and the translation is the integer-coefficient linear form

$$T(w; a, b, c) = \sum_{j=1}^{k} t_{w_j} \prod_{\ell=j+1}^{k} w_\ell. \tag{1}$$

Because every nonempty word has $M(w) \ge 2$ while $M(\varepsilon) = 1$, no nonempty word can equal ε; hence freeness of the semigroup $\langle f_2, f_3, f_6 \rangle \subset \mathrm{Aff}(K)$ on $\{f_2, f_3, f_6\}$ is identical to injectivity of the monoid homomorphism $w \mapsto f_w$ on Σ^*.

A *collision* (or nontrivial relation) is an unordered pair of distinct words $u \ne v$ with $f_u = f_v$, meaning $M(u) = M(v)$ and $T(u; a, b, c) = T(v; a, b, c)$. For a reduced projective rational $r = p/q \in \mathbb{P}^1(\mathbb{Q})$ ($\gcd(p, q) = 1$, $q \ge 0$, with ∞ written uniquely as $1/0$), its *height* is $H(p/q) := \max(|p|, q)$. When $S(a, b, c)$ is not free, $M^*(r)$ denotes the minimum multiplier $M(u)$ over all collisions $u \ne v$ at ratio r.

The *inverse buttons* (backward maps) are the contractions $g_m(y) := f_m^{-1}(y) = (y - t_m)/m$ for $m \in \{2, 3, 6\}$, with fixed points $x_m^* = -t_m/(m - 1)$. The smallest closed interval preserved by all three contractions is their fixed-point hull $I = [u, v]$ with $u = \min(x_2^*, x_3^*, x_6^*)$ and $v = \max(x_2^*, x_3^*, x_6^*)$. For any word $w = w_1 \cdots w_k$, its *inverse piece* (cylinder interval) is

$$I_w := f_w^{-1}(I) = (g_{w_1} \circ g_{w_2} \circ \cdots \circ g_{w_k})(I),$$

which has length $|I|/M(w)$ and is the exact set of points carried onto I by f_w.

Worked example 1: $(a, b, c) = (1, 1, 1)$ ($r = 4$). Here $r = (5 - 1)/(2 - 1) = 4$. The five words of multiplier 12 yield:

$$\begin{aligned} f_{2.6}(x) &= 6(2x + 1) + 1 = 12x + 7, \\ f_{3.2.2}(x) &= 2(2(3x + 1) + 1) + 1 = 12x + 7, \\ f_{6.2}(x) &= 12x + 3, \quad f_{2.3.2}(x) = 12x + 9, \\ f_{2.2.3}(x) &= 12x + 10. \end{aligned}$$

Thus $2.6 = 3.2.2$ is a collision at multiplier 12 (both carry $x = 1$ to 19: $1 \to 3 \to 19$ and $1 \to 4 \to 9 \to 19$). Scanning the 9 words with $M(w) < 12$ confirms $M^*(4) = 12$. This exact relation was used by Crampin and Hilton (unpublished, recorded in Lagarias [7], entry 97) to answer Erdős's question on the density of the orbit of 1 under $\langle 2x + 1, 3x + 1, 6x + 1 \rangle$.

Worked example 2: $(a, b, c) = (1, 0, 8)$ ($r = -3/2$). Here $r = (5 - 8)/(2 - 0) = -3/2$, which lies at distance $1/4$ from the free pole $r = -5/4$. No collision occurs among the 1,146 words with multiplier $< 1{,}296$. At $M^*(-3/2) = 1{,}296 = 2^4 \cdot 3^4$, the 6-letter word 3.6.2.6.2.3 and the 8-letter word 2.3.3.3.2.3.2.2 both compose to $1296x + 663$ (sending $x = 1$ to 1,959 via $1 \to 3 \to 26 \to 53 \to 326 \to 653 \to 1{,}959$ and $1 \to 3 \to 9 \to 27 \to 81 \to 163 \to 489 \to 979 \to 1{,}959$).

Worked example 3: Klarner's six triples and the six stick orders. Table 1 lists the six permutations of laying pieces of lengths $1/2$, $1/3$, and $1/6$ along $[0, 1]$ alongside Klarner's six integer triples [7]. Reversing the left-to-right order of the three pieces corresponds to reflecting the interval ($x \mapsto 1 - x$), which preserves r and pairs the six permutations into exactly three ratios: $3/2$, $10/3$, and $-5/4$.

Table 1. The six interval orders of $(1/2, 1/3, 1/6)$ along $[0, 1]$, their exact forward affine triples, Klarner's integer representatives, and the three invariant ratios $r \in \{3/2, 10/3, -5/4\}$ (verified in `recompute.py`).

Order on $[0, 1]$	Forward (a, b, c) on $[0, 1]$	Klarner integer triple	Ratio r
2 | 6 | 3	$(0, -2, -3)$	$(0, 2, 3)$	$3/2$
3 | 6 | 2	$(-1, 0, -2)$	$(1, 0, 2)$	$3/2$
2 | 3 | 6	$(0, -3/2, -5)$	$(0, 3, 10)$	$10/3$
6 | 3 | 2	$(-1, -1/2, 0)$	$(2, 1, 0)$	$10/3$
3 | 2 | 6	$(-2/3, 0, -5)$	$(2, 0, 15)$	$-5/4$
6 | 2 | 3	$(-1/3, -2, 0)$	$(1, 6, 0)$	$-5/4$

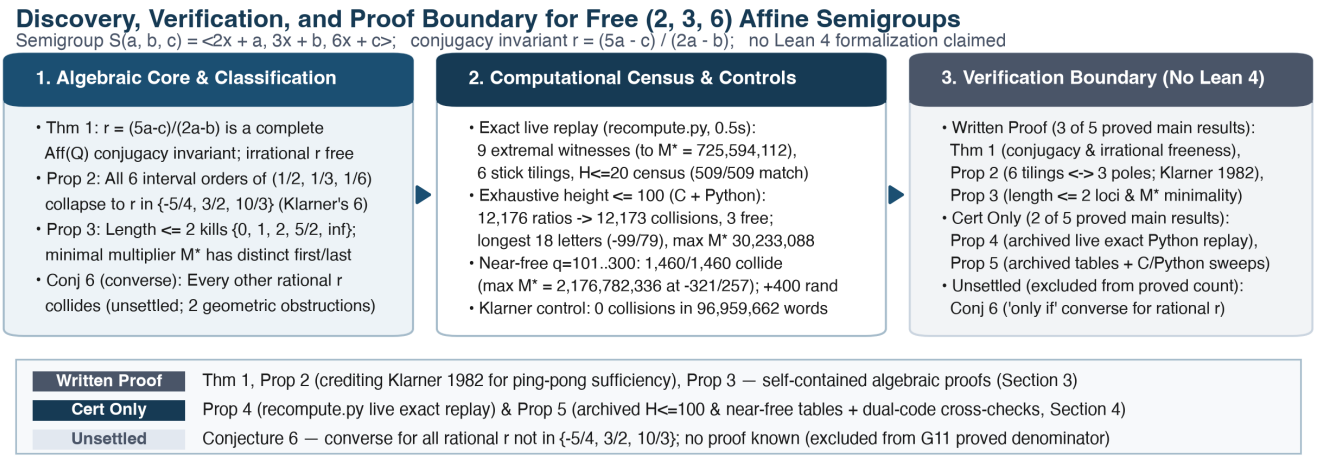


Figure 1. Discovery, verification, and proof boundary for Theorem 1, Propositions 2–5, and Conjecture 6. No Lean 4 formalization is claimed (0 of 5 proved main results in Lean 4; 3 of 5 by written algebraic proof; 2 of 5 by external certificate, stratified between archived live replay in `recompute.py` and external C/Python sweeps). Conjecture 6 (1 unsettled claim) is excluded from the proved denominator.

3. PROOFS OF THE ALGEBRAIC RESULTS

We give complete self-contained proofs of Theorem 1 and Propositions 2 and 3. Because Theorem 1(3) uses the existence of at least one free triple (such as $(0,2,3)$), we establish Proposition 2 immediately after parts (1)–(2) of Theorem 1, ensuring a strictly acyclic proof order.

Proof of Theorem 1, parts (1) and (2). Let $\phi(x) = \lambda x + \mu$ be an invertible affine transformation over K ($\lambda \in K^\times$, $\mu \in K$). For any generator $f_m(x) = mx + t_m$, direct substitution yields

$$(\phi^{-1} \circ f_m \circ \phi)(x) = \frac{m(\lambda x + \mu) + t_m - \mu}{\lambda} = mx + \frac{t_m + (m-1)\mu}{\lambda}.$$

Thus conjugation by ϕ replaces (a,b,c) with

$$(a',b',c') = \left(\frac{a+\mu}{\lambda}, \frac{b+2\mu}{\lambda}, \frac{c+5\mu}{\lambda}\right).$$

Computing the linear invariants $\Delta_1' = 2a' - b'$ and $\Delta_2' = 5a' - c'$ gives

$$\Delta_1' = \frac{2(a+\mu) - (b+2\mu)}{\lambda} = \frac{\Delta_1}{\lambda},$$
$$\Delta_2' = \frac{5(a+\mu) - (c+5\mu)}{\lambda} = \frac{\Delta_2}{\lambda}.$$

Hence the degenerate condition $\Delta_1 = \Delta_2 = 0$ is conjugacy-invariant, and on the non-degenerate locus $(\Delta_1, \Delta_2) \neq (0,0)$ the projective ratio $r(a',b',c') = [\Delta_2' : \Delta_1'] = [\Delta_2 : \Delta_1] = r(a,b,c) \in \mathbb{P}^1(K)$ is invariant under $\mathrm{Aff}(K)$.

For part (1), if $\Delta_1 = \Delta_2 = 0$, then $b = 2a$ and $c = 5a$. Choosing $\lambda = 1$ and $\mu = -a$ conjugates $(a, 2a, 5a)$ to $(0,0,0)$, whose generators $x \mapsto 2x$, $3x$, $6x$ commute pairwise (in particular $2.3 = 3.2 = 6$), so $S(a,b,c)$ is not free.

For the converse direction of part (2), suppose $(a,b,c) \in K^3$ is non-degenerate.

- If $\Delta_1 \neq 0$ (so $r = \Delta_2/\Delta_1 \in K$), set $\mu = -a$ and $\lambda = -\Delta_1 \in K^\times$. Then $a' = 0$, $b' = (b-2a)/(-\Delta_1) = 1$, and $c' = (c-5a)/(-\Delta_1) = \Delta_2/\Delta_1 = r$, so (a,b,c) is conjugate to $(0,1,r)$. If $r = p/q \in \mathbb{Q}$ in lowest terms ($q \geq 1$), choosing instead $\lambda = -\Delta_1/q$ conjugates (a,b,c) to the integer triple $(0,q,p)$.
- If $\Delta_1 = 0$ and $\Delta_2 \neq 0$ ($r = \infty$), set $\mu = -a$ and $\lambda = -\Delta_2 \in K^\times$. Then $(a',b',c') = (0,0,1)$, where $f_2(x) = 2x$ and $f_3(x) = 3x$ satisfy $2.3 = 3.2$ ($6x$).

Because every non-degenerate triple is conjugate to a canonical representative depending only on $r(a,b,c)$, two non-degenerate triples are $\mathrm{Aff}(K)$-conjugate if and only if they have the same ratio $r \in \mathbb{P}^1(K)$. Since conjugation by ϕ induces an isomorphism $S(a,b,c) \cong S(a',b',c')$ preserving generators, freeness depends solely on $r(a,b,c)$. □

Proof of Proposition 2. Suppose the three inverse contractions $g_m(y) = (y - t_m)/m$ ($m \in \{2,3,6\}$) partition $I = [0,1]$ end-to-end into pieces of lengths $1/2$, $1/3$, and $1/6$ with pairwise disjoint interiors. Let $\pi = (m_1, m_2, m_3)$ be the left-to-right ordering of $\{2,3,6\}$ along $[0,1]$, with left endpoints $s_1 = 0$, $s_2 = 1/m_1$, and $s_3 = 1/m_1 + 1/m_2$ (since $1/2 + 1/3 + 1/6 = 1$, the third piece ends at 1). For each $j \in \{1,2,3\}$, the tiling condition $g_{m_j}([0,1]) = [s_j, s_j + 1/m_j]$ holds if and only if $-t_{m_j}/m_j = s_j$, that is, $t_{m_j} = -m_j s_j$. Evaluating $(a,b,c) = (t_2, t_3, t_6)$ over all $3! = 6$ permutations yields the six triples in Table 1.

Reversing the order $(m_1, m_2, m_3) \mapsto (m_3, m_2, m_1)$ corresponds to conjugating the interval $[0,1]$ by the reflection $\rho(x) = 1 - x$, which sends $f_m(x) = mx + t_m$ to $(\rho^{-1} \circ f_m \circ \rho)(x) = mx + (1 - m - t_m)$ and therefore preserves r by Theorem 1(2). Computing $r = (5a - c)/(2a - b)$ directly on the six triples confirms that the three reflected pairs take the exact values

$$r = \frac{3}{2}, \qquad r = \frac{10}{3}, \qquad r = -\frac{5}{4},$$

and rescaling/reflecting each triple to minimal nonnegative integers yields Klarner's six triples $(0,2,3)$, $(1,0,2)$, $(0,3,10)$, $(2,1,0)$, $(2,0,15)$, and $(1,6,0)$.

Conversely, by Theorem 1(2), every triple having ratio $r \in \{-5/4, 3/2, 10/3\}$ is conjugate to one of these six interval tilings on $I = [0,1]$. For completeness we record the standard ping-pong verification (Klarner [4], Kolpakov–Talambutsa [5] `[to check]`): suppose two distinct words $u \neq v$ satisfy $f_u = f_v$. Since each f_m is bijective on $\mathbb{R}$, canceling any common initial letters leaves $u' \neq v'$ with $f_{u'} = f_{v'}$. Neither word can be empty (since $M(\varepsilon) = 1$ whereas every nonempty word has multiplier ≥ 2), so u' and v' begin with distinct letters $m \neq m'$. Then the closed interval $I_{u'} = f_{u'}^{-1}(I) = f_{v'}^{-1}(I) = I_{v'}$ has positive length $1/M(u')$ and lies in both $g_m(I)$ and $g_{m'}(I)$, contradicting the disjointness of the interiors of $g_m(I)$ and $g_{m'}(I)$. Thus $S(a,b,c)$ is free. □

Proof of Theorem 1, part (3). Let $u \neq v \in \Sigma^*$ be distinct words with $M(u) = M(v) = M$. Under a shift $\phi(x) = x + \mu$, the translation $T(w;a,b,c)$ transforms as

$$T(w;a+\mu,b+2\mu,c+5\mu) = T(w;a,b,c) + (M(w)-1)\mu$$

because $(\phi^{-1} \circ f_w \circ \phi)(x) = M(w)x + T(w;a,b,c) + (M(w)-1)\mu$. Consequently, the difference $D_{u,v}(a,b,c) := T(u;a,b,c) - T(v;a,b,c)$ is an integer-coefficient linear form in (a,b,c) that vanishes identically along the shift line $(1,2,5)$. Since the kernel of $(a,b,c) \mapsto (\Delta_1, \Delta_2) = (2a-b, 5a-c)$ is precisely $\mathbb{Q}\cdot(1,2,5)$, there exist unique integers $A_{u,v}, B_{u,v} \in \mathbb{Z}$ (specifically $A_{u,v} = D_{u,v}(0,-1,0)$ and $B_{u,v} = D_{u,v}(0,0,-1)$) such that

$$T(u;a,b,c) - T(v;a,b,c) = A_{u,v}(2a-b) + B_{u,v}(5a-c). \quad (2)$$

We must have $(A_{u,v}, B_{u,v}) \neq (0,0)$: otherwise $D_{u,v}$ would vanish identically for every (a,b,c), contradicting the freeness of $(0,2,3)$ established in Proposition 2. Therefore a non-degenerate triple (a,b,c) satisfies $f_u = f_v$ if and only if $A_{u,v}\Delta_1 + B_{u,v}\Delta_2 = 0$, which determines the single rational projective value $r = -A_{u,v}/B_{u,v} \in \mathbb{P}^1(\mathbb{Q})$. In particular, no pair $u \neq v$ can collide when $r \in \mathbb{R}\setminus\mathbb{Q}$, so every irrational r generates a free semigroup. □

Proof of Proposition 3. For part (1), the $3+9=12$ nonempty words of length at most 2 have multipliers: 2 (word 2); 3 (word 3); 4 (2.2); 6 (6, 2.3, 3.2); 9 (3.3); 12 (2.6, 6.2); 18 (3.6, 6.3); and 36 (6.6). Only the classes $M \in \{6, 12, 18\}$ contain more than one word, giving $\binom{3}{2} + 1 + 1 = 5$ unordered pairs. Evaluating (1) directly gives:

$$\begin{aligned}
T(6) - T(2.3) &= c - 3a - b = \Delta_1 - \Delta_2 && (r = 1),\\
T(6) - T(3.2) &= c - 2b - a = 2\Delta_1 - \Delta_2 && (r = 2),\\
T(2.3) - T(3.2) &= 2a - b = \Delta_1 && (r = \infty),\\
T(2.6) - T(6.2) &= 5a - c = \Delta_2 && (r = 0),\\
T(3.6) - T(6.3) &= 5b - 2c = 5\Delta_1 - 2\Delta_2 && (r = 5/2),
\end{aligned}$$

where each vanishing locus is shown in parentheses. Thus length ≤ 2 kills exactly $\{0, 1, 2, 5/2, \infty\}$.

For part (2), let $u \neq v$ be a collision achieving the minimal multiplier $M^*(r)$. Because each affine map f_m is bijective on $\mathbb{R}$, if $u = mu'$ and $v = mv'$ shared an initial letter $m \in \Sigma$, then $f_{u'} \circ f_m = f_u = f_v = f_{v'} \circ f_m$ would imply $f_{u'} = f_{v'}$ with $u' \neq v'$ and multiplier $M^*(r)/m < M^*(r)$, contradicting minimality. Symmetrically, if $u = u'm$ and $v = v'm$ shared a terminal letter m, left-canceling f_m would yield $f_{u'} = f_{v'}$ at multiplier $M^*(r)/m < M^*(r)$.

Now specialize to the integer normal form $(0,q,p)$ for $r = p/q$, so $(t_2, t_3, t_6) = (0, q, p)$. For each $N = 2^i 3^j \leq B$, prepending letter $m \in \{2,3,6\}$ to a word w' of multiplier N/m and translation t yields the word mw' of multiplier N and translation

$$T(mw';0,q,p) = T(w';0,q,p) + \frac{N}{m}\, t_m.$$

Therefore the set $T(N)$ of all translations realized at multiplier N satisfies the exact recursion

$$\begin{aligned}
T(N) = T(N/2) &\cup (T(N/3) + (N/3)q)\\
&\cup (T(N/6) + (N/6)p),
\end{aligned}$$

where terms with non-integer N/m are omitted. Because no collision occurs at any multiplier $< M^*(r)$, the three shifted sets at every $N < M^*(r)$ are pairwise disjoint and each translation has a unique word; at $N = M^*(r)$, the minimal collision has distinct initial letters and therefore manifests as an exact intersection between two of the three shifted sets. Hence the algorithm detects $M^*(p/q)$ whenever $B \geq M^*(p/q)$. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Exact Live Replay and Extremal Collision Witnesses. Every calculation in Proposition 4 is reproduced deterministically from scratch in under 1 second by `recompute.py` using only Python's standard library and exact arbitrary-precision integer and rational arithmetic (`fractions.Fraction`). Table 2 collects the nine worked and extremal collision identities replayed verbatim in `recompute.py`, where each pair (u,v) is verified simultaneously on the integer normal form $(0,q,p)$, on the shifted integer conjugate $(1, q+2, p+5)$, and on the non-normalized rational conjugate $-\frac{5}{2}(\frac{7}{3}, q + \frac{14}{3}, p + \frac{35}{3})$.

Backward Stick Geometry: Tiling Versus Exact Overlap. Figure 2 illustrates the geometric dichotomy separating Klarner's three conjugacy classes from every colliding ratio. For Klarner's representative $(0,2,3)$ $(r = 3/2)$, the fixed points of $g_2(y) = y/2$, $g_3(y) = (y-2)/3$, and $g_6(y) = (y-3)/6$ are 0, -1, and $-3/5$, giving hull $I = [-1, 0]$. The three depth-1 pieces $[-1, -2/3]$, $[-2/3, -1/2]$, and $[-1/2, 0]$ sit end-to-end with exact total length $1/3 + 1/6 + 1/2 = 1$, and iterating produces 3^k pairwise disjoint-interior intervals of total length 1 at every depth k (verified in exact rationals through $k = 3$ in `recompute.py`). By contrast, for $(1,1,1)$ $(r = 4)$, the fixed points are -1, $-1/2$, and $-1/5$, giving hull $I = [-1, -1/5]$ of length $4/5$. Pieces $g_2(I) = [-1, -3/5]$ and $g_3(I) = [-2/3, -2/5]$ overlap on $[-2/3, -3/5]$ (length $1/15$) while leaving an equal gap $[-2/5, -1/3]$ before $g_6(I) = [-1/3, -1/5]$. Inside the overlap band, the depth-2 cylinder $I_{2.6} = g_2(g_6(I))$ and the depth-3 cylinder $I_{3.2.2} = g_3(g_2(g_2(I)))$ coincide identically on $[-2/3, -3/5]$, expressing the algebraic collision $2.6 = 3.2.2 = 12x + 7$.

Boundary Crowding, Word-Length Growth, and Stratified Provenance. How rapidly do collisions grow as r approaches one of the three free poles $r_0 \in \{-5/4, 3/2, 10/3\}$? Figure 3 plots both the word-length profile across height

Table 2. Worked and extremal collision witnesses verified in exact rational arithmetic across three affine conjugates each in `recompute.py` (words written left-to-right; lengths given as $|u|, |v|$).

Ratio $r = p/q$	Multiplier M^*	Lengths	Pole / $d(r)$	Witness words ($u = v$, compact notation)	Role
4	12	2, 3	10/3 (0.667)	26 = 322	Worked example ($12x + 7$)
−3/2	1,296	6, 8	−5/4 (0.250)	362623 = 23332322	Worked example ($1296x + 663$)
−29/23	373,248	15, 11	−5/4 (0.0109)	233333322222222 = 36662222263	Longest word at $H \le 30$
−99/79	15,116,544	18, 12	−5/4 (0.00316)	2333333323232332222 = 366662332663	Longest word at $H \le 100$
−86/69	30,233,088	14, 15	−5/4 (0.00362)	26666222333336 = 633333322266622	Largest M^* at $H \le 100$
1353/904	214,990,848	23, 20	3/2 (0.00332)	23333332222223232222222 = 62222222263326223323	Hardest of 400 random ratios
320/213	429,981,696	21, 21	3/2 (0.00235)	322222222223623623336 = 633333222622222222622	Near-free pole 3/2
543/163	483,729,408	18, 19	10/3 (0.00204)	366663623232222322 = 6222222222336336363	Near-free pole 10/3
−201/161	725,594,112	16, 21	−5/4 (0.00155)	2666626363236322 = 633333332222222262233	Near-free pole −5/4

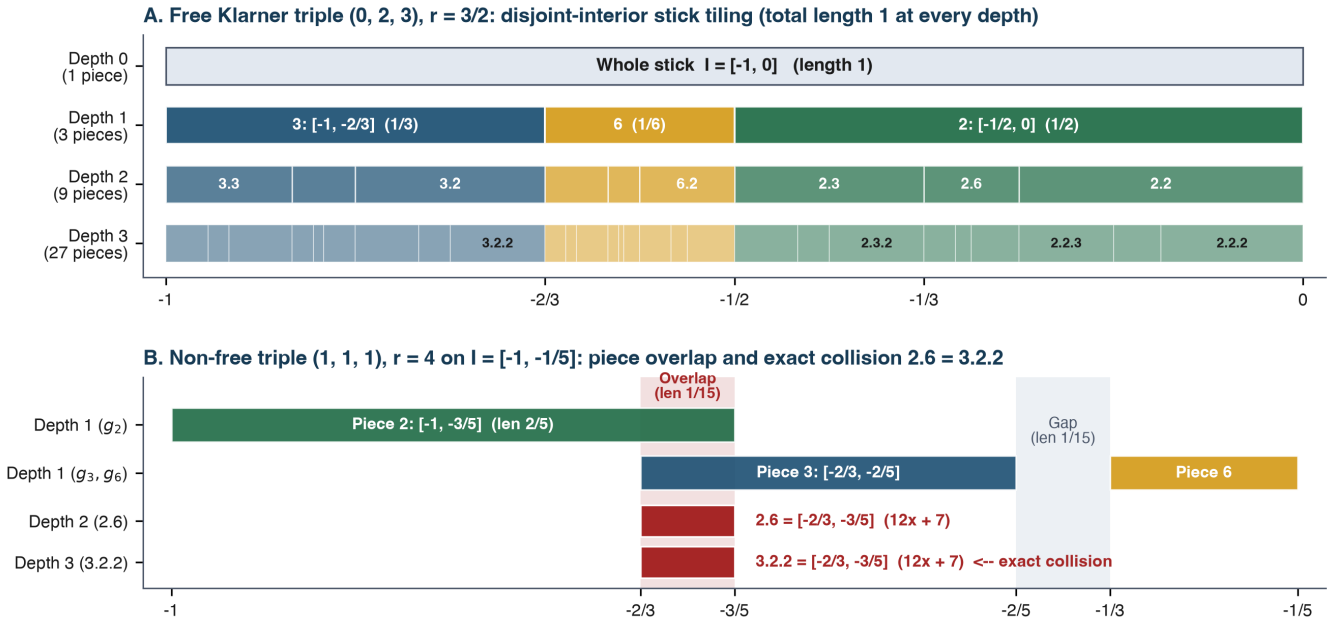


Figure 2. Backward cylinder intervals $I_w = f_w^{-1}(I)$ for a free Klarner triple versus a colliding triple (verified in exact rationals by `recompute.py`). Panel A: for $(0, 2, 3)$ ($r = 3/2$) on $I = [-1, 0]$, depths $0, 1, 2, 3$ partition the stick into $1, 3, 9, 27$ disjoint-interior intervals of exact total length 1, so every address is unique. Panel B: for $(1, 1, 1)$ ($r = 4$) on $I = [-1, -1/5]$, pieces $g_2(I)$ and $g_3(I)$ overlap on $[-2/3, -3/5]$ (length $1/15$) while leaving gap $[-2/5, -1/3]$, and the cylinders $I_{2.6}$ and $I_{3.2.2}$ coincide identically on $[-2/3, -3/5]$.

≤ 30 (Panel A, 963 colliding ratios in $[-3, 5]$ from `census.tsv`, peaking at 15 letters at $r = -29/23$) and the log-log growth of $M^*(r)$ against distance $d(r) := \min_{r_0 \in \{-5/4, 3/2, 10/3\}} |r - r_0|$ across all 12,172 finite colliding height-≤ 100 ratios together with the 1,460 boundary-crowding ratios of denominators $101 \le q \le 300$ (Panel B). On the 1,460 near-free ratios (reaching minimum distance $d = 0.000836$ and maximum multiplier $M^* = 2{,}176{,}782{,}336$ at $r = -321/257$), ordinary least squares yields the empirical scaling law

$$\ln M^*(r) \approx 1.771 \ln(1/d(r)) + 7.008 \qquad (n = 1{,}460),$$

so $M^*(r)$ scales roughly like $d(r)^{-1.77}$ near the three poles, whereas away from the poles ($d(r) > 0.1$, height ≤ 100) no ratio exceeds $M^* = 1.7 \times 10^6$ (and probing $r = -1 \pm 1/q$ for $q \le 501$ finds $M^* \le 559{,}872$ throughout). A complementary short-word enumeration (`kills.out`) confirms that pairs of words of length at most $L = 2, 3, 4, 5, 6, 7, 8$ kill 5, 18, 142, 1,119, 8,974, 71,480, and 573,684 distinct ratios, respectively, closing in to distances 0.0590 from 3/2 (611/424), 0.0603 from 10/3 (851/260), and 0.0291 from −5/4 (−55/43) by $L = 8$, while leaving all three Klarner poles untouched.

Table 3 stratifies the provenance of all sweeps in Propositions 4 and 5, separating live replay in `recompute.py` and SHA-256-pinned tables (`archived_replayable`) from larger C/Python runs kept only in local run records (`regenerable_deterministic`). Notice also one arithmetic bookkeeping correction caught during independent review and verified in `recompute.py`: an unreduced loop over $\gcd(p, q) = 1$ with $|p| \le 60$, $0 \le q \le 60$ emits 4,409 pairs because ∞ appears twice (as $-1/0$ and $1/0$); counting ∞ once yields 4,408 distinct projective ratios and 4,405 non-free ratios (not 4,406), and identically 12,176 distinct ratios (12,173 non-free) at height ≤ 100.

Negative-mutation controls. Five negative-mutation controls in `recompute.py` (5/5 caught) verify sensitivity of the exact checker:

1. Replacing 3.2.2 with 6.2 at $(1, 1, 1)$ breaks the collision ($12x + 7 \ne 12x + 3$).

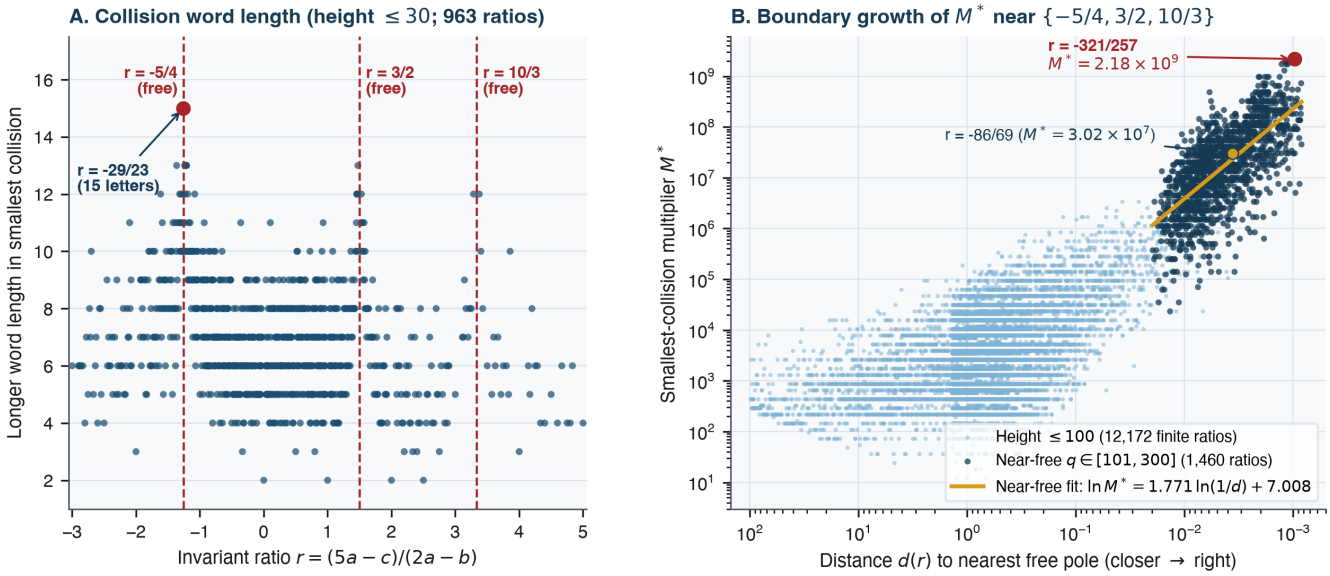


Figure 3. Collision complexity across the ratio line (archived tables `census.tsv`, `falsifier-allM.txt`, and `falsifier-nearfree.tsv`, audited by `recompute.py`). Panel A: length of the longer colliding word across all 963 colliding ratios of height $\leq$ 30 in $[-3, 5]$, spiking symmetrically toward the three free poles $r \in \{-5/4,\ 3/2,\ 10/3\}$ (longest 15 letters at $r = -29/23$, $M^* = 373{,}248$). Panel B: smallest-collision multiplier M^* versus distance $d(r)$ to the nearest free pole (log-log, closer to the right) across all 12,172 finite colliding height-$\leq$ 100 ratios and 1,460 near-free ratios ($q \in [101, 300]$), with least-squares fit $\ln M^* = 1.771 \ln(1/d) + 7.008$ and callouts for $r = -86/69$ ($M^* = 30{,}233{,}088$) and $r = -321/257$ ($M^* = 2{,}176{,}782{,}336$).

Table 3. Honest provenance stratification of all computational sweeps supporting Propositions 4 and 5.

Scope	Method / cross-check	Result	Archive status
$H \leq 20$ (512) + 9 witnesses	Live Python (`recompute.py`)	509/509 collide (0 diffs vs $H \leq 100$); 3 free ($M \leq 10^5$)	`archived_replayable`
$H \leq 30$ (1,112 ratios)	Python (`census.py` → `census.tsv`)	1,109 collide (max 15 letters, $-29/23$); 3 free ($M \leq 10^7$)	`archived_replayable`
$H \leq 60$ (4,408 ratios)	Initial scan + C cross-check	4,405 collide (0 diffs on 4,406 rows with $\pm 1/0$)	`regenerable_deterministic`
$H \leq 100$ (12,176 ratios)	C (`falsifier-allM.txt`) + Fractions	12,173 collide (max 18 letters, $-99/79$; $M^* \leq 30{,}233{,}088$)	`archived_replayable` (table)
Near-free $q \in [101, 300]$ (1,460)	C (`falsifier-nearfree.tsv`) + 15-row C	1,460/1,460 collide (max 2,176,782,336 at $-321/257$)	`archived_replayable` (table)
Random $q \in [101, 1000]$ (400)	C DFS + banded search + 76 witnesses	400/400 collide (max $M^* = 214{,}990{,}848$ at 1353/904)	`regenerable_deterministic`
Klarner $(-5/4, 3/2, 10/3)$	C (10^8) + Python (10^7) + C DFS (10^6)	0 collisions in 96,959,662 words ($B = 10^8$) per ratio	`regenerable_deterministic`

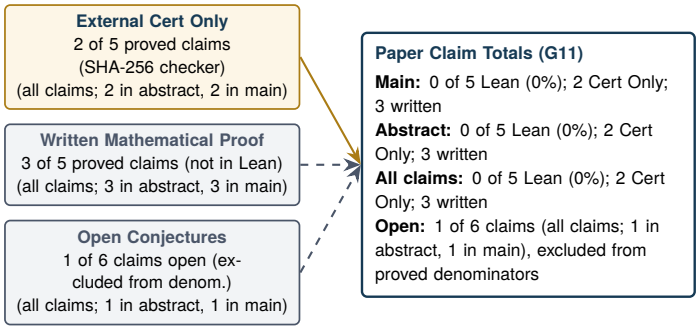
2. Perturbing Klarner's $(0, 2, 3)$ ($r = 3/2$) to $(0, 2, 4)$ ($r = 2$) creates a piece-3/piece-6 overlap of length 1/6 and triggers the length-$\leq$ 2 collision $6 = 3.2$.
3. Mutating the final letter of the 18-letter witness at $r = -99/79$ breaks the affine identity.
4. Counting both $-1/0$ and $1/0$ reproduces the 4,405 → 4,406 height-$\leq$ 60 inflation caught during review.
5. Perturbing $r = 3/2$ to $r = 31/20$ immediately triggers a collision at $M^* = 10{,}368$ (3.2.2.2.2.2.3.2.2.3.3 = 6.3.3.6.2.2.2.2.2).

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 4 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Outside the commuting locus 2a = b, 5a = c, the projective ratio r = (5a - c)/(2a - b) is a complete Aff(K) conjugacy invariant deciding freeness of S(a, b, c), and every irrational r yields a free semigroup	Written	Written proof	Lean: none; fidelity: n/a	Self-contained algebraic proof in Section 3 (part 3 uses existence of one free rational triple from Proposition 2 / Klarner 1982). No Lean 4 formalization.
Prop. 2: The six interval partitions of [0, 1] into lengths 1/2, 1/3, 1/6 collapse under reflection to exactly r in {-5/4, 3/2, 10/3}, matching Klarner's six free integer triples (sufficiency credited to Klarner 1982 via interval ping-pong)	Written	Written proof	Lean: none; fidelity: n/a	Self-contained permutation and reflection-pairing proof in Section 3, explicitly crediting Klarner (1982, read via Lagarias 2003 entry 97) and Kolpakov-Talambutsa for interval ping-pong freeness. No Lean 4 formalization.
Prop. 3: Pairs of distinct equal-multiplier words of length <= 2 kill exactly r in {0, 1, 2, 5/2, infinity}, and any collision attaining minimal multiplier M*(r) has distinct initial and terminal letters, making the multiplier-class recurrence exact	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof in Section 3 via direct translation evaluation and bijectivity of affine generators. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 4: Exact live Python replay (recompute.py) verifies worked and extremal witnesses up to M* = 725,594,112, all six stick tilings and depth-3 [-1, 0] partition vs r = 4 overlap, exact ratio and [0, 12]^3 counts, word counts f(B), live height-<=20 census (509/509 match), and archived table audits	Cert Only	recompute.py (9f72349fbf4b) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Directly replayed in pure exact integer/rational arithmetic (zero floats in collision decisions) by recompute.py against SHA-256-pinned archived tables.
Prop. 5: Exhaustive C and Python searches verify collisions at all 12,173 non-free height-<=100 ratios (longest 18 letters at -99/79, max M* = 30,233,088 at -86/69), all 1,460 near-free ratios (q in 101..300, max M* = 2,176,782,336 at -321/257), and 400 random ratios (q <= 1,000), with 0 collisions in 96,959,662 words (M <= 10^8) on each Klarner ratio	Cert Only	recompute.py (9f72349fbf4b) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Stratified provenance (Table 3): height-<=20 live in recompute.py; height-<=30, height-<=100, and near-free tables archived and audited in recompute.py; full C search logs, 400-ratio random sample, and 96,959,662-word control runs sit in local non-archived run records (regenerable_deterministic).
Conj. 6: Every rational r not in {-5/4, 3/2, 10/3} admits a collision (so S(a, b, c) is free on Q^3 if and only if r in {-5/4, 3/2, 10/3}), together with two geometric obstructions, scoped orbit-density corollary, and distinct-slope extension	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open conjecture (excluded from G11 proved denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 5ced809cdddd | Fidelity SHA-256: 1b0574b49bd8.

6. PRIOR WORK AND OPEN QUESTIONS

The freeness and orbit-density literature for affine semigroups provides the context and credits for every known ingredient:

- *Klarner* [4] (*[to check], via Lagarias* [7]*, entry 97*) proved $\sum 1/m_i \leq 1$ is necessary for freeness, gave the sufficient lexicographic/interval-order criterion, and identified the six free $(2,3,6)$ integer triples in Table 1. The "if" direction of our classification is Klarner's theorem.
- *Crampin and Hilton* (*unpublished, via Lagarias* [7]*, entry 97*) resolved Erdős's positive-density question for $\langle 2x+1, 3x+1, 6x+1\rangle$ from seed 1: any nontrivial relation forces shortlex-minimal words to avoid a factor, pushing the Dirichlet exponent below 1 and forcing density zero.
- *Konyagin, Protasov, and Talambutsa* [6] (*[to check]*) form the nearest relative: for equal-slope generators $\{nx+a_i\}$ (n maps), prime n gives freeness if and only if the digits are distinct modulo n, whereas composite n admits non-standard free digit sets (Theorem 3 [to check]). For unequal slopes they cite Klarner and note (footnote 2 [to check]) that freeness remains open for integer affine functions.
- *Kolpakov and Talambutsa* [5] (*[to check]*) derive Klarner-type sufficiency via ping-pong and analyze the $\sum 1/m_i > 1$ relation threshold, noting that proving relations at exact balance is harder.
- *Shamazov and Talambutsa* [9] (*[to check]*) study orbit sets of one-dimensional affine semigroups, citing Klarner's six triples without stating or asking the $(2,3,6)$ converse.
- *Kalmynin* [2] (*[to check]*) proves density zero for Klarner's six exact triples from any density-zero seed set $B \subseteq \mathbb{Z}_{\geq 0}$ (Theorem 3 [to check]), noting that the tiling locus is Zariski closed in a suitable sense.
- *Related tiling and decidability work.* Kenyon [3] ([to check]) settles the equal-ratio $(1/3, 1/3, 1/3)$ exact-overlap dichotomy for rational slopes p/q; Zaitseva [10] ([to check]) classifies one-dimensional finite-union-of-intervals self-similar tiles for equal ratios only; Cassaigne and Nicolas [1] ([to check]) study freeness decidability for upper-triangular 2×2 matrix semigroups.

Prior-work searches logged. Two independent literature-search logs (both passing the repository search gate) queried arXiv, zbMATH Open, OEIS (words and $M^*(n)$ for integer $r = n$ and $1/n$), Lagarias's annotated bibliography [7] (entry 97 read in full), and Lagarias's 2015 talk slides (24–26). Keyed CORE searches on 2026-11-19 (7 requests, 23 records, 6 relatives, 0 same-claim) and a 5-query Google Scholar sweep on 2026-11-22 (41 records) followed. The "only if" converse was *not found in the searches logged*.

Unread and unsearched items (`[to check]`). Remaining unread or unsearched items (marked `[to check]` in `references-checked.md`): (1) Klarner [4] full text (paywalled); (2) Lagarias [8], Section 8.9 (paywalled, likeliest location for a published converse remark); (3) Guy's *Unsolved Problems in Number Theory* (Problem E36) and Guy (1983, Problem 4); (4) full texts of [1, 2, 3, 5, 6, 9, 10]; and (5) direct Math Stack Exchange and MathOverflow API queries (throttled).

Conjecture 6 (Converse Characterization of Free $(2,3,6)$ Affine Semigroups and Open Questions). *The following questions remain unsettled:*

1. (Main converse conjecture.) *For $(a,b,c) \in \mathbb{Q}^3$, $S(a,b,c) = \langle 2x+a,\ 3x+b,\ 6x+c\rangle$ is free if and only if $r(a,b,c) \in \{-5/4,\ 3/2,\ 10/3\}$; equivalently, every rational $r \in \mathbb{P}^1(\mathbb{Q})\backslash\{-5/4,\ 3/2,\ 10/3\}$ collides, and over $\mathbb{R}$ the non-free locus is $\mathbb{P}^1(\mathbb{Q})\backslash\{-5/4,\ 3/2,\ 10/3\}$.*
2. (Two geometric obstructions.) *At exact balance $1/2 + 1/3 + 1/6 = 1$, pigeonhole counting on multiplier classes 2^i3^j falls short by $\sqrt{\ln M}$ to $\ln M$. Proving part (1) requires ruling out: (a) any non-interval self-similar tile of $\mathbb{R}$ with ratios $(1/2, 1/3, 1/6)$, and (b) any rational r whose inverse attractor is Lebesgue-null with no exact overlaps (the unequal-ratio analogue of Kenyon [3] `[to check]`).*
3. (Scoped orbit-density corollary.) *Conditional on part (1), every non-free $(2,3,6)$ triple with $a,b,c \geq 0$ has density-zero orbit sets from every seed $s \geq 1$ (Crampin–Hilton [7]), while the free triples are the Klarner classes, where Kalmynin [2] (`[to check]`) proves density zero for the six exact triples (leaving off-lattice conjugates such as $(0,4,6)$ from seed 1 separate).*
4. (Other distinct unit-fraction slope sets.) *Determine whether interval tiling characterizes freeness for other distinct slope sets with $\sum 1/m_i = 1$ (at height 9 on $\{2,3,10,15\}$, 2,880 of 2,881 vectors collide and the single free vector tiles, whereas repeated slopes such as $\{2,4,4\}$ admit non-tiling free classes via exact covering systems).*

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, and `recompute.py` live in `papers/free-2-3-6-machines/draft1/`. Running `python3 papers/free-2-3-6-machines/draft1/recompute.py` finishes in under 1 second (Python standard library only) and verifies Proposition 4, Tables 1 and 2, the Section 1 snippet, all three archived data tables (`falsifier-allM.txt`, `falsifier-nearfree.tsv`, `census.tsv`), and 5/5 negative-mutation controls. Larger C and Python sweep logs for Proposition 5 (including the 96,959,662-word control runs and 400-ratio random sample) reside in local non-archived records (`regenerable_deterministic`; Table 3). No Lean 4 formalization is claimed.

AI research agents built and directed by Tyler B. Hughes discovered the conjugacy formulation and converse conjecture, proved the algebraic results, ran and cross-checked the computational sweeps and exact verification scripts, and drafted this manuscript. No human mathematician has yet refereed the proof.

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