MATHDISCOVERIES
Mathematical Preprint

Working Draft

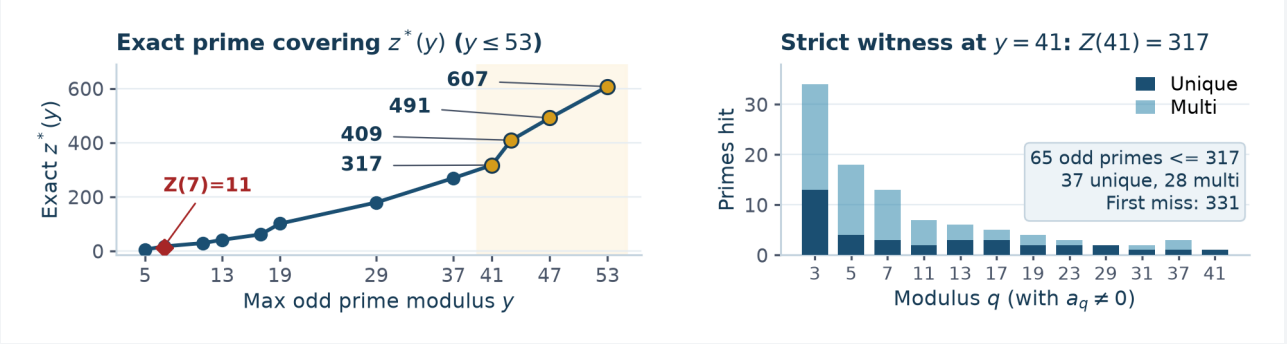
Exact Residue Coverings of the Odd Primes for Moduli up to 53

Tyler B. Hughes*,†

†Independent Researcher, United States

Ⓢ *Supporting Information:* **`recompute.py`** *and* **`replay_exhaustive.py`** *(standalone exact Python verifiers and exhaustive branch-and-bound replay)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For an even $n \ge 4$, let $p(n)$ be the least prime p with $n - p$ prime. Choosing one residue class a_q mod q per odd prime $q \le y$ covering every odd prime $p \le z$ forces $p(n) > z$ via the Chinese Remainder Theorem; let $z^*(y)$ be the largest such z (allowing $a_q = 0$ to cover $p = q$) and $Z(y) \le z^*(y)$ the strict variant ($q \neq p$). Combining explicit strict covering witnesses with partitioned branch-and-bound infeasibility proofs, we establish $z^*(41) = Z(41) = 317$, $z^*(43) = Z(43) = 409$, $z^*(47) = Z(47) = 491$, and $z^*(53) = Z(53) = 607$ (65,79,93,110 odd primes covered; infeasibility at 331,419,499,613 proved in 47,640 to 1,066,513 nodes). For smaller moduli $y \in \{5,7,11,13,17,19,29,37\}$, we record exact $z^*(y) = 5,17,29,41,61,101,179,269$ and $Z(7) = 11 < 17 = z^*(7)$. We prove the CRT lifting proposition (composite $n - p$ for all odd primes $p \le z$), branch-and-bound soundness, and $z^*(p_n) < h_2(n)$ against Ziller and Morack's paired Jacobsthal function. These finite-y results do not move asymptotic Goldbach bounds; no Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)
Abstract claims: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written

Full Trust Boundary: Section 5.

MSC 2020: 11P32, 11A07, 11Y99 **Keywords:** Goldbach partitions, residue coverings, prime moduli, Jacobsthal function, Chinese remainder theorem, branch-and-bound

1. INTRODUCTION AND MAIN RESULTS

For an even integer $n \ge 4$, let $p(n)$ denote the least prime p such that $n - p$ is also prime (the *least Goldbach prime* of n, defined whenever n admits a binary Goldbach representation). A classical congruence mechanism for forcing $p(n)$ to be large—parallel to the Erdős–Rankin covering construction for large gaps between consecutive primes [1, 4]—selects a single residue class $a_q \in \{0, 1, \dots, q-1\}$ for each odd prime modulus $q \le y$ so that every odd prime $p \le z$ falls into at least one chosen class a_q mod q. Lifting the congruence system $n \equiv 0 \pmod 2$ and $n \equiv a_q \pmod q$ ($3 \le q \le y$) by the Chinese Remainder Theorem to any even integer $n > z + y$ guarantees that $q \mid (n - p)$ with $n - p > q$ for every odd prime $p \le z$, making every difference $n - p$ composite and therefore forcing $p(n) > z$.

Two natural conventions arise for the prime $p = q$ itself. In the *relaxed* covering function $z^*(y)$, choosing $a_q = 0$ is permitted and counts as covering the target $p = q$: then $q \mid n$, so $n - q = q(n/q - 1)$ is still composite whenever $n > 2q$. In the *strict* covering function $Z(y)$, every odd prime $p \le z$ must be covered by some modulus $q \le y$ with $q \neq p$ (equivalently, $a_q = 0$ does not cover $p = q$, and since no other prime is divisible by q, one may restrict to $a_q \in \{1, \dots, q-1\}$). Clearly $Z(y) \le z^*(y)$, and strict inequality can occur at small moduli: for instance, $Z(3) = 0 < 3 = z^*(3)$ and $Z(7) = 11 < 17 = z^*(7)$.

While asymptotic covering bounds for prime gaps and progressions have been extensively studied [1, 2, 4, 6], and exact values have been tabulated for Jacobsthal's function $h(Q(y))$ on consecutive integers [3] (`[to check]`) and for Ziller and Morack's paired Jacobsthal function $h_2(n)$ [7, 8] (`[to check]`), exact values of the odd-prime covering functions $z^*(y)$ and $Z(y)$ were not found in the literature searches logged in Section 6. We determine $z^*(y)$ and $Z(y)$ exactly for the headline prime moduli $y \in \{41, 43, 47, 53\}$, together with the earlier moduli $y \in \{5,7,11,13,17,19,29,37\}$. Throughout, we emphasize that these are finite-y congruence-covering values: they establish exact upper and lower covering limits at fixed $y \le 53$, and do not move asymptotic bounds on Goldbach's conjecture.

Theorem 1 (Exact Coverings for $y \in \{41, 43, 47, 53\}$). *For each modulus bound $y \in \{41, 43, 47, 53\}$, the relaxed and strict covering functions coincide:*

$$z^*(41) = Z(41) = 317, \qquad z^*(43) = Z(43) = 409,$$
$$z^*(47) = Z(47) = 491, \qquad z^*(53) = Z(53) = 607.$$

In each case, an explicit strict residue vector $(a_q)_{3 \le q \le y}$ with $a_q \in \{1, \dots, q-1\}$ covers all 65, 79, 93, and 110 odd primes up to 317, 409, 491, and 607 respectively, while exhaustive partitioned branch-and-bound search proves that no residue vector (relaxed or strict) covers every odd prime up to the least uncovered prime $p^+ \in \{331, 419, 499, 613\}$.

At $y = 41$, the maximizing covering produced by the initial exhaustive run took $a_{37} = 0$ (covering 37 only by itself), leaving the strict value $Z(41)$ unsettled below 317. An explicit strict witness with all 12 classes nonzero resolves $Z(41) = 317$, while at $y \in \{43, 47, 53\}$ the maximizing coverings produced by the search already have $a_q \neq 0$ for every $q \le y$.

Proposition 2 (Small Moduli and Separation at $y = 7$). *Across the smaller moduli $y \in \{5, 7, 11, 13, 17, 19, 29, 37\}$, the exact values of $z^*(y)$ are*

$$5,\ 17,\ 29,\ 41,\ 61,\ 101,\ 179,\ 269,$$

established by unpruned brute-force enumeration of every residue vector for $y \le 17$ (up to 255,255 vectors at $y = 17$) and by exact branch-and-bound for $y \in \{19, 29, 37\}$. Strict equality $Z(y) = z^*(y)$ holds for all seven moduli other than $y = 7$, whereas at $y = 7$ strict covering is strictly weaker:

$$Z(7) = 11 < 17 = z^*(7)$$

(and at the degenerate base modulus $y = 3$, $Z(3) = 0 < 3 = z^*(3)$).

Proposition 3 (Strict–Relaxed Monotonicity and CRT Lifting). *Let $y \ge 3$ be an integer, and let $Q(y) = 2 \prod_{3 \le q \le y} q$ be the primorial of y.*

1. *$Z(y) \le z^*(y)$, with equality whenever some maximizing relaxed covering $(a_q)_{3 \le q \le y}$ covers every odd prime $p \le z^*(y)$ by a modulus $q \neq p$ (in particular, whenever $a_q \not\equiv 0 \pmod q$ for every odd prime $q \le y$).*
2. *If $(a_q)_{3 \le q \le y}$ covers every odd prime $p \le z$ (even under the relaxed convention), then for every even integer $n > z + y$ satisfying $n \equiv 0 \pmod 2$ and $n \equiv a_q \pmod q$ $(3 \le q \le y)$, the difference $n - p$ is composite for every odd prime $p \le z$. In particular, such an even integer exists in $(z + y, z + y + Q(y)]$, and whenever n admits a Goldbach representation its least Goldbach prime satisfies $p(n) > z$.*

The upper bounds in Theorem 1 and Proposition 2 rest on a deterministic partitioned branch-and-bound infeasibility tree whose soundness is established algebraically in Lemma 4 (Section 3). In addition, Proposition 5 (Section 3) relates $z^*(p_n)$ rigorously to Ziller and Morack's paired Jacobsthal function $h_2(n)$ [7, 8] (`[to check]`), showing $z^*(p_n) < h_2(n)$ whenever $h_2(n)$ is even and $h_2(n) + 1 < p_{n+1}^2$. Figure 1 summarizes the discovery workflow, independent verification controls, and explicit verification boundary.

Instant Witness Check: Strict Covering $Z(41) = 317$

```
qs = [3,5,7,11,13,17,19,23,29,31,37,41]
aq = [2,3,3, 6, 8, 7,10, 1,10, 6,19,38]
ps = [p for p in range(3, 332, 2)
      if all(p % d for d in range(3, int(p**0.5)+1, 2))]
miss = [p for p in ps if not any(p%q == a and p != q
                                 for q, a in zip(qs, aq))]
assert len(ps) == 66 and miss == [331]  # 65 primes <= 317
```

Roadmap. Section 2 fixes notation and works through the boundary phenomena at $y = 3$, $y = 7$ (where $Z(7) < z^*(7)$), and $y = 41$ (comparing the relaxed and strict maximizing coverings). Section 3 proves Proposition 3, Lemma 4 (soundness of the partitioned branch-and-bound tree), and Proposition 5 (the h_2 upper bound). Section 4 presents the explicit strict covering vectors for $y \le 53$, the exhaustive infeasibility logs and independent cross-checks establishing Theorem 1 and Proposition 2, and empirical comparisons against Jacobsthal's $h(Q(y))$ and $h_2(n)$. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 details prior-work searches, unread sources (`[to check]`), and open questions (Conjecture 6), and Section 7 documents data availability and AI disclosure.

2. DEFINITIONS, NOTATION, AND WORKED EXAMPLES

For any real $x \ge 3$, let $\mathcal{P}_x = \{q \text{ odd prime} : q \le x\}$ denote the set of odd primes up to x, with cardinality $k(x) = |\mathcal{P}_x| = \pi(x) - 1$. A *residue covering vector* at modulus bound $y \ge 3$ is a tuple $\mathbf{a} = (a_q)_{q \in \mathcal{P}_y}$ with $a_q \in \{0, 1, \dots, q-1\}$ for each $q \in \mathcal{P}_y$.

- Under the *relaxed* covering rule, an odd prime p is *covered* by $\mathbf{a}$ if there exists $q \in \mathcal{P}_y$ such that $p \equiv a_q \pmod q$. (In particular, choosing $a_q = 0$ covers $p = q$ and no other prime.)
- Under the *strict* covering rule, an odd prime p is *strictly covered* by $\mathbf{a}$ if there exists $q \in \mathcal{P}_y$ with $q \neq p$ and $p \equiv a_q \pmod q$.

For a given vector $\mathbf{a}$, its *relaxed reach* $\rho^*(\mathbf{a})$ is the largest odd prime z such that every odd prime $p \le z$ is covered by $\mathbf{a}$ (or 0 if $p = 3$ is uncovered); its *strict reach* $\rho(\mathbf{a})$ is defined identically using strict coverage. Equivalently, if $p^+(\mathbf{a})$ is the least odd prime not covered by $\mathbf{a}$, then $\rho^*(\mathbf{a})$ is the odd prime immediately preceding $p^+(\mathbf{a})$ (or 0 when $p^+(\mathbf{a}) = 3$). We define

$$z^*(y) := \max_{\mathbf{a}} \rho^*(\mathbf{a}), \qquad Z(y) := \max_{\mathbf{a}} \rho(\mathbf{a}),$$

where the maximum ranges over all $\prod_{q \in \mathcal{P}_y} q$ residue vectors $\mathbf{a}$. Note that both functions depend only on the largest prime $y_0 \le y$, so $z^*(20) = z^*(19)$, $z^*(30) = z^*(29)$, and $z^*(40) = z^*(37)$.

Worked example 1: $y = 3$ and $y = 7$ (why $Z(y)$ can fall below $z^*(y)$). At $y = 3$, there is a single modulus $q = 3$. Choosing $a_3 = 0$ covers $p = 3$ under the relaxed rule (giving $z^*(3) = 3$ and CRT lift $n = 12$, where $12 - 3 = 9 = 3^2$ is composite), whereas no modulus $q \neq 3$ exists to cover $p = 3$ strictly, so $Z(3) = 0$.

A nontrivial separation occurs at $y = 7$ ($\mathcal{P}_7 = \{3, 5, 7\}$, $3 \cdot 5 \cdot 7 = 105$ vectors). Under the relaxed rule, $\mathbf{a} = (2, 3, 0)$

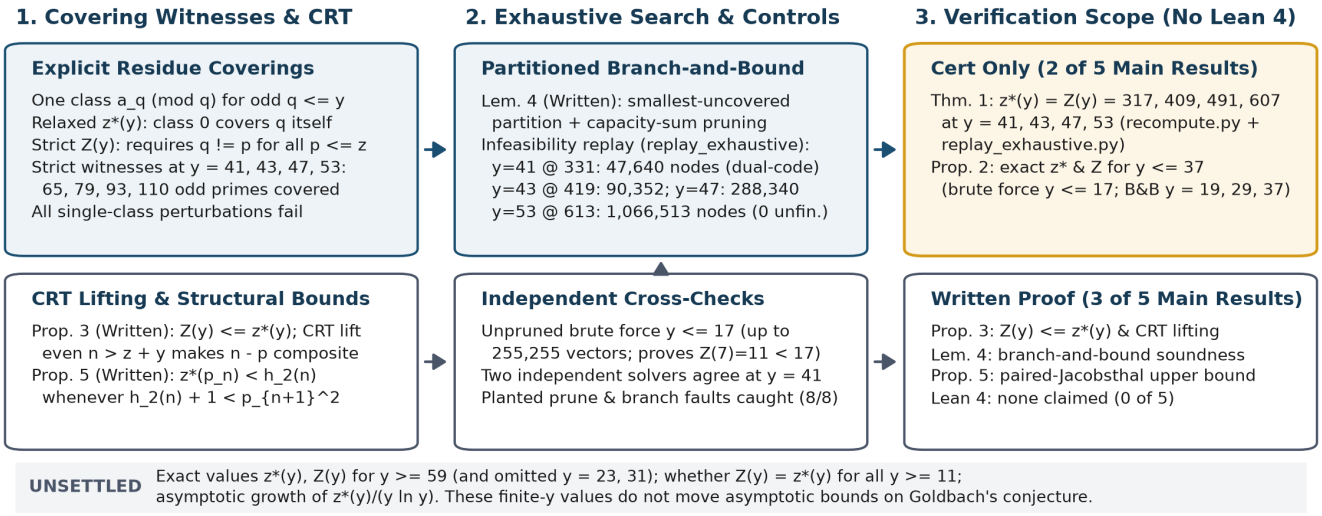


Figure 1. Discovery and verification architecture for Theorem 1 and Propositions 2–5. No Lean 4 formalization is claimed (0 of 5 proved main results in Lean 4; 2 of 5 by archived deterministic certificate scripts `recompute.py` and `replay_exhaustive.py`; 3 of 5 by written proof). Two independent search codes agree on infeasibility at $y = 41$, unpruned brute force confirms $y \le 17$, and planted-fault controls verify sensitivity. Exact values for $y \ge 59$ (and omitted $y \in \{23, 31\}$) and asymptotic scaling remain unsettled.

covers all 6 odd primes up to 17: $a_3 = 2$ covers $\{5, 11, 17\}$, $a_5 = 3$ covers $\{3, 13\}$, and $a_7 = 0$ covers $\{7\}$, with least uncovered prime 19. Lifting $n \equiv 0 \pmod 2$, $n \equiv 2 \pmod 3$, $n \equiv 3 \pmod 5$, $n \equiv 0 \pmod 7$ above $17 + 7 = 24$ gives $n = 98$, for which $98 - p \in \{95, 93, 91, 87, 85, 81\}$ are all composite and $98 - 19 = 79$ is prime, realizing the empirical record $p(98) = 19$. Under the strict rule, however, $q = 7$ cannot cover 7: every choice of $a_3 \in \{1, 2\}$ leaves 3 or 4 primes in $\{3, 5, 7, 11, 13, 17\}$ whose residues modulo 5 and 7 cannot be covered by two nonzero classes without self-covering. Explicitly, if $a_3 = 2$ (covering $5, 11, 17$), the surviving set $\{3, 7, 13\}$ forces $a_5 = 2$ to cover 7, leaving $\{3, 13\}$ which are distinct modulo 7 (3 and 6); if $a_3 = 1$ (covering $7, 13$), the surviving set $\{3, 5, 11, 17\}$ has $5, 11$ distinct modulo 5 $(0, 1)$ and only $\{3, 17\}$ sharing a class modulo 7 (3). Hence no strict covering reaches 13 or 17, while $\mathbf{a} = (2, 2, 3)$ strictly covers $\{3, 5, 7, 11\}$ and misses 13, establishing $Z(7) = 11 < 17 = z^*(7)$.

Worked example 2: Relaxed versus strict optima at $y = 41$. At $y = 41$, there are $k(41) = 12$ odd prime moduli. The initial exhaustive search returned the maximizing vector

$$\mathbf{a}_{41}^{\mathrm{rel}} = (2, 3, 3, 6, 8, 7, 10, 1, 10, 19, 0, 38),$$

which covers all 65 odd primes up to 317 and misses 331, but assigns $a_{37} = 0$ (and no other modulus covers 37, so its strict reach drops to 31, missing 37). Replacing only the two coordinates $(a_{31}, a_{37}) = (19, 0)$ by $(6, 19)$ produces the strict witness

$$\mathbf{a}_{41}^{\mathrm{str}} = (2, 3, 3, 6, 8, 7, 10, 1, 10, 6, 19, 38),$$

in which every $a_q \in \{1, \dots, q-1\}$. Here 37 is covered by $a_{31} = 6$ ($37 \equiv 6 \pmod{31}$), while 19, 167, and 241 are covered by $a_{37} = 19$ (19 uniquely, since $19 \not\equiv a_q \pmod q$ for all other $q \le 41$). All 65 odd primes up to 317 are strictly covered (37 uniquely and 28 by two or more moduli), and 331 is the least uncovered prime, establishing $Z(41) = 317$ once $z^*(41) \le 317$ is proved.

3. STRUCTURAL PROOFS AND SEARCH SOUNDNESS

This section proves the three written results of the paper: monotonicity and CRT lifting (Proposition 3), branch-and-bound soundness (Lemma 4), and the paired-Jacobsthal bound (Proposition 5).

Proof of Proposition 3. For part (1), every strict cover relation ($q \ne p$ and $p \equiv a_q \pmod q$) is a fortiori a relaxed cover relation, so $\rho(\mathbf{a}) \le \rho^*(\mathbf{a})$ for every residue vector $\mathbf{a}$. Taking the maximum over all vectors gives $Z(y) \le z^*(y)$. If a maximizing vector $\mathbf{a}$ with $\rho^*(\mathbf{a}) = z^*(y)$ covers every odd prime $p \le z^*(y)$ by some $q \ne p$—which holds automatically when $a_q \not\equiv 0 \pmod q$ for every $q \in \mathcal{P}_y$, because $p \equiv a_q \pmod q$ cannot hold with $p = q$—then $\rho(\mathbf{a}) = z^*(y)$, forcing $Z(y) \ge z^*(y)$ and hence $Z(y) = z^*(y)$.

For part (2), let $\mathbf{a} = (a_q)_{q \in \mathcal{P}_y}$ satisfy $\rho^*(\mathbf{a}) \ge z$. Since the moduli $\{2\} \cup \mathcal{P}_y$ are pairwise coprime with product $Q(y) = 2 \prod_{q \in \mathcal{P}_y} q$, the Chinese Remainder Theorem determines a unique residue class $n_0 \in \{0, 1, \dots, Q(y) - 1\}$ satisfying $n_0 \equiv 0 \pmod 2$ and $n_0 \equiv a_q \pmod q$ for all $q \in \mathcal{P}_y$. Translating by an appropriate integer multiple of $Q(y)$ yields a unique even integer $n \in (z + y, z + y + Q(y)]$ in this congruence class. For any even $n > z + y$ in this class and any odd prime $p \le z$, there exists $q \in \mathcal{P}_y$ with $p \equiv a_q \equiv n \pmod q$. Hence $q \mid (n - p)$. Because $p \le z$ and $q \le y$, we have

$$n - p > (z + y) - z = y \ge q,$$

so $n - p$ is an integer multiple of the prime q strictly exceeding q (note this holds equally when $p = q$ and $a_q = 0$, since $n - q > z \ge 3$ and $q \mid n - q$ with $n - q > y \ge q$). Thus $n - p$ is composite for every odd prime $p \le z$. Since n is even and $n - 2 > 2$ is even and composite, if n admits a Goldbach

representation $n = p + p'$ with $p \le p'$ prime, then necessarily $p > z$, so $p(n) > z$. □

Partitioned branch-and-bound formulation. Fix an odd prime bound $y \ge 3$ and a target odd prime $T \ge 3$, with moduli $\mathcal{P}_y = (q_1 < q_2 < \cdots < q_k)$ and targets $\mathcal{T}_T = \mathcal{P}_T$. A search state is a triple (L, A, B) where $L \subseteq \mathcal{T}_T$ is the set of currently uncovered targets, $A : D \to \mathbb{Z}$ is a partial residue assignment on $D \subseteq \mathcal{P}_y$ with $0 \le A(q) < q$, and B maps each free modulus $q \in F := \mathcal{P}_y \setminus D$ to a set $B(q) \subseteq \{0, 1, \dots, q-1\}$ of forbidden classes. A full residue vector $\mathbf{a}$ is *compatible* with (L, A, B) if $a_q = A(q)$ for all $q \in D$, $a_q \notin B(q)$ for all $q \in F$, and every $p \in L$ is covered by $(a_q)_{q \in F}$. Notice that at the root state $(\mathcal{T}_T, \emptyset, \emptyset)$, a vector is compatible if and only if it covers every odd prime $p \le T$.

Lemma 4 (Soundness of Branch-and-Bound). *Let (L, A, B) be a search state for (y, T) with free moduli $F = \mathcal{P}_y \setminus D$.*

1. (Partition branching.) *Suppose $L \ne \emptyset$ and let $t = \min L$. Enumerate the candidate free moduli whose residue $t \bmod q$ is not forbidden in fixed increasing order as $q_{i_1} < q_{i_2} < \cdots < q_{i_r}$. For each $j \in \{1, \dots, r\}$, form the child state (L_j, A_j, B_j) by assigning $A_j(q_{i_j}) = t \bmod q_{i_j}$, removing $\{p \in L : p \equiv t \pmod{q_{i_j}}\}$ from L, and adding $t \bmod q_{i_\ell}$ to $B_j(q_{i_\ell})$ for every $\ell < j$. Then the set of full vectors compatible with (L, A, B) is the disjoint union of the sets of full vectors compatible with $(L_1, A_1, B_1), \dots, (L_r, A_r, B_r)$. In particular, if $r = 0$, no compatible vector exists.*
2. (Admissible capacity pruning.) *For each $q \in F$, set*
$$\mathrm{best}(q; L, B) := \max_{\substack{0 \le a < q \\ a \notin B(q)}} |\{p \in L : p \equiv a \pmod q\}|$$
(with the convention $\max \emptyset = 0$). If
$$\sum_{q \in F} \mathrm{best}(q; L, B) < |L|,$$
then no vector is compatible with (L, A, B).

Consequently, if the finite search tree rooted at $(\mathcal{T}_T, \emptyset, \emptyset)$ terminates with every branch pruned and no state reaching $L = \emptyset$, then no residue covering covers all odd primes up to T, and hence $z^*(y) < T$.

Proof. For part (1), let $\mathbf{a}$ be any residue vector compatible with (L, A, B). Since $t \in L$ must be covered by some free modulus $q \in F$ with $a_q \notin B(q)$ and $a_q \equiv t \pmod q$, the set $\{j \in \{1, \dots, r\} : a_{q_{i_j}} = t \bmod q_{i_j}\}$ is nonempty. Let j^* be its least element. Then $a_{q_{i_{j^*}}} = t \bmod q_{i_{j^*}}$ and $a_{q_{i_\ell}} \ne t \bmod q_{i_\ell}$ for every $\ell < j^*$, while all targets in $L \setminus L_{j^*}$ are already covered by $q_{i_{j^*}}$. Hence $\mathbf{a}$ is compatible with $(L_{j^*}, A_{j^*}, B_{j^*})$, and for $j \ne j^*$ the constraints at $q_{i_{\min(j, j^*)}}$ are mutually exclusive, making the child solution sets pairwise disjoint. Conversely, every vector compatible with (L_j, A_j, B_j) obeys all constraints of (L, A, B) and covers $L = (L \setminus L_j) \cup L_j$.

For part (2), if $\mathbf{a}$ is compatible with (L, A, B), then $L \subseteq \bigcup_{q \in F} \{p \in L : p \equiv a_q \pmod q\}$ with $a_q \notin B(q)$ for every $q \in F$. By the union bound,
$$|L| \le \sum_{q \in F} |\{p \in L : p \equiv a_q \pmod q\}| \le \sum_{q \in F} \mathrm{best}(q; L, B).$$
Hence $\sum_{q \in F} \mathrm{best}(q; L, B) < |L|$ implies no compatible vector exists.

Finally, each branching step increases $|D|$ by 1, so the tree has depth at most $k = |\mathcal{P}_y|$ and finitely many nodes. By induction from leaves to root using parts (1) and (2), if no leaf reaches $L = \emptyset$, the root admits no compatible covering of $\mathcal{T}_T$, proving $z^*(y) < T$. □

Notice two practical consequences of Lemma 4: (i) because the child sets partition the solution space without overlap, every covering is explored at most once (eliminating permutation symmetry across moduli); and (ii) splitting the tree at a fixed frontier depth d (we use $d = 3$) partitions the root space into independent subproblems whose union is exact provided zero tasks are left unfinished.

Next, we relate $z^*(p_n)$ to Ziller and Morack's paired Jacobsthal function $h_2(n)$ [7, 8] (`[to check]`), where p_n denotes the n-th prime ($p_1 = 2, p_2 = 3, p_3 = 5, \dots$). Under the standard reading matching OEIS `A288815` and `A072753`, $h_2(n)$ is the least positive integer m such that for every integer pair (a, b) with $b - a$ even, the run of m consecutive pairs $(a+i, b+i)$ $(1 \le i \le m)$ contains at least one pair with *both* coordinates coprime to the primorial $p_n\# = Q(p_n)$.

Proposition 5 (Paired-Jacobsthal Upper Bound). *Let $n \ge 3$ so that $p_n \ge 5$. Suppose $h_2(n)$ is even and satisfies $h_2(n) + 1 < p_{n+1}^2$. Then $z^*(p_n) < h_2(n)$.*

Proof. Suppose for contradiction that $z^*(p_n) \ge h_2(n)$. Because $h_2(n)$ is even while $z^*(p_n)$ is an odd prime, we have $z^*(p_n) \ge h_2(n) + 1$. Let $\mathbf{a} = (a_q)_{q \in \mathcal{P}_{p_n}}$ be a maximizing relaxed covering reaching $z^*(p_n)$, and let N be an even integer satisfying $N \equiv 0 \pmod 2$ and $N \equiv a_q \pmod q$ for all $q \in \mathcal{P}_{p_n}$. Set $a = 1$ and $b = 1 - N$, so $b - a = -N$ is even, and consider the $h_2(n)$ consecutive pairs
$$(a + i,\ b + i) = (x,\ x - N), \qquad x \in \{2, 3, \dots, h_2(n) + 1\}.$$
We verify that every $x \in [2, h_2(n) + 1]$ has $\gcd(x, p_n\#) > 1$ or $\gcd(x - N, p_n\#) > 1$:

- If x is even, then $2 \mid x$ and $2 \mid p_n\#$.
- If x is an odd prime with $x \le p_n$, then $x \mid p_n\#$.
- If x is odd and composite, then since $x \le h_2(n) + 1 < p_{n+1}^2$, its least prime factor r satisfies $r < p_{n+1}$, hence $r \le p_n$ and $r \mid \gcd(x, p_n\#)$.
- If x is an odd prime with $p_n < x \le h_2(n) + 1 \le z^*(p_n)$, then by the covering property there exists $q \in \mathcal{P}_{p_n}$ with $x \equiv a_q \equiv N \pmod q$, so $q \mid \gcd(x - N, p_n\#)$.

Thus none of the $h_2(n)$ pairs $(a+i, b+i)$ has both coordinates coprime to $p_n\#$, contradicting the definition of $h_2(n)$. Hence $z^*(p_n) < h_2(n)$. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Explicit Strict Coverings. Table 1 records the explicit residue covering vectors establishing the lower bounds in Theorem 1 and Proposition 2. For every tabulated modulus other than $y = 3$ and $y = 7$, all entries of the strict vector satisfy $1 \le a_q \le q - 1$, so no prime covers itself and $Z(y) \ge z^*(y)$ follows from Proposition 3(1).

Table 1. Explicit maximizing residue covering vectors $(a_q)_{3\le q\le y}$ (listed in increasing odd prime order $q = 3, 5, \dots, y$), verified reach z, strictness, covered odd-prime count $\pi_{\text{odd}}(z)$, and least uncovered prime p^+. All vectors and their CRT lifts $n > z + y$ are verified by `recompute.py`.

y	$k(y)$	z	$\pi_{\text{odd}}(z)$	p^+	Strict?	Residue vector $(a_3, a_5, \dots, a_y)$
3	1	3	1	5	No ($Z = 0$)	(0)
5	2	5	2	7	Yes	(2, 3)
7	3	17	6	19	No ($Z = 11$)	(2, 3, 0) [strict max: (2, 2, 3) → 11]
11	4	29	9	31	Yes	(2, 3, 5, 7)
13	5	41	12	43	Yes	(2, 2, 3, 2, 6)
17	6	61	17	67	Yes	(2, 3, 5, 4, 5, 7)
19	7	101	25	103	Yes	(2, 2, 3, 2, 6, 9, 4)
29	9	179	40	181	Yes	(2, 2, 5, 7, 5, 9, 3, 13, 18)
37	11	269	56	271	Yes	(2, 3, 4, 9, 1, 3, 2, 15, 7, 19, 16)
41	12	317	65	331	No ($a_{37} = 0$)	(2, 3, 3, 6, 8, 7, 10, 1, 10, 19, 0, 38)
41	12	317	65	331	**Yes**	(2, 3, 3, 6, 8, 7, 10, 1, 10, 6, 19, 38)
43	13	409	79	419	**Yes**	(1, 3, 1, 4, 11, 13, 17, 18, 21, 14, 5, 19, 11)
47	14	491	93	499	**Yes**	(2, 2, 5, 2, 4, 3, 14, 7, 2, 8, 3, 29, 30, 40)
53	15	607	110	613	**Yes**	(2, 2, 6, 10, 8, 11, 8, 18, 19, 3, 14, 31, 34, 23, 8)

Figure 2 visualizes the fine combinatorial structure of the $y = 41$ strict witness and compares its per-modulus load profile with the $y = 53$ maximizing covering. Every single modulus in all four headline strict coverings is load-bearing: in `recompute.py`, shifting any single residue $a_q \mapsto (a_q+1) \bmod q$ drops the reach strictly below $z^*(y)$ in 12/12 coordinates at $y = 41$, 13/13 at $y = 43$, 14/14 at $y = 47$, and 15/15 at $y = 53$. At $y = 41$, 37 of the 65 covered odd primes are covered uniquely by a single modulus and 28 are multi-covered; the small moduli carry most of the weight ($q = 3$ covers 34 primes, 13 uniquely; $q = 5$ covers 18, 4 uniquely; $q = 7$ covers 13, 3 uniquely), while the largest modulus $q = 41$ ($a_{41} = 38$) covers a single prime (79) uniquely. At $y = 53$, 68 of the 110 covered odd primes are uniquely covered and 42 are multi-covered ($q = 3$ covers 57 primes, 25 uniquely; $q = 5$ covers 29, 9 uniquely; and even $q = 53$ covers 3 primes, 2 uniquely).

Exhaustive Infeasibility Proofs and Independent Controls. To prove that no residue covering reaches p^+, we combine three layers of verification backed by `recompute.py` and `replay_exhaustive.py`:

1. *Unpruned brute force* ($y \le 17$). Enumerating every residue vector directly from the definitions (3, 15, 105, 1,155, 15,015, and 255,255 vectors for $y = 3, 5, 7, 11, 13, 17$) settles both $z^*(y)$ and $Z(y)$ unconditionally with zero pruning logic, confirming $Z(7) = 11 < 17 = z^*(7)$ and agreement at every modulus $y \le 17$.
2. *Dual-code independence at $y = 41$ and planted-fault controls.* Two independent exact solvers written from the definitions and sharing no code proved infeasibility of covering $\mathcal{P}_{331}$ at $y = 41$: an initial unpartitioned depth-first solver required 18,449,209 nodes (323.0 CPU seconds), whereas the partitioned branch-and-bound solver of Lemma 4 proved infeasibility in 47,640 nodes (1.3 CPU seconds in the original run; 0.8 seconds single-threaded in `recompute.py`). As negative controls, two deliberately corrupted variants of the branch-and-bound solver—an over-eager pruning rule cutting nodes where room = need (when need > 1) and an incomplete brancher dropping the last candidate modulus—each fail to find known coverings across all 8/8 tested moduli $y \in \{5, 7, 11, 13, 17, 19, 29, 37\}$.
3. *Partitioned infeasibility replay* ($y \le 53$). Table 2 reports the exact search statistics reproduced deterministically by `replay_exhaustive.py` for $y \in \{41, 43, 47, 53\}$ (frontier depth $d = 3$, zero unfinished tasks; total wall time under 9 seconds on 16 cores). Before these exact runs, heuristic searches had reached only 347 and 359 at $y = 43, 401$ and 419 at $y = 47$, and 599 at $y = 53$ (where the unpartitioned solver exhausted its 1,200-second, 56,226,000-node budget at target 601).

Table 2. Exact branch-and-bound infeasibility receipts for $y \in \{19, 29, 37, 41, 43, 47, 53\}$ (depth-3 partitioned tasks for $y \ge 41$ replayed in `replay_exhaustive.py`; 0 unfinished tasks in every run).

y	$z^*(y)$	Target p^+	Tasks	Nodes	Unfin.
19	101	103	—	222	0
29	179	181	—	780	0
37	269	271	—	13,031	0
41	317	331	1,310	47,640	0
43	409	419	1,705	90,352	0
47	491	499	2,172	288,340	0
53	607	613	2,717	1,066,513	0

Comparison with Jacobsthal Relatives and Empirical Records. Table 3 and Figure 3 place all 12 exact values of $z^*(y)$ alongside two distinct classical relatives and empirical least-Goldbach-prime records:

- *Jacobsthal's function* $h(Q(y))$ (OEIS `A048670`, computed for $n < 50$ by Hagedorn [3] `[to check]`), which covers *every* integer in an interval using one class per prime up to y. Because $z^*(y)$ needs to cover only odd primes rather than composites, $z^*(y)$ exceeds $h(Q(y))$ substantially once $y \ge 7$ (reaching 607 versus 106 at $y = 53$).
- *Ziller and Morack's paired Jacobsthal function* $h_2(n)$ (OEIS `A288815`, tabulated through $p_{21} = 73$ in [8] `[to check]`), which uses *two* classes per odd prime to cover all integers in a paired window. The hypothesis $h_2(n) + 1 < p_{n+1}^2$ of Proposition 5 holds at every row of Table 3 (for instance, $1{,}422 + 1 = 1{,}423 < 59^2 = 3{,}481$ at $y = 53$), giving an explicit upper bound $z^*(p_n) < h_2(n)$ that sits roughly 2.3× to 3.6× above $z^*(p_n)$ for $y \in [13, 53]$.
- *Empirical records for* $p(n)$ (OEIS `A025018/A025019`; Oliveira e Silva, Herzog, and Pardi [5] `[to check]`). As witnesses

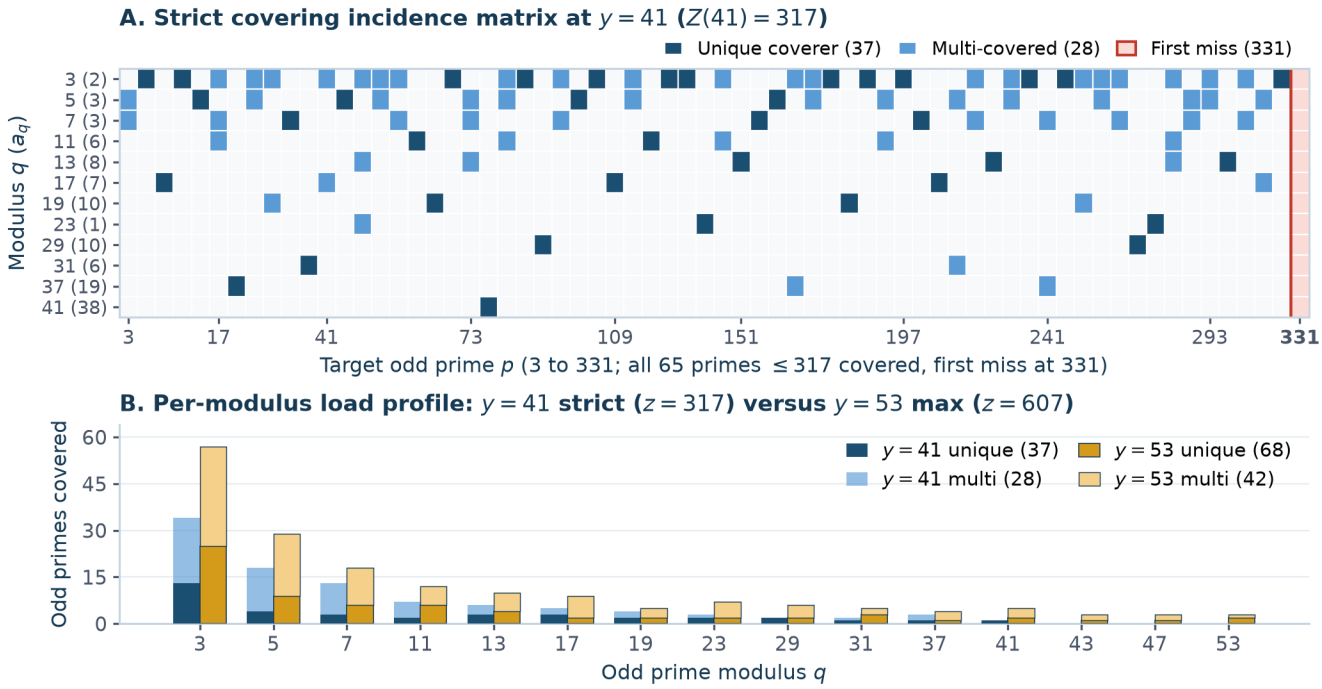


Figure 2. Prime-covering structure for $y = 41$ (strict witness) and $y = 53$ (maximizing covering). Panel A: incidence matrix of the 12 odd prime moduli $q \le 41$ across all 66 odd primes $3 \le p \le 331$, showing 37 uniquely covered primes (dark navy), 28 multi-covered primes (blue), zero self-covers, and the least uncovered prime $p^+ = 331$ (coral border). Panel B: per-modulus total and unique coverage counts for $y = 41$ (65 odd primes ≤ 317) alongside $y = 53$ (110 odd primes ≤ 607, 68 unique and 42 multi-covered).

Table 3. Complete 12-value summary of exact $z^*(y)$ and $Z(y)$ for $y \le 53$, normalized ratios, Chebyshev log-primorial $\theta(y) = \ln Q(y)$, distinct relatives $h(Q(y))$ (`A048670` [3] `[to check]`) and $h_2(n)$ (`A288815` [8] `[to check]`), and earliest empirical record $(n, p(n))$ with $p(n) > z^*(y)$ (`A025018/A025019` [5] `[to check]`). All arithmetic verified by `recompute.py`.

y	$k(y)$	$z^*(y)$	$Z(y)$	p^+	$\frac{z^*}{y \ln y}$	$\frac{z^*}{y \ln^2 y}$	$\theta(y)$	$h(Q(y))$	$h_2(n)$	Earliest record $p(n) > z^*$
5	2	5	5	7	0.621	0.386	3.40	6	18	$n = 30$ ($p = 7$)
7	3	17	11	19	1.248	0.641	5.35	10	30	$n = 98$ ($p = 19$)
11	4	29	29	31	1.099	0.459	7.75	14	66	$n = 308$ ($p = 31$)
13	5	41	41	43	1.230	0.479	10.31	22	150	$n = 556$ ($p = 47$)
17	6	61	61	67	1.266	0.447	13.14	26	192	$n = 992$ ($p = 73$)
19	7	101	101	103	1.805	0.613	16.09	34	258	$n = 2{,}642$ ($p = 103$)
29	9	179	179	181	1.833	0.544	22.59	46	450	$n = 43{,}532$ ($p = 211$)
37	11	269	269	271	2.013	0.558	29.64	66	708	$n = 63{,}274$ ($p = 293$)
41	12	317	317	331	2.082	0.561	33.35	74	894	$n = 128{,}168$ ($p = 331$)
43	13	409	409	419	2.529	0.672	37.11	90	1,044	$n = 503{,}222$ ($p = 523$)
47	14	491	491	499	2.713	0.705	40.96	100	1,284	$n = 503{,}222$ ($p = 523$)
53	15	607	607	613	2.885	0.727	44.93	106	1,422	$n = 3{,}526{,}958$ ($p = 727$)

for large $p(n)$, finite-y coverings sit far above the smallest empirical records: for example, $p(n) > 607$ already occurs at $n = 3{,}526{,}958$ (where $p(n) = 727$), whereas the CRT lift of the $y = 53$ covering has size comparable to the primorial $Q(53) \approx 3.26 \times 10^{19}$ ($\theta(53) = \ln Q(53) \approx 44.93$).

Over the finite range $y \le 53$, the ratio $z^*(y)/(y \ln y)$ rises from 0.621 at $y = 5$ to 2.885 at $y = 53$, while $z^*(y)/(y \ln^2 y)$ stays in $[0.386, 0.727]$ (Figure 3, Panel B). Twelve finite values cannot determine an asymptotic rate.

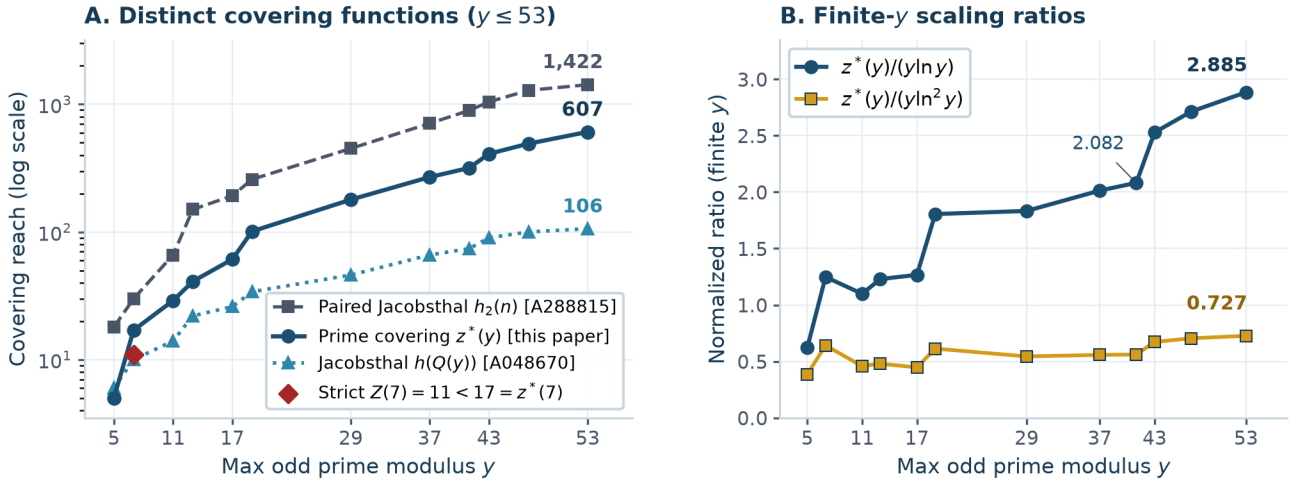


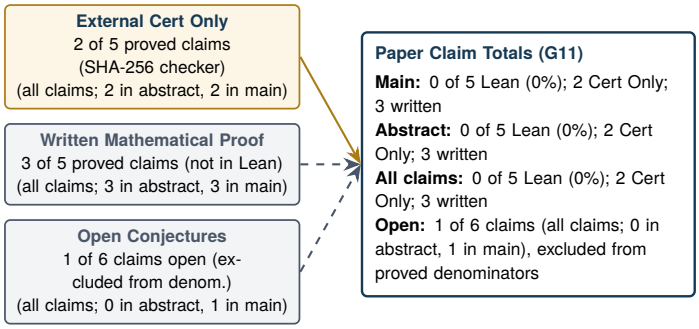
Figure 3. Comparison of $z^*(y)$ against distinct covering functions and finite-y scaling ratios. Panel A: log-scale comparison of Jacobsthal's $h(Q(y))$ (`A048670` [3] `[to check]`), exact odd-prime covering $z^*(y)$ (with strict separation $Z(7) = 11$ marked by a red diamond), and Ziller–Morack's paired Jacobsthal $h_2(n)$ (`A288815` [8] `[to check]`), illustrating Proposition 5 ($z^*(p_n) < h_2(n)$). Panel B: finite-y ratios $z^*(y)/(y \ln y)$ (reaching 2.885 at $y = 53$) and $z^*(y)/(y \ln^2 y)$ (reaching 0.727 at $y = 53$); these 12 finite values do not determine asymptotic behavior.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 2 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 4 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Exact values z*(41) = Z(41) = 317, z*(43) = Z(43) = 409, z*(47) = Z(47) = 491, and z*(53) = Z(53) = 607	Cert Only	recompute.py (d2e44f3f17e0) [archived; replay pending]; replay_exhaustive.py (ba 4112957c16) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Lower bounds certified by explicit strict residue vectors in recompute.py; upper bounds certified by depth-3 partitioned B&B infeasibility replay in replay_exhaustive.py (47,640 to 1,066,513 nodes, 0 unfinished tasks) plus Lemma 4 soundness and dual-code check at y = 41.
Prop. 2: Exact z*(y) = 5, 17, 29, 41, 61, 101, 179, 269 for y in {5, 7, 11, 13, 17, 19, 29, 37}, with Z(7) = 11 < 17 = z*(7) and Z(y) = z*(y) otherwise	Cert Only	recompute.py (d2e44f3f17e0) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Certified by unpruned brute force (y <= 17, up to 255,255 vectors) and exact B&B replay (y = 19, 29, 37) in recompute.py.
Prop. 3: Monotonicity Z(y) <= z*(y) and Chinese Remainder Theorem lifting to even n > z + y making n - p composite for every odd prime p <= z	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via the Chinese Remainder Theorem (Section 3); finite-y instances also checked in recompute.py. No Lean 4 formalization.
Lem. 4: Soundness of smallest-uncovered partition branching and capacity-sum pruning for proving z*(y) < T	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof via disjoint solution-space partitioning and the union bound (Section 3). No Lean 4 formalization.
Prop. 5: Upper bound z*(p_n) < h_2(n) whenever paired Jacobsthal h_2(n) is even and h_2(n) + 1 < p_{n+1}^2	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof under the both-coordinates-coprime reading of h_2(n) matching OEIS A288815/A072753 (Section 3; full PDFs of Ziller and Morack 2017 flagged [to check]). No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Conj. 6: Exact values for y >= 59 (and omitted y in {23, 31}), strict equality Z(y) = z*(y) for all y >= 11, and asymptotic growth of z*(y)/(y ln y)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open questions (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 7d712e7b539b | Fidelity SHA-256: 302f54f2a086.

6. PRIOR WORK AND OPEN QUESTIONS

The covering mechanism behind Proposition 3 is classical in the prime-gaps and progressions literature: Pomerance [6] applied Jacobsthal-type coverings to the least prime in an arithmetic progression, Granville and Pomerance [2] refined bounds for progressions, Maier and Pomerance [4] studied large prime gaps and conjectured $x(\log x)^{2+o(1)}$ scaling for interval coverings, and Ford, Green, Konyagin, Maynard, and Tao [1] established major asymptotic improvements on Erdős–Rankin coverings for large gaps between consecutive primes. On the exact computational side, the nearest tabulated functions are all distinct: Hagedorn [3] (`[to check]`) computed Jacobsthal's $h(n)$ for primorials with $n < 50$ (OEIS `A048670`), Ziller and Morack [7, 8] (`[to check]`) defined and tabulated the paired Jacobsthal function $h_2(n)$ through $p_{21} = 73$ (OEIS `A288815`, linked to Morack's two-stage sieve `A072753` via $h_2(n) = 6 \cdot \mathrm{A072753}(n) + 6$), and Oliveira e Silva, Herzog, and Pardi [5] (`[to check]`) verified the even Goldbach conjecture through $4 \cdot 10^{18}$ alongside minimal-prime records (OEIS `A025018`, `A025019`).

Prior-work searches logged. Thirty-three plain requests (OEIS term searches on consecutive runs $5, 17, 29, 41, 61, 101$ and $269, 317, 409, 491, 607$ and their least-uncovered counterparts $7, 19, 31, 43, 67, 103$ and $271, 331, 419, 499, 613$, OEIS word searches, six arXiv API queries, and six web searches under Jacobsthal-type functions, Erdős–Rankin coverings, admissible sets, and large values of the least Goldbach prime) returned zero matching tabulations of $z^*(y)$ or $Z(y)$. A subsequent 5-query Google Scholar sweep on 2026-11-22 (50 records inspected) likewise found no prior work stating the exact values $z^*(41) = 317$, $z^*(43) = 409$, $z^*(47) = 491$, $z^*(53) = 607$, or $Z(41) = 317$.

Unread and unsearched items (`[to check]`). The following sources remain unread or unsearched and are flagged `[to check]` in `references-checked.md`: (1) MathSciNet and zbMATH full-text searches; (2) full text and ancillary data file of Hagedorn [3]; (3) full text of Oliveira e Silva, Herzog, and Pardi [5] and full PDFs of Ziller and Morack [7, 8] (inspected via `ar5iv` extracts and OEIS); (4) two 2026 `preprints.org` manuscripts surfaced in the 2026-11-22 Scholar sweep whose landing pages returned HTTP 403 (B. Bahboui, `202610.2486`, and T. T. K. Nguyen, `202608.1299`); (5) the gated ProQuest dissertation by Stumpf (2023); and (6) contiguous OEIS term searches once $z^*(y)$ is computed at the omitted primes $y \in \{3, 23, 31\}$, where the logged OEIS runs split.

Conjecture 6 (Open Exact Values and Asymptotic Scaling of $z^*(y)$). *The following questions remain unsettled:*

1. *Determine the exact values of $z^*(y)$ and $Z(y)$ for prime moduli $y \geq 59$ (as well as the omitted intermediate primes $y \in \{23, 31\}$).*
2. *Decide whether $Z(y) = z^*(y)$ holds for every prime $y \geq 11$ (so that $y \in \{3, 7\}$ are the only prime moduli with strict separation $Z(y) < z^*(y)$).*
3. *Determine whether $z^*(y)/(y \ln y) \to \infty$ as $y \to \infty$, and whether $z^*(y) = y(\ln y)^{2+o(1)}$ (analogous to the Maier–Pomerance heuristic [4] for interval coverings).*

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure generation scripts, and standalone exact-arithmetic verifiers live in `papers/goldbach-residue-coverings/draft1/`. Running `python3 papers/goldbach-residue-coverings/draft1/recompute.py` completes in under 4 seconds and deterministically verifies all explicit covering witnesses ($y \leq 53$, including the strict witness at $y = 41$ and strictness at $43, 47, 53$), exact CRT lifts, single-class and unique-cover negative-mutation checks, unpruned brute-force determination of $z^*(y)$ and $Z(y)$ for $y \leq 17$, branch-and-bound infeasibility through $y = 43$, planted solver-fault controls, Tables 1–3, and the Section 1 Python snippet. Running `python3 papers/goldbach-residue-coverings/draft1/replay_exhaustive.py` replays the full depth-3 partitioned branch-and-bound infeasibility proofs for $y \in \{41, 43, 47, 53\}$ at targets $331, 419, 499, 613$ in under 9 wall seconds on 16 cores, reproducing the exact node counts (47,640, 90,352, 288,340, and 1,066,513) with zero unfinished tasks. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes discovered the exact covering values and strict witness, wrote and independently rechecked the exhaustive search and verification programs, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 24, 2026

REFERENCES

(1) Ford, K.; Green, B.; Konyagin, S.; Maynard, J.; Tao, T. Long gaps between primes. *Journal of the American Mathematical Society* **2018**, *31* (1), 65–105. DOI: 10.1090/jams/876. arXiv:1412.5029.

(2) Granville, A.; Pomerance, C. On the least prime in certain arithmetic progressions. *Journal of the London Mathematical Society* **1990**, *s2-41* (2), 193–200. DOI: 10.1112/jlms/s2-41.2.193.

(3) Hagedorn, T. R. Computation of Jacobsthal's function $h(n)$ for $n < 50$. *Mathematics of Computation* **2009**, *78* (266), 1073–1087. DOI: 10.1090/S0025-5718-08-02166-2.

(4) Maier, H.; Pomerance, C. Unusually large gaps between consecutive primes. *Transactions of the American Mathematical Society* **1990**, *322* (1), 201–237. DOI: 10.2307/2001529.

(5) Oliveira e Silva, T.; Herzog, S.; Pardi, S. Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$. *Mathematics of Computation* **2014**, *83* (288), 2033–2060. DOI: 10.1090/S0025-5718-2013-02787-1.

(6) Pomerance, C. A note on the least prime in an arithmetic progression. *Journal of Number Theory* **1980**, *12* (2), 218–223. DOI: 10.1016/0022-314X(80)90056-6.

(7) Ziller, M.; Morack, J. F. Divisibility in paired progressions, Goldbach's conjecture, and the infinitude of prime pairs. Preprint, **2017**, arXiv:1706.00317.

(8) Ziller, M.; Morack, J. F. A short note on the computation of the generalised Jacobsthal function for paired progressions. Preprint, **2017**, arXiv:1706.03668.