MATHDISCOVERIES
Mathematical Preprint

Working Draft

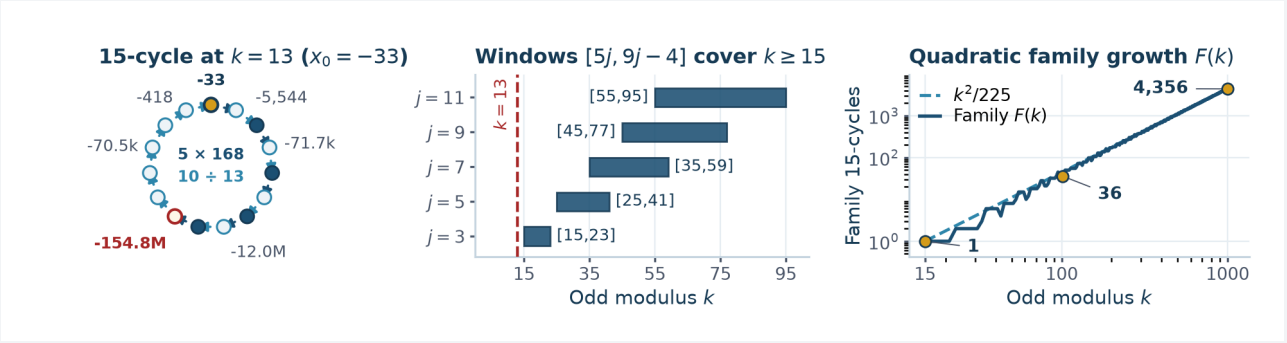
The 15-Step Loops of Hajari's Maps: A Counterexample at $k = 13$ and a Quadratic Family for Every Odd $k \geq 15$

Tyler B. Hughes*,†

†Independent Researcher, United States

Ⓢ *Supporting Information:* `recompute.py` *(standalone exact-integer Python verifier for listed cycles, witnesses, family cycles, and sweeps)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For odd $k \geq 3$, Hajari's map $t_k : \mathbb{Z} \to \mathbb{Z}$ sends odd x to $(k^2-1)x$ and even x to $\lfloor x/k \rfloor$. Keith Matthews's web page (16 December 2020) states that "it seems certain" every negative start reaches one of $(k^2-1)/4$ listed 3-cycles. We prove this fails for every odd $k \geq 13$: (1) all $(k^2-1)/4$ listed 3-cycles are valid and disjoint; (2) at $k = 13$, -33 generates an unlisted primitive 15-cycle peaking at $-154{,}847{,}784$; (3) for odd k, odd $j \geq 3$, and even $c \geq 2$ with $\max(5j, 5c+3) \leq k \leq 9j-4$, $-(jk-c)$ is the unique least-magnitude element of a primitive 15-cycle, yielding $F(k) \geq 1$ disjoint unlisted 15-cycles for every odd $k \geq 15$ with $F(k) = k^2/225 + O(k)$; and (4) every negative cycle with $b \geq 2a$ even steps per a odd steps visits $|y| < 2(k-1)k^2$. Standalone replay verifies all 735,691 family cycles ($k \leq 1{,}001$), 44,200 listed 3-cycles ($k \leq 101$), and sweeps ($k \leq 25$); extended sweeps ($k \leq 101$) record $E(101) = 100$ and $b < 2a$ length floors ($k \leq 11$). Length divisibility by 3, $E(k) = O(k^2)$, and completeness for $k \leq 11$ remain unsettled; no Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 6 in Lean Kernel (0%) · 2 Cert Only · 4 written · 1 open (not counted)
Abstract claims: 0 of 6 in Lean Kernel (0%) · 2 Cert Only · 4 written · 1 open (not counted)
Full Trust Boundary: Section 5.

MSC 2020: 11B83, 11B37, 11Y55 **Keywords:** Hajari's map, generalized Collatz mappings, floor-division dynamics, negative periodic orbits, 15-cycles, Keith Matthews

1. INTRODUCTION AND MAIN RESULTS

Fix an odd integer $k \geq 3$. Following a question posed by Pruthviraj Hajari on MathOverflow (Question 378293, 16 December 2020) and the companion web pages of Keith R. Matthews, consider the two-branch integer map $t_k : \mathbb{Z} \to \mathbb{Z}$ defined by

$$t_k(x) := \begin{cases} (k^2-1)\,x & \text{if } x \text{ is odd,} \\ \lfloor x/k \rfloor & \text{if } x \text{ is even,} \end{cases} \tag{1}$$

where $\lfloor u \rfloor$ denotes the ordinary floor function rounding toward $-\infty$ (so for negative x, $q = \lfloor x/k \rfloor$ is the unique integer satisfying $0 \leq x - kq \leq k-1$; for instance, $\lfloor -5{,}544/13 \rfloor = -427$, not -426). Hajari asked whether every positive integer $x > 0$ eventually reaches 0 under t_k.

On his web page "Conjecture of Pruthviraj Hajari" (located at `numbertheory.org/php/hajari3.html`, last modified 16 December 2020), Matthews extended t_k to negative integers and wrote verbatim:

> It seems certain that if $x < 0$, then $t^n(x)$ will eventually reach one of the $(k^2-1)/4$ cycles with starting points $-(ki+2s)$, where $1 \leq i \leq k-2$, t odd, and $0 \leq s \leq (k-1)/2$; each cycle is of length 3: $-(ki+2s) \to -(ki+2s)(k^2-1) \to -k^2i - 2sk + i \to -(ki+2s)$.

(Here "t odd" is read as "i odd", the unique reading under which the counts of i and s multiply to $(k^2-1)/4$.) Two points of framing deserve emphasis up front: Matthews's statement appeared on a research web page about generalized $3x+1$ mappings rather than in a peer-reviewed journal paper and was phrased cautiously as "it seems certain"; moreover, negative periodic orbits do not touch Hajari's own question concerning positive starts.

Nevertheless, Matthews's list is put forward as a complete inventory of negative attractors for every odd $k \geq 3$. In this paper, we confirm that all $(k^2-1)/4$ listed 3-cycles are genuine and pairwise disjoint, and we prove unconditionally that the completeness statement fails for every odd $k \geq 13$: at $k = 13$ via a hand-checkable 15-cycle through -33, and for every odd $k \geq 15$ via an explicit two-parameter poly-

nomial family yielding $\sim k^2/225$ pairwise disjoint unlisted 15-cycles. We also prove an elementary magnitude bound showing that every negative cycle with at least two divisions per multiplication visits an even integer of size strictly below $2(k-1)k^2$, certifying that exhaustive negative sweeps find all such cycles. We state the main results and unsettled questions upfront.

Proposition 1 (Matthews's Listed 3-Cycles and Non-Membership Criterion). *For every odd integer $k \geq 3$, every odd integer i with $1 \leq i \leq k-2$, and every integer s with $0 \leq s \leq (k-1)/2$, the negative odd integer $x_0 = -(ki+2s)$ lies on a primitive 3-cycle of t_k:*

$$-(ki+2s) \longmapsto -(ki+2s)(k^2-1) \longmapsto -k^2 i - 2sk + i \longmapsto -(ki+2s).$$

These $(k^2-1)/4$ cycles are pairwise disjoint. Moreover, a negative odd integer $-m$ $(m \geq 1)$ lies on a listed 3-cycle if and only if its Euclidean division $m = ki + r$ $(0 \leq r \leq k-1)$ has i odd, $1 \leq i \leq k-2$, and r even.

Proposition 2 (Counterexample at $k = 13$). *For $k = 13$ ($k^2-1 = 168$), the integer $x_0 = -33$ generates a primitive 15-cycle of t_{13} with parity word* OEOEOEOEOEEEEEE *(5 multiplications by 168 and 10 floor divisions by 13), peaking at $x_9 = -154{,}847{,}784$ and returning to $x_{15} = -33$. This cycle is distinct from all 42 listed 3-cycles at $k = 13$. Hence Matthews's completeness claim fails at $k = 13$.*

Theorem 3 (Quadratic Family of Primitive 15-Cycles for Every Odd $k \geq 15$). *Let $k, j, c \in \mathbb{Z}$ with k odd, $j \geq 3$ odd, $c \geq 2$ even, and*

$$\max(5j,\, 5c+3) \leq k \leq 9j - 4. \tag{2}$$

Set $x_0 := -(jk - c)$. Then:

1. (Exact 15-step return and converse carry equivalence.) *The 15-step t_k-orbit of x_0 follows the parity word* OEOEOEOEOEEEEEE *(5 odd steps, 10 even steps), with each x_n given by an explicit polynomial in k linear in (j, c) and the ten floor-division carries $(-1,-1,0,0,-1,0,0,-2,-1,0)$, returning to $x_{15} = x_0$. Conversely, for any odd $k \geq 1$, odd $j \in \mathbb{Z}$, and even $c \in \mathbb{Z}$, all ten carry inequalities hold simultaneously if and only if $j \geq 3$, $c \geq 2$, and (2) holds.*
2. (Primitivity and pairwise disjointness.) *The orbit of x_0 has primitive period 15 (hence is omitted from Matthews's list of 3-cycles), and x_0 is the unique element of strictly smallest absolute value on its cycle. Distinct pairs (j, c) therefore generate pairwise disjoint 15-cycles.*
3. (Coverage and quadratic growth.) *For every odd $k \geq 15$, the exact number of valid pairs (j, c) is*

$$F(k) = \left(\left\lfloor \frac{k-5}{10} \right\rfloor - \left\lceil \frac{k-5}{18} \right\rceil + 1 \right) \left\lfloor \frac{k-3}{10} \right\rfloor \geq 1, \tag{3}$$

satisfying $F(k) = k^2/225 + O(k)$ as odd $k \to \infty$. Consequently, Matthews's list is incomplete for every odd $k \geq 13$, and the number of unlisted negative cycles grows at least quadratically in k.

To interpret exhaustive computer sweeps rigorously, we separate cycles according to the ratio of even steps b to odd steps a.

Proposition 4 (Magnitude Bound for Balanced and Contracting Negative Cycles). *Let $k \geq 3$ be odd, and let $C \subset \mathbb{Z}_{<0}$ be any negative cycle of t_k containing $a \geq 1$ odd elements and b even elements with $b \geq 2a$. Then C contains an even element y with $|y| < 2(k-1)k^2$. In particular, an exhaustive sweep of $[-2(k-1)k^2, -1]$ detects every negative cycle of t_k satisfying $b \geq 2a$.*

On the computational side, we separate finite checks replayed directly by our standalone script `recompute.py` from extended sweeps resting on committed receipts.

Proposition 5 (Standalone Exact Replay Through $k = 1{,}001$ and Sweeps Through $k = 25$). *Standalone exact-integer Python arithmetic (`recompute.py`, zero floats in any predicate) certifies:*

1. *The 15-step cycles of -33 at $k = 13$ and -43 at $k = 15$, the symbolic polynomial derivation of $x_0, \ldots, x_{15}$, and exact equivalence between the ten carry windows and (2) across 473,536 triples (odd $1 \leq k \leq 301$, odd $-11 \leq j \leq 99$, even $-12 \leq c \leq 98$: 0 disagreements, 19,532 family members).*
2. *All 735,691 family members across odd $15 \leq k \leq 1{,}001$ (cumulative counts 5,727 through $k = 201$ and 46,627 through $k = 401$; individual values $F(15) = 1$, $F(45) = 8$, $F(101) = 36$, $F(201) = 171$, $F(401) = 702$, $F(1{,}001) = 4{,}356$) form pairwise disjoint primitive 15-cycles with unique least-magnitude element $-(jk - c)$, 0 failures, and exact agreement with (3).*
3. *All 44,200 Matthews listed 3-cycles across the 50 odd moduli $3 \leq k \leq 101$ (2 at $k = 3$, 42 at $k = 13$, 2,550 at $k = 101$; 0 failures).*
4. *Exhaustive negative sweeps over $[-2(k-1)k^2, -1]$ for all odd $3 \leq k \leq 25$: combined with Proposition 4, for $k \in \{3, 5, 7, 9, 11\}$ there are 0 unlisted negative cycles with $b \geq 2a$; at $k = 13$ the sole unlisted $b \geq 2a$ cycle is -33 (length 15); at $k = 15$ the sole unlisted $b \geq 2a$ cycle is -43 (length 15); and every extra cycle found through $k = 25$ satisfies $b = 2a$.*

Proposition 6 (Extended Multi-Program Sweeps Through $k = 101$ and $b < 2a$ Length Floors). *Committed exact-arithmetic sweep receipts (`growth.out` and the audit file `verdict.md`, independently reproduced for odd $3 \leq k \leq 31$ in blind re-review) establish across all odd $3 \leq k \leq 101$ swept up to $\max(2(k-1)k^2, 10^6)$ (reaching 2,040,200 at $k = 101$):*

1. *Every unlisted negative cycle with $b \geq 2a$ satisfies $b = 2a$ (lengths 15 through 135 at $k = 101$). Letting $E(k)$ denote the count of unlisted $b \geq 2a$ negative cycles, $E(k) = 0$ for $k \leq 11$, $E(13) = 1$, $E(15) = 1$, $E(101) = 100$, $E(k) \geq F(k)$ throughout, and $0.0064 \leq E(k)/k^2 \leq 0.0113$ across $33 \leq k \leq 101$ (minimum at $k = 61$, maximum at $k = 95$).*
2. *For $k \in \{3, 5, 7, 9, 11\}$, any hypothetical unlisted negative cycle must satisfy $b < 2a$ and have length at least 1,565, 821, 1,411, 530, and 866, respectively.*
3. *No positive cycle and no unresolved orbit appeared among positive starts $1 \leq x \leq 10^5$ for odd $3 \leq k \leq 101$.*

Conjecture 7 (Unsettled Questions on Cycle Multiples, Upper Growth, and Small Moduli). *The following statements remain unsettled:*

1. (Length divisibility by 3.) *Every nonzero cycle of t_k (odd*

$k \geq 3$) *satisfies* $b = 2a$*, hence has length divisible by* 3 (*unsettled for* $b < 2a$*; excluding positive cycles also bears on Hajari's positive-start question*).

2. (Upper growth and finiteness of $E(k)$.) *The total number of unlisted negative cycles of* t_k *satisfies* $E(k) = \Theta(k^2)$ *as odd* $k \to \infty$ (*Theorem 3 gives the lower half* $E(k) \geq k^2/225 + O(k)$*; both* $E(k) = O(k^2)$ *and finiteness of* $E(k)$ *for fixed odd* $k \geq 13$ *are unsettled*).
3. (Matthews's completeness claim restricted to $k \leq 11$.) *For* $k \in \{3, 5, 7, 9, 11\}$*, Matthews's* $(k^2 - 1)/4$ *listed 3-cycles exhaust all negative cycles of* t_k (*proved above for* $b \geq 2a$*; unsettled for* $b < 2a$*; this is Matthews's own statement restricted to* $k \leq 11$*, not an independent conjecture of this paper*).

Figure 1 summarizes the discovery and verification structure across written proofs, standalone and extended certificates, and the unsettled conjecture boundary.

Instant Witness Check: The 15-Cycles at $k = 13$ (-33) and $k = 15$ (-43)

```
def t(x, k): return (k * k - 1) * x if x % 2 else x // k
def loop15(x0, k):
    s = [x0]
    for _ in range(15): s.append(t(s[-1], k))
    return s
c13 = loop15(-33, 13)
c15 = loop15(-43, 15)
assert c13[-1] == -33 and len(set(c13[:-1])) == 15
assert c15[-1] == -43 and len(set(c15[:-1])) == 15
```

Roadmap. Section 2 fixes floor-division conventions and exhibits the $k = 3$ baseline and the full hand-checkable $k = 13$ and $k = 15$ 15-cycles (Table 1, Figure 2). Section 3 gives complete proofs of Propositions 1, 2, 4, and Theorem 3, including the symbolic polynomial cascade (Table 2), strict minimum-magnitude uniqueness, and exact counting formula (3). Section 4 reports standalone verification through $k = 1{,}001$, extended sweeps through $k = 101$ (Table 3, Figure 3), and the heuristic balance behind 15-cycles. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 documents logged prior-work searches, unread caveats (`[to check]`), and open directions, and Section 7 provides reproducibility and AI disclosure.

2. DEFINITIONS, FLOOR CONVENTIONS, AND WORKED EXAMPLES

Throughout the paper, $k \geq 3$ is an odd integer, so $k^2 - 1$ is even (in fact divisible by 8). For any integer x and positive integer k, the floor quotient $q = \lfloor x/k \rfloor$ is characterized by the Euclidean division identity

$$x = kq + r, \qquad 0 \leq r \leq k - 1. \tag{4}$$

For negative x, rounding toward $-\infty$ differs from truncation toward zero whenever $k \nmid x$: for instance, $\lfloor -7/2 \rfloor = -4$ (since $-7 = 2(-4) + 1$) and $\lfloor -5{,}544/13 \rfloor = -427$ (since $-5{,}544 = 13(-427) + 7$). In Python, integer division `x // k` computes exact floor division for every $x \in \mathbb{Z}$ and $k > 0$.

An *orbit* of t_k is a sequence $x_0, x_1, x_2, \ldots$ with $x_{n+1} = t_k(x_n)$. A *cycle* (or *loop*) of *primitive length* (or *least period*) $L \geq 1$ is an orbit satisfying $x_L = x_0$ with $x_0, x_1, \ldots, x_{L-1}$ pairwise distinct. Its *parity word* is the length-L string over $\{O, E\}$ recording whether x_n is odd (O, multiplication by $k^2 - 1$) or even (E, floor division by k) for $0 \leq n \leq L - 1$.

Baseline at $k = 3$. When $k = 3$, $t_3(x) = 8x$ for odd x and $\lfloor x/3 \rfloor$ for even x. A positive start such as 7 descends rapidly to the fixed point 0:

$$7 \to 56 \to 18 \to 6 \to 2 \to 0 \to 0.$$

For negative starts, Matthews's $k = 3$ page (`php/hajari.html`, last modified 5 June 2026) records the $(3^2 - 1)/4 = 2$ listed 3-cycles corresponding to $(i, s) \in \{(1, 0), (1, 1)\}$:

$$-3 \to -24 \to -8 \to -3, \qquad -5 \to -40 \to -14 \to -5.$$

Each listed cycle has parity word OEE: one multiplication followed by two divisions.

The $k = 13$ counterexample by hand. At $k = 13$, $k^2 - 1 = 168$. Table 1 records every step of the orbit from $x_0 = -33$ together with the exact Euclidean remainder $r = x_n - 13x_{n+1} \in \{0, \ldots, 12\}$ on each division step. After 15 steps (5 multiplications by 168 and 10 floor divisions by 13), the orbit returns to -33.

Table 1. Complete 15-step orbit of $x_0 = -33$ under t_{13} ($k^2 - 1 = 168$). Every division step $x_n \mapsto x_{n+1} = \lfloor x_n/13 \rfloor$ is certified by $0 \leq x_n - 13x_{n+1} \leq 12$.

Step	x_n	Move	x_{n+1}	r
1	-33	$\times 168$	$-5{,}544$	—
2	$-5{,}544$	$\lfloor \cdot/13 \rfloor$	-427	7
3	-427	$\times 168$	$-71{,}736$	—
4	$-71{,}736$	$\lfloor \cdot/13 \rfloor$	$-5{,}519$	11
5	$-5{,}519$	$\times 168$	$-927{,}192$	—
6	$-927{,}192$	$\lfloor \cdot/13 \rfloor$	$-71{,}323$	7
7	$-71{,}323$	$\times 168$	$-11{,}982{,}264$	—
8	$-11{,}982{,}264$	$\lfloor \cdot/13 \rfloor$	$-921{,}713$	5
9	$-921{,}713$	$\times 168$	$-154{,}847{,}784$	—
10	$-154{,}847{,}784$	$\lfloor \cdot/13 \rfloor$	$-11{,}911{,}368$	0
11	$-11{,}911{,}368$	$\lfloor \cdot/13 \rfloor$	$-916{,}260$	12
12	$-916{,}260$	$\lfloor \cdot/13 \rfloor$	$-70{,}482$	6
13	$-70{,}482$	$\lfloor \cdot/13 \rfloor$	$-5{,}422$	4
14	$-5{,}422$	$\lfloor \cdot/13 \rfloor$	-418	12
15	-418	$\lfloor \cdot/13 \rfloor$	-33	11

Figure 2 (Panel A) contrasts the staircase profile of the -33 loop with five laps of the listed 3-cycle $-25 \to -4{,}200 \to -324 \to -25$ at $k = 13$. Note that $-33 = -(3 \cdot 13 - 6)$ has the same parity word OEOEOEOEOEEEEEE as the family of Theorem 3, yet $(j, c, k) = (3, 6, 13)$ violates (2) (since $\max(5j, 5c + 3) = 33 > 13$); its carries differ from the family's carry vector, making $k = 13$ a sporadic precursor just below the family's onset at $k = 15$.

The smallest family member at $k = 15$. At $k = 15$, taking $j = 3$ and $c = 2$ yields $5j = 15$, $5c + 3 = 13$, and $9j - 4 = 23$, so (2) holds and $x_0 = -(3 \cdot 15 - 2) = -43$. Under t_{15} ($k^2 - 1 = 224$), the orbit visits

$$\begin{aligned} -43 &\to -9{,}632 \to -643 \to -144{,}032 \to -9{,}603 \\ &\to -2{,}151{,}072 \to -143{,}405 \to -32{,}122{,}720 \\ &\to -2{,}141{,}515 \to -479{,}699{,}360 \to -31{,}979{,}958 \\ &\to -2{,}131{,}998 \to -142{,}134 \to -9{,}476 \to -632 \to -43, \end{aligned}$$

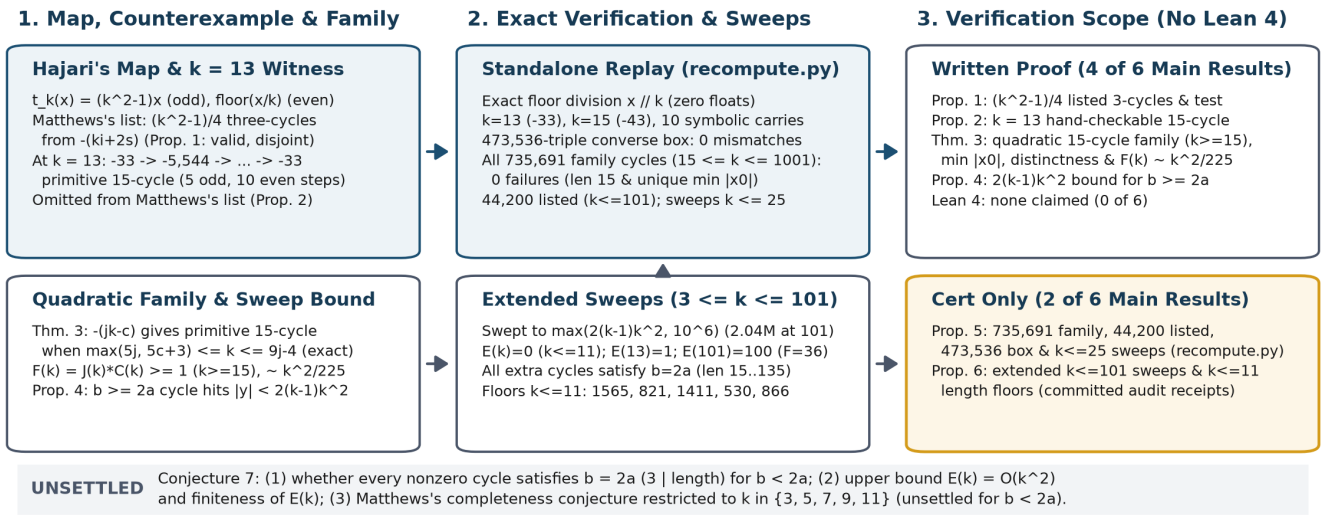


Figure 1. Discovery and verification architecture for Propositions 1, 2, 4, 5, 6, Theorem 3, and Conjecture 7. No Lean 4 formalization is claimed (0 of 6 proved main results in Lean 4; 4 of 6 by written proof; 2 of 6 by standalone exact Python replay and committed extended sweep receipts). Conjecture 7 remains unsettled.

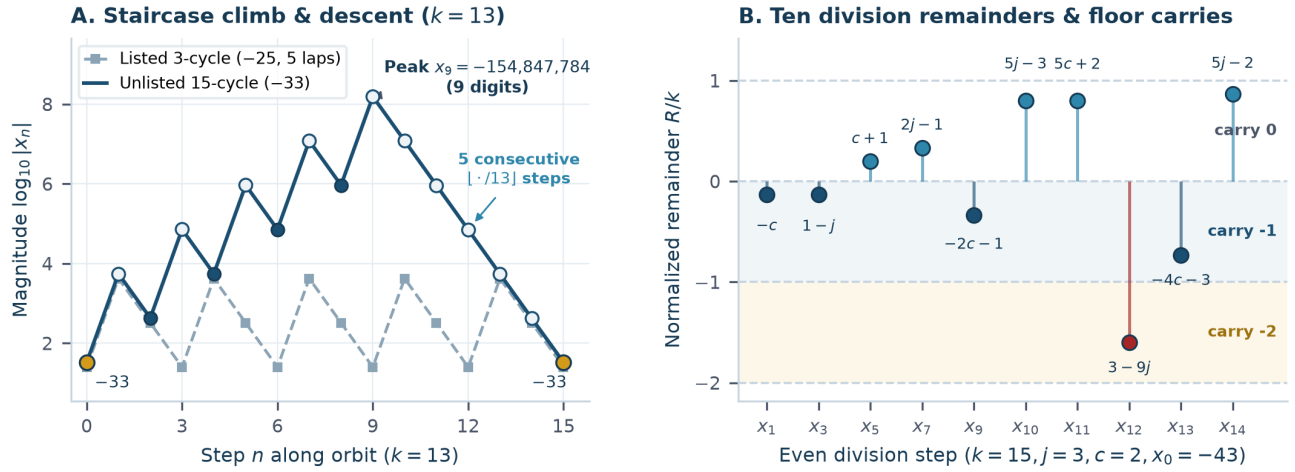


Figure 2. Anatomy of the 15-step loops. Panel A: magnitude $\log_{10}|x_n|$ along the unlisted $k = 13$ 15-cycle through -33 (navy, filled circles for odd $\times 168$ steps, light circles for even $\lfloor \cdot/13 \rfloor$ steps) versus five laps of the Matthews listed 3-cycle through -25 (dashed slate); five alternating climb steps reach the 9-digit peak $x_9 = -154{,}847{,}784$ before five consecutive floor divisions return to -33. Panel B: normalized symbolic remainders R/k for the ten even division steps of the smallest family member $(k, j, c) = (15, 3, 2)$ ($x_0 = -43$), sitting inside the integer carry bands 0, -1, and -2 enforced by $\max(5j, 5c+3) \le k \le 9j - 4$.

with exact Euclidean remainders $13, 13, 3, 5, 10, 12, 12, 6, 4, 13$ and normalized symbolic remainders $R/15$ plotted in Figure 2 (Panel B).

3. PROOFS OF THE MAIN RESULTS

This section gives complete, self-contained written proofs of Propositions 1, 2, 4, and Theorem 3.

Proof of Proposition 1. Let $k \ge 3$ be odd, $1 \le i \le k-2$ odd, and $0 \le s \le (k-1)/2$. Set $x_0 = -(ki + 2s)$. Since k and i are odd and $2s$ is even, x_0 is negative and odd. Therefore

$$x_1 = t_k(x_0) = -(ki + 2s)(k^2 - 1) = k(-k^2 i - 2sk + i) + 2s.$$

Because $k^2 - 1$ is even, x_1 is even. Moreover, $0 \le s \le (k-1)/2$ implies $0 \le 2s \le k - 1$, so by (4),

$$x_2 = t_k(x_1) = \lfloor x_1/k \rfloor = -k^2 i - 2sk + i = -(k^2 - 1)i - 2sk.$$

Since both $(k^2 - 1)i$ and $2sk$ are even, x_2 is even. Writing $x_2 = k(-ki - 2s) + i$ and noting $0 < i \le k - 2 < k$, (4) gives

$$t_k(x_2) = \lfloor x_2/k \rfloor = -ki - 2s = x_0.$$

Because the parity word of (x_0, x_1, x_2) is OEE, its primitive period cannot be 1, so it is exactly 3. Each listed 3-cycle contains a unique odd element $x_0 = -(ki + 2s)$. Writing $-x_0 = ki + 2s$ with $0 \le 2s \le k - 1$, Euclidean division by k recovers i and $2s$ uniquely. Hence distinct pairs (i, s) yield

pairwise disjoint 3-cycles, and the characterization of listed odd elements follows immediately. There are $(k-1)/2$ odd integers $i \in [1, k-2]$ and $(k+1)/2$ integers $s \in [0, (k-1)/2]$, giving $\frac{k-1}{2} \cdot \frac{k+1}{2} = \frac{k^2-1}{4}$ cycles. □

Proof of Proposition 2. Direct arithmetic verified in Table 1 checks every step $x_{n+1} = t_{13}(x_n)$ for $0 \le n \le 14$: at each odd step $n \in \{0, 2, 4, 6, 8\}$, $x_{n+1} = 168x_n$; at each even step $n \in \{1, 3, 5, 7, 9, 10, 11, 12, 13, 14\}$, $x_n = 13x_{n+1} + r$ with $r \in \{7, 11, 7, 5, 0, 12, 6, 4, 12, 11\} \subset [0, 12]$, confirming $x_{n+1} = \lfloor x_n/13 \rfloor$ by (4). The 15 elements $x_0, \dots, x_{14}$ are pairwise distinct and $x_{15} = -33 = x_0$, so -33 lies on a primitive 15-cycle. It cannot appear on Matthews's list of $(13^2 - 1)/4 = 42$ cycles of length 3 (and directly, $33 = 2 \cdot 13 + 7$ fails Proposition 1's criterion because quotient 2 is even and remainder 7 is odd). □

Proof of Theorem 3. Let $k, j, c \in \mathbb{Z}$ with k odd, j odd, and c even. We trace the orbit from $x_0 = -jk + c$ symbolically as polynomials in k whose coefficients are linear forms in (j, c). Because $k \equiv 1 \pmod 2$, $j \equiv 1 \pmod 2$, and $c \equiv 0 \pmod 2$, any polynomial $P(k) = \sum_{e \ge 0} (a_e j + b_e c + d_e) k^e$ has fixed parity

$$P(k) \equiv \sum_{e \ge 0} (a_e + d_e) \pmod 2,$$

independent of the specific odd k, odd j, and even c. Moreover, if $P(k) = Q(k) \cdot k + R$ where $R = a_0 j + b_0 c + d_0$ is the k-free constant term, then

$$\lfloor P(k)/k \rfloor = Q(k) + \lfloor R/k \rfloor = Q(k) + m$$

holds if and only if $mk \le R < (m+1)k$.

Part (1): Symbolic cascade and converse carry equivalence. Starting from $x_0 = -jk + c$ (odd, since $-1 \equiv 1 \pmod 2$), we apply $x \mapsto (k^2 - 1)x$ on odd states and $x \mapsto Q(k) + m$ on even states using the ten carries m listed in Table 2. Explicitly, the 15 symbolic iterates are:

$$\begin{aligned}
x_0 &= -jk + c \quad [\mathrm{O}],\\
x_1 &= -jk^3 + ck^2 + jk - c \quad [\mathrm{E}],\\
x_2 &= -jk^2 + ck + (j-1) \quad [\mathrm{O}],\\
x_3 &= -jk^4 + ck^3 + (2j-1)k^2 - ck + (1-j) \quad [\mathrm{E}],\\
x_4 &= -jk^3 + ck^2 + (2j-1)k + (-c-1) \quad [\mathrm{O}],\\
x_5 &= -jk^5 + ck^4 + (3j-1)k^3 + (-2c-1)k^2\\
&\quad + (1-2j)k + (c+1) \quad [\mathrm{E}],\\
x_6 &= -jk^4 + ck^3 + (3j-1)k^2 + (-2c-1)k + (1-2j) \quad [\mathrm{O}],\\
x_7 &= -jk^6 + ck^5 + (4j-1)k^4 + (-3c-1)k^3\\
&\quad + (2-5j)k^2 + (2c+1)k + (2j-1) \quad [\mathrm{E}],\\
x_8 &= -jk^5 + ck^4 + (4j-1)k^3 + (-3c-1)k^2\\
&\quad + (2-5j)k + (2c+1) \quad [\mathrm{O}],\\
x_9 &= -jk^7 + ck^6 + (5j-1)k^5 + (-4c-1)k^4 + (3-9j)k^3\\
&\quad + (5c+2)k^2 + (5j-2)k + (-2c-1) \quad [\mathrm{E}],\\
x_{10} &= -jk^6 + ck^5 + (5j-1)k^4 + (-4c-1)k^3\\
&\quad + (3-9j)k^2 + (5c+2)k + (5j-3) \quad [\mathrm{E}],\\
x_{11} &= -jk^5 + ck^4 + (5j-1)k^3 + (-4c-1)k^2\\
&\quad + (3-9j)k + (5c+2) \quad [\mathrm{E}],\\
x_{12} &= -jk^4 + ck^3 + (5j-1)k^2 + (-4c-1)k + (3-9j) \quad [\mathrm{E}],\\
x_{13} &= -jk^3 + ck^2 + (5j-1)k + (-4c-3) \quad [\mathrm{E}],\\
x_{14} &= -jk^2 + ck + (5j-2) \quad [\mathrm{E}],\\
x_{15} &= -jk + c = x_0.
\end{aligned}$$

One verifies term by term that each odd iterate $(x_0, x_2, x_4, x_6, x_8)$ has odd parity sum $\sum (a_e + d_e) \equiv 1 \pmod 2$, each even iterate $(x_1, x_3, x_5, x_7, x_9$ and $x_{10}, \dots, x_{14})$ has even parity sum $0 \pmod 2$, and adding the carry m of Table 2 to the linear coefficient of each even x_n produces x_{n+1} identically, ending at $x_{15} = -jk + c = x_0$.

Table 2. The ten symbolic floor-division remainders R and carries $m = \lfloor R/k \rfloor$ along the 15-step orbit of $x_0 = -(jk - c)$.

Div. after	Remainder R	Carry window	Carry m
x_1	$-c$	$-k \le -c < 0$	-1
x_3	$1 - j$	$-k \le 1 - j < 0$	-1
x_5	$c + 1$	$0 \le c + 1 < k$	0
x_7	$2j - 1$	$0 \le 2j - 1 < k$	0
x_9	$-2c - 1$	$-k \le -2c - 1 < 0$	-1
x_{10}	$5j - 3$	$0 \le 5j - 3 < k$	0
x_{11}	$5c + 2$	$0 \le 5c + 2 < k$	0
x_{12}	$3 - 9j$	$-2k \le 3 - 9j < -k$	-2
x_{13}	$-4c - 3$	$-k \le -4c - 3 < 0$	-1
x_{14}	$5j - 2$	$0 \le 5j - 2 < k$	0

We now show that for odd $k \ge 1$, odd $j \in \mathbb{Z}$, and even $c \in \mathbb{Z}$, the ten carry windows of Table 2 hold simultaneously if and only if $j \ge 3$, $c \ge 2$, and $\max(5j, 5c + 3) \le k \le 9j - 4$:

- *Necessity.* From $-c < 0$ and c even, $c \ge 2$. From $1 - j < 0$ and j odd, $j \ge 3$. From $5c + 2 < k$, $5c + 3 \le k$. From $5j - 2 < k$, since $5j - 2$ and k are both odd, $5j - 2 \le k - 2$, i.e., $5j \le k$. From $3 - 9j < -k$, $k \le 9j - 4$.
- *Sufficiency.* Assume $j \ge 3$ odd, $c \ge 2$ even, and $\max(5j, 5c + 3) \le k \le 9j - 4$. Then $c \ge 2$ gives $-c < 0$,

$c+1 \geq 3 > 0$, $-2c-1 < 0$, $5c+2 > 0$, and $-4c-3 < 0$; $j \geq 3$ gives $1-j \leq -2 < 0$, $2j-1 \geq 5 > 0$, $5j-3 \geq 12 > 0$, and $5j-2 \geq 13 > 0$. From $5c+3 \leq k$, we get $c < 2c+1 < 4c+3 < 5c+2 < k$, which discharges the remaining bounds at $x_1, x_5, x_9, x_{11}, x_{13}$. From $5j \leq k$, we get $j-1 < 2j-1 < 5j-3 < 5j-2 < k$ and $9j-3 < 10j \leq 2k$ (hence $-2k < 3-9j$), while $k \leq 9j-4$ gives $3-9j < -k$. Thus all ten carry windows hold.

Part (2): Primitive period 15, strict minimum magnitude, and pairwise disjointness. Because $x_{15} = x_0$, the primitive period L of x_0 divides 15, so $L \in \{1,3,5,15\}$. Any periodic orbit of length L must have an L-periodic parity word, yet the word $W = \text{OEOEOEOEOEEEEEE}$ satisfies $W_0 = \text{O} \neq \text{E} = W_1$, $W_0 = \text{O} \neq \text{E} = W_9$ (ruling out $L = 3$), and $W_0 = \text{O} \neq \text{E} = W_{10}$ (ruling out $L = 5$). Hence $L = 15$, and in particular no family orbit belongs to Matthews's list of 3-cycles.

To prove that $|x_n| > |x_0|$ for all $1 \leq n \leq 14$, note first that $j \geq 3$ and $c \leq (k-3)/5$ imply

$$x_0 = -(jk-c) \leq -\left(3k - \frac{k-3}{5}\right) = -\frac{14k+3}{5} < -2$$

(since $k \geq 5j \geq 15$). On the climbing half, whenever $x_{2m} < 0$ ($0 \leq m \leq 4$), we have $x_{2m+1} = (k^2-1)x_{2m} < x_{2m} < 0$, so $|x_{2m+1}| = (k^2-1)|x_{2m}| > |x_{2m}|$. Moreover, for $0 \leq m \leq 3$,

$$x_{2m+2} = \left\lfloor \frac{x_{2m+1}}{k} \right\rfloor \leq \frac{x_{2m+1}}{k} = \frac{k^2-1}{k}\, x_{2m} < x_{2m} < 0$$

since $(k^2-1)/k > 1$ and $x_{2m} < 0$. Induction yields

$$|x_0| < |x_2| < |x_4| < |x_6| < |x_8| \quad \text{and} \quad |x_{2m+1}| > |x_{2m}| \geq |x_0|$$

for all $0 \leq m \leq 4$. On the descending half $x_9 \to x_{10} \to \ldots \to x_{15} = x_0$, observe that for every integer $y \leq -2$ and every $k \geq 3$,

$$k(y+1) - y = (k-1)y + k \leq -2(k-1) + k = 2 - k < 0,$$

so $y+1 < y/k$, hence $y < y+1 \leq \lfloor y/k \rfloor < 0$, i.e., $0 < |\lfloor y/k \rfloor| < |y|$. Because $x_9 < 0$ and $\lfloor y/k \rfloor < 0$ whenever $y < 0$, every x_r ($9 \leq r \leq 14$) is negative; because $x_9, \ldots, x_{14}$ are even, none can equal -1, so $x_r \leq -2$ for every $9 \leq r \leq 14$. Applying $0 < |\lfloor y/k \rfloor| < |y|$ inductively gives

$$|x_9| > |x_{10}| > |x_{11}| > |x_{12}| > |x_{13}| > |x_{14}| > |x_{15}| = |x_0|.$$

Combining both halves, $|x_n| > |x_0|$ for all $1 \leq n \leq 14$: x_0 is the unique element of strictly smallest absolute value on its 15-cycle. Finally, because $0 < c \leq (k-3)/5 < k$, writing $|x_0| = (j-1)k + (k-c)$ with $1 \leq k-c \leq k-1$ recovers (j,c) uniquely from x_0 via Euclidean division by k. Distinct pairs (j,c) therefore have distinct unique minimum-magnitude elements and hence generate pairwise disjoint 15-cycles.

Part (3): Exact count, coverage for all odd $k \geq 15$, and quadratic growth. For fixed odd $k \geq 15$, the inequalities (2) decouple completely into

$$\max\left(3, \frac{k+4}{9}\right) \leq j \leq \frac{k}{5} \quad (j \text{ odd}),$$

$$2 \leq c \leq \frac{k-3}{5} \quad (c \text{ even}).$$

Writing $c = 2u$ ($u \geq 1$), the condition $5(2u)+3 \leq k$ is equivalent to $1 \leq u \leq \lfloor (k-3)/10 \rfloor$, giving $C(k) = \lfloor (k-3)/10 \rfloor \geq 1$ valid values of c. Writing $j = 2m+1$ ($m \geq 1$), the condition $5(2m+1) \leq k \leq 9(2m+1) - 4$ becomes $10m \leq k-5 \leq 18m$, i.e.,

$$\left\lceil \frac{k-5}{18} \right\rceil \leq m \leq \left\lfloor \frac{k-5}{10} \right\rfloor.$$

Since $k \geq 15$ gives $(k-5)/18 \geq 10/18 > 0$, the lower endpoint satisfies $\lceil (k-5)/18 \rceil \geq 1$ automatically (so $j \geq 3$ is redundant for $k \geq 15$). To see that at least one odd $j \geq 3$ fits for every odd $k \geq 15$, note that each odd $j \geq 3$ covers the interval $[5j, 9j-4]$; consecutive intervals for j and $j+2$ have gap

$$5(j+2) - (9j-4) = 14 - 4j \leq 2 \qquad (\text{for all } j \geq 3),$$

with equality at $j = 3$ ($[15,23]$ followed directly by odd $25 \in [25,41]$) and overlap for $j \geq 5$. Hence no odd $k \geq 15$ is skipped, so $J(k) = \lfloor (k-5)/10 \rfloor - \lceil (k-5)/18 \rceil + 1 \geq 1$ and $F(k) = J(k)\, C(k) \geq 1$, establishing (3). Finally,

$$J(k) = \left(\frac{1}{10} - \frac{1}{18}\right)k + O(1) = \frac{2}{45}k + O(1),$$

$$C(k) = \frac{1}{10}k + O(1),$$

so $F(k) = J(k)\, C(k) = k^2/225 + O(k)$ as odd $k \to \infty$. Combined with Proposition 2 at $k = 13$, Matthews's list is incomplete for every odd $k \geq 13$. □

Proof of Proposition 4. Let $C \subset \mathbb{Z}_{<0}$ be a negative cycle of t_k ($k \geq 3$ odd) with odd subset O ($|O| = a \geq 1$) and even subset E ($|E| = b = 2a+d$, $d \geq 0$). For every $x \in O$, $|t_k(x)|/|x| = k^2 - 1$. For every $y \in E$, writing $y = k\lfloor y/k \rfloor + r_y$ with $0 \leq r_y \leq k-1$ gives $-t_k(y) = (-y + r_y)/k$, so

$$\frac{|t_k(y)|}{|y|} = \frac{1}{k}\left(1 + \frac{r_y}{|y|}\right).$$

Multiplying $|t_k(z)|/|z|$ around the cycle C telescopes to 1:

$$1 = \prod_{z \in C} \frac{|t_k(z)|}{|z|} = (k^2-1)^a k^{-(2a+d)} \prod_{y \in E}\left(1 + \frac{r_y}{|y|}\right),$$

or, after taking natural logarithms,

$$a \ln\left(\frac{k^2}{k^2-1}\right) + d \ln k = \sum_{y \in E} \ln\left(1 + \frac{r_y}{|y|}\right). \tag{5}$$

Let $M := \min_{y \in E} |y| \geq 2$. Since $0 \leq r_y \leq k-1$, the right-hand side of (5) is at most $(2a+d)\ln(1 + (k-1)/M)$. On the left-hand side, for every $k \geq 3$ we have $\ln k \geq \ln 3 > 1 > \frac{1}{2}\ln(9/8) \geq \frac{1}{2}\ln(k^2/(k^2-1))$, so $d \geq 0$ implies

$$a \ln\left(\frac{k^2}{k^2-1}\right) + d \ln k \geq \frac{2a+d}{2} \ln\left(\frac{k^2}{k^2-1}\right).$$

Dividing by $2a+d > 0$ and applying the elementary inequalities $-\ln(1-u) > u$ (for $u = 1/k^2 \in (0,1)$) and $\ln(1+v) < v$ (for $v = (k-1)/M > 0$) yields

$$\frac{1}{2k^2} < \frac{1}{2}\ln\left(\frac{k^2}{k^2-1}\right) \leq \ln\left(1 + \frac{k-1}{M}\right) < \frac{k-1}{M}.$$

Rearranging gives $M < 2(k-1)k^2$, so C visits an even negative integer in $[-2(k-1)k^2, -1]$. □

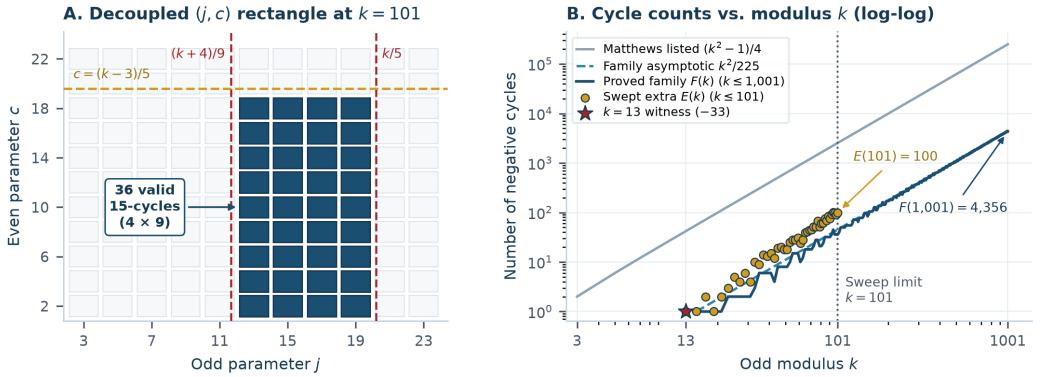


Figure 3. Parameter geometry and quadratic growth of unlisted negative cycles. Panel A: the decoupled (j, c) parameter grid at $k = 101$, where $(k+4)/9 = 11.67 \le j \le k/5 = 20.20$ (4 odd values $13, 15, 17, 19$) and $2 \le c \le (k-3)/5 = 19.60$ (9 even values $2, 4, \ldots, 18$) form a 4×9 rectangle of 36 pairwise disjoint 15-cycles. Panel B: log-log comparison of Matthews's listed $(k^2-1)/4$ 3-cycles (slate), total swept unlisted $b \ge 2a$ cycles $E(k)$ for odd $13 \le k \le 101$ (gold circles; vertical dotted boundary at the sweep limit $k = 101$), the proved family count $F(k)$ through $k = 1{,}001$ (navy), and the asymptotic curve $k^2/225$ (dashed teal), with the $k = 13$ witness (-33) marked by a star.

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Standalone Exact Replay. Every finite claim of Proposition 5 is verified from scratch by `recompute.py` using Python standard-library integer arithmetic (zero floats in any predicate; runtime roughly 9 seconds; log in `recompute.out`):

- *Listed 3-cycles* $(3 \le k \le 101)$. Across all 50 odd moduli, all 44,200 listed starts $-(ki + 2s)$ yield pairwise disjoint 3-cycles (2 at $k = 3$, 42 at $k = 13$, 56 at $k = 15$, 2,550 at $k = 101$; 0 failures).
- *Witnesses and converse box.* The orbits of -33 $(k = 13)$ and -43 $(k = 15)$ match Section 2; symbolic polynomial arithmetic reproduces Table 2, $x_{15} = x_0$, and non-periodicity at divisors $1, 3, 5$, and tests all 10 carry windows against (2) over 473,536 triples (odd $1 \le k \le 301$, odd $-11 \le j \le 99$, even $-12 \le c \le 98$: 0 disagreements, 19,532 members).
- *All 735,691 family cycles* $(15 \le k \le 1{,}001)$. Direct iteration on every valid (j, c) verifies all 735,691 cycles (5,727 through $k = 201$; 46,627 through $k = 401$; 0 failures) for primitive length 15, unique least $|x_0|$, and (3).
- *Exhaustive sweeps* $(3 \le k \le 25)$ *and controls.* Sweeping $[-2(k-1)k^2, -1]$ (up to 30,000 at $k = 25$) gives $E(k) = 0$ for $k \le 11$, $E(13) = 1$ (-33), $E(15) = 1$ (-43), $E(17) = 2$ $(-49, -43)$, $E(19) = 1$ (-55), $E(21) = 2$ $(-61, -53)$, $E(23) = 3$ (adding a 24-cycle through -51), and $E(25) = 5$ (adding a 24-cycle through -55 and 27-cycle through -69), all with $b = 2a$. Four negative-mutation checks (truncation at $k = 13$; $k = 25 > 9j - 4$; $k = 13 < 5j$; $k = 15 < 5c + 3$) are all caught (4/4), and the Section 1 Python box passes verbatim.

Extended Sweeps $(3 \le k \le 101)$ **and Growth Behavior.** In independent exact-arithmetic sweeps documented in `growth.out` and `verdict.md` (128-bit arithmetic with multiprecision fallback above 2^{100}; independently re-checked in Python for odd $3 \le k \le 31$), every negative start in $[-\max(2(k-1)k^2, 10^6), -1]$ and positive start in $[1, 10^5]$ was followed for all 50 odd moduli $3 \le k \le 101$ (2,040,200 at $k = 101$). By Proposition 4, these capture every negative cycle with $b \ge 2a$. Table 3 and Figure 3 compare $(k^2-1)/4$, $F(k)$, and $E(k)$.

Table 3. Selected moduli comparing Matthews's listed 3-cycles $(k^2 - 1)/4$, family count $F(k)$, and swept unlisted $b \ge 2a$ count $E(k)$ (swept through $k = 101$; rows $k \le 25$ and all $F(k)$ replayed directly by `recompute.py`).

k	Listed	$F(k)$	$E(k)$	$F(k)/k^2$	$E(k)/k^2$
11	30	0	0	0.00000	0.0000
13	42	0	1	0.00000	0.0059
15	56	1	1	0.00444	0.0044
45	506	8	19	0.00395	0.0094
61	930	10	24	0.00269	0.0064
95	2,256	45	102	0.00499	0.0113
101	2,550	36	100	0.00353	0.0098
201	10,100	171	—	0.00423	—
401	40,200	702	—	0.00437	—
1,001	250,500	4,356	—	0.00435	—

Across $33 \le k \le 101$, $E(k)/k^2$ stays within $[0.0064, 0.0113]$ (min $k = 61$, max $k = 95$), roughly 1.5–2.5 times $1/225 \approx 0.00444$, and every extra cycle has $b = 2a$ (lengths 15–135 at $k = 101$). For $k \in \{3, 5, 7, 9, 11\}$, any missed cycle must have $b < 2a$ and satisfy $a \ln(k^2/(k^2-1)) \approx (2a-b) \ln k$ outside 10^6; linear-form floor checks (`verdict.md`) give minimum lengths 1,565, 821, 1,411, 530, and 866.

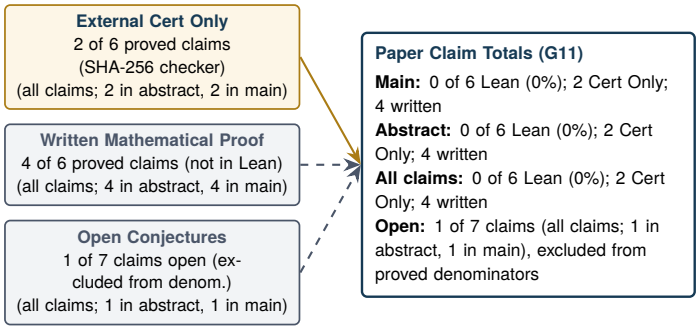
Why 15-Cycles Occur (Heuristic Balance). One multiplication by $k^2 - 1$ and two divisions by k scale by $(k^2-1)/k^2 = 1 - 1/k^2$. On negative integers, floor rounding toward $-\infty$ increases $|x|$ on every non-exact division, offsetting $(1 - 1/k^2)^a$: at $k = 13$ (-33), $168^5/13^{10} \approx 0.970762$, and ten floor roundings supply the exact reciprocal 1.030118. As k grows, the per-triple contraction weakens like $1/k^2$, opening quadratically many (j, c) rounding balances.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 2 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 4 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 1: For every odd k >= 3, all (k^2 - 1)/4 Matthews listed starts -(ki + 2s) (1 <= i <= k - 2 odd, 0 <= s <= (k - 1)/2) generate pairwise disjoint primitive 3-cycles, with an exact Euclidean-division membership criterion	Written	Written proof	Lean: none; fidelity: n/a	Elementary written proof (Section 3); finite check of all 44,200 listed 3-cycles across odd 3 <= k <= 101 in recompute.py. No Lean 4 formalization.
Prop. 2: At k = 13 (k^2 - 1 = 168), x_0 = -33 generates a primitive 15-cycle of t_13 (parity word OEOEOEOEOEEEEEE, peak -154,847,784) omitted from all 42 listed 3-cycles, refuting Matthews's completeness claim at k = 13	Written	Written proof	Lean: none; fidelity: n/a	Hand-checkable 15-step arithmetic proof (Table 1 and Section 3); also verified in recompute.py. No Lean 4 formalization.
Thm. 3: For odd k, odd j >= 3, and even c >= 2 with max(5j, 5c + 3) <= k <= 9j - 4, x_0 = -(jk - c) lies on a primitive 15-cycle (converse carry equivalence), with unique strict least |x_0| giving pairwise disjoint cycles and exact count F(k) >= 1 for all odd k >= 15 with F(k) = k^2/225 + O(k)	Written	Written proof	Lean: none; fidelity: n/a	Self-contained symbolic polynomial proof with 10 floor-division carries, parity primitivity, strict monotonicity of |x_n| on climb and descent, and window-gap counting (Section 3); corroborated on all 735,691 members through k = 1,001 in recompute.py. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 4: Every negative cycle of t_k (odd k >= 3) with a >= 1 odd steps and b >= 2a even steps contains an even element y with |y| < 2(k - 1)k^2, certifying exhaustiveness of negative sweeps over [-2(k - 1)k^2, -1] for b >= 2a	Written	Written proof	Lean: none; fidelity: n/a	Elementary written logarithmic-product proof (Section 3). No Lean 4 formalization.
Prop. 5: Standalone exact Python replay certifies k = 13 (-33), k = 15 (-43), the 473,536-triple converse box (0 disagreements), all 735,691 family cycles (15 <= k <= 1,001; 0 failures), all 44,200 listed 3-cycles (3 <= k <= 101), and exhaustive sweeps over [-2(k-1)k^2, -1] for odd 3 <= k <= 25	Cert Only	recompute.py (f491eb9962ea) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Certified by standalone exact-integer Python script recompute.py (standard library only, zero floats in predicates, ~9 s runtime, 4/4 negative-mutation controls).
Prop. 6: Committed extended exact-arithmetic sweeps through max(2(k-1)k^2, 10^6) for odd 3 <= k <= 101 confirm E(k) = 0 for k <= 11, E(101) = 100 (0.0064 <= E(k)/k^2 <= 0.0113 for 33 <= k <= 101), b = 2a on every extra cycle found, b < 2a length floors (1,565, 821, 1,411, 530, 866) for k <= 11, and zero positive cycles up to 10^5	Cert Only	recompute.py (f491eb9962ea) [archived; replay pending]; growth.py (d3e850291fb2) [regenerable; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Backed by committed exact-arithmetic sweep and falsifier audit receipts (growth.out, verdict.md; corroborated for 3 <= k <= 31 in blind re-review; raw per-modulus C logs retired outside git and regenerable deterministically); recompute.py replays odd 3 <= k <= 25 directly.
Conj. 7: Unsettled questions on whether every nonzero cycle satisfies b = 2a (length divisible by 3), whether E(k) = Theta(k^2) (upper bound and finiteness), and Matthews's completeness claim restricted to k in {3, 5, 7, 9, 11} (unsettled for b < 2a)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open questions (excluded from G11 denominators).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: d404cb8a6041 | Fidelity SHA-256: 6422949a6270.

6. PRIOR WORK AND OPEN QUESTIONS

Primary sources and context. The map t_k was posed for positive starts by Pruthviraj Hajari on MathOverflow (Question 378293, `https://mathoverflow.net/questions/378293/`, 16 December 2020, 0 answers; comments do not mention negative cycles). Keith R. Matthews's web pages "Conjecture of Pruthviraj Hajari" for general odd $k \geq 3$ (`http://www.numbertheory.org/php/hajari3.html`, last modified 16 December 2020) and for $k = 3$ (`http://www.numbertheory.org/php/hajari.html`, last modified 5 June 2026) formulated the negative-start completeness claim answered here; both pages were read in full from archived copies, together with Matthews's index and companion pages (`hajari1.html`, `hajari2.html`), none of which mentions any negative cycle beyond the $(k^2 - 1)/4$ listed 3-cycles. Broader frameworks for generalized $3x + 1$ mappings and cycle searches appear in Matthews's survey chapter in J. C. Lagarias's edited volume [3] (`[to check]` in print; web version read), G. Venturini [8] (`[to check]`), Lagarias's two annotated bibliographies [1, 2], and R. Tremblay's three cycle-search preprints [5, 6, 7]. The nearest OEIS relative found was Clark Kimberling's multiply-and-divide sequence [4], which uses 8 and 3 under a different greedy permutation rule.

Logged prior-work searches. Two gate-passing literature search logs plus a Google Scholar sweep were recorded:

- *Initial search log (2026-11-19):* OEIS queries by the $k = 13$ cycle terms with both signs (`33,5544,427,71736` and negated), the $k = 15$ cycle terms (`43,9632,643,144032,9603`), "Hajari", and "Matthews 3x+1 cycles" (0 hits); arXiv queries including

"Hajari Collatz floor" (0 hits); full-text grep of Lagarias's bibliographies [1, 2] and Tremblay's three preprints [5, 6, 7] (0 Hajari mentions); hand inspection of Matthews's site and MathOverflow 378293; and 3 web searches (0 hits).

- *Keyed CORE search (2026-11-19/20):* 7 CORE API requests returning 13 records (1 irrelevant ordinary $3x+1$ paper, 10 non-mathematical namesakes of Hajari, 0 relevant hits; CORE does not index Matthews's pages and searches titles and abstracts only).
- *Google Scholar sweep (2026-11-22):* 4 targeted queries returning 37 records total across Hajari/Matthews, floor-division generalized Collatz cycles, and Matthews follow-up queries (0 mathematical papers mentioning t_k, the -33 cycle, or the 15-cycle family).

Consequently, the $k = 13$ 15-cycle and the quadratic family of Theorem 3 were not found in the searches logged.

Unread caveats and still-owed sources (`[to check]`). The following sources remain unverified or unread and are flagged `[to check]` in `references-checked.md`: the throttled Math Stack Exchange and MathOverflow search APIs (MathOverflow 378293 was read directly from a cached copy); the printed AMS edition of Matthews's survey chapter in Lagarias [3] (pp. 79–103; its web version was read); the paywalled journal body of Venturini [8] (zbMATH review `Zbl 0773.11019` and Lagarias's entry 176 read); R. K. Guy's *Unsolved Problems in Number Theory* (in print); G. J. Wirsching's *The Dynamical System Generated by the* $3n+1$ *Function* (LNM 1681, in print); and S. Kohl's PhD thesis.

Open directions. Beyond the three parts of Conjecture 7, two concrete extensions suggest themselves: (i) classify additional symbolic families of parity words with $b = 2a$ (such as the 24-cycle at $k = 23$ and 27-cycle at $k = 25$) to test whether finitely many polynomial families account for the empirical $E(k)/k^2 \approx 0.0064$–0.0113 band; and (ii) examine Matthews's companion even-k ceiling map (`hajari2.html`), whose printed cycle list was not tested here.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, and the standalone exact verifier live in `papers/hajari-15-cycles/draft1/`. Running `python3 papers/hajari-15-cycles/draft1/recompute.py` finishes in roughly 9 seconds using only the Python standard library and deterministically reproduces `recompute.out` (all 44,200 listed 3-cycles for $3 \le k \le 101$, the -33 and -43 15-cycles, the symbolic 10-carry derivation and 473,536-triple converse box, all 735,691 family 15-cycles through $k = 1{,}001$, exhaustive negative sweeps through $k = 25$, 4/4 negative-mutation controls, and the Section 1 Python snippet). Committed summary receipts for the extended $k \le 101$ sweeps live in the repository's `hajari-15-cycles` documentation archive (`growth.out`, `verdict.md`, SHA-256 pinned in Section 5); raw per-modulus C sweep logs are retired outside the git tree and are deterministically regenerable. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes discovered the $k = 13$ counterexample and two-parameter family, proved Propositions 1, 2, 4 and Theorem 3, wrote and independently re-verified the exact Python and C sweep programs, conducted the logged prior-work searches, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 24, 2026

REFERENCES

(1) Lagarias, J. C. The $3x+1$ problem: An annotated bibliography (1963–1999) (sorted by author). Preprint, **2003**, arXiv:math/0309224.

(2) Lagarias, J. C. The $3x+1$ problem: An annotated bibliography, II (2000–2009). Preprint, **2006**, arXiv:math/0608208.

(3) Lagarias, J. C., Ed. *The Ultimate Challenge: The* $3x+1$ *Problem* (including K. R. Matthews, "Generalized $3x+1$ mappings: Markov chains and ergodic theory", pp. 79–103). American Mathematical Society: Providence, RI, **2010**; DOI: 10.1090/mbk/078.

(4) OEIS Foundation Inc. Entry A050096: Multiply-and-divide sequence for $(8, 3)$ (C. Kimberling). https://oeis.org/A050096.

(5) Tremblay, R. Generalized $3x+1$ mappings: searching for cycles. Preprint, **2019**, arXiv:1907.13302.

(6) Tremblay, R. Generalized $3x+1$ mappings: counting cycles. Preprint, **2019**, arXiv:1909.00213.

(7) Tremblay, R. Generalized $3x+1$ mappings: convergence and divergence. Preprint, **2019**, arXiv:1910.11798.

(8) Venturini, G. Iterates of number theoretic functions with periodic rational coefficients (generalization of the $3x+1$ problem). *Studies in Applied Mathematics* **1992**, *86* (3), 185–218. DOI: 10.1002/sapm1992863185.