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Working Draft

How Few Edges Preserve a Hypercube's Diameter? Exact Conservation Laws and Kernel-Checked Bounds

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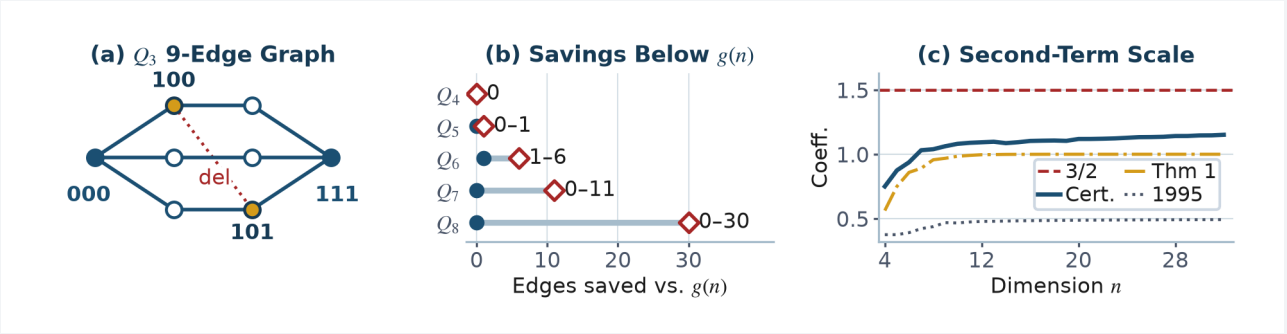
Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For the n-cube Q_n ($N = 2^n$), let $f(n)$ be the fewest edges in a diameter-n spanning subgraph; counting edges across BFS layers gives a global potential law and an antipodal conservation law bounding each root's weighted merges by $\mu = (n-1)(|E| - N) - N + 2n$. Averaging yields $f(n) \geq N + \lceil (N-2n)/(n-1) \rceil$ ($n \geq 2$), beating the 1995 bound $N - 1 + \lceil (N-1)/(2n-1) \rceil$ for all $n \geq 4$ and refuting five entries of Bass–Sudborough Table 1 ($n = 4, \ldots, 8$). Sphere-width caps raise the asymptotic coefficient to $\liminf_{n\to\infty}(n-1)(f(n)-N)/N \geq 3/2$ alongside $f(n) \leq g(n) := N + \binom{n}{\lfloor n/2 \rfloor} - 2$. We verify $f(4) = 20$, $39 \leq f(5) \leq 40$, $f(6) \leq 81 < g(6) = 82$, and kernel-checked bounds for $17 \leq n \leq 32$ reaching $f(32) \geq 4{,}454{,}492{,}243$ ($n = 9, \ldots, 16$ from $f(9) \geq 580$). Certificate evaluations give $f(6) \geq 76$, $f(8) \geq 294$, and a three-radius bound gives $f(7) \geq 150$. Whether $f(5) = 39$ or 40 and the order of $f(n) - N$ remain open.

Verification Scope (G11): WARN: Human referee: pending
Main results: 11 of 14 in Lean Kernel (79%) · 1 Lean + Cert · 2 Cert Only · 1 open (not counted)
Abstract claims: 9 of 11 in Lean Kernel (82%) · 1 Lean + Cert · 1 Cert Only · 1 open (not counted)
Full Trust Boundary: Section 5.

MSC 2020: 05C12, 05C35 **Keywords:** hypercube, diameter-preserving spanning subgraph, distance spheres, antipodal conservation, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Opposite corners of the n-dimensional hypercube Q_n lie at Hamming distance n, and no spanning subgraph of Q_n can have diameter below n. Removing an edge between two neighboring vertices may stretch the shortest path between those endpoints without enlarging the overall diameter: in the nine-edge spanning subgraph of Q_3 built from three vertex-disjoint 3-step paths between 000 and 111 (Figure 1), deleting the cube edge $\{100, 101\}$ triples the distance between 100 and 101 from 1 to 3 while keeping every pair within distance 3. Any lower bound on the minimum edge count $f(n)$ of a diameter-n spanning subgraph of Q_n must therefore permit substantial local detours.

Write $N = 2^n$, $h(u,v)$ for Hamming distance on $Q_n = \{0,1\}^n$, and $d(u,v)$ for graph distance in a diameter-n spanning subgraph G of Q_n (called *valid*). Every vertex of a valid graph has degree at least 2, so its degree excess $x_v = \deg_G(v) - 2$ is nonnegative, and writing $|E(G)| = N + t$ gives $\sum_v x_v = 2t$. For a root $w \in Q_n$ with bitwise complement $\bar{w}$, write $\delta(w) = \sum_v (d(w,v) - h(w,v))$ for total distance detour from w. Orienting each edge away from w across BFS layers $S_j = \{v : d(w,v) = j\}$, each nonroot vertex needs one parent edge; if M_j counts extra parent edges entering S_j, let $M(w) = \sum_{j=2}^{n-1} (n-j) M_j$ be the depth-weighted merge count and set $\mu = (n-1)t - (N - 2n)$.

Theorem 1 (Universal Lower Bound). *For every integer $n \geq 2$, the minimum edge count $f(n)$ of a diameter-n spanning subgraph of Q_n satisfies*

$$f(n) \geq 2^n + \left\lceil \frac{2^n - 2n}{n-1} \right\rceil = 2^n + \left\lfloor \frac{2^n - 2n + n - 2}{n-1} \right\rfloor.$$

For $n \geq 4$, Theorem 1 strictly exceeds the classical lower bound of Bouabdallah, Delorme, and Djelloul [3] (recalled in Wang et al. [8]) and rules out five values printed in Bass and Sudborough's removable-edge tables [1, 2].

Corollary 2 (Strict Improvement over the 1995 Bound). *For every*

integer $n \geq 4$,

$$2^n - 1 + \left\lceil \frac{2^n - 1}{2n - 1} \right\rceil < f(n).$$

In particular, evaluating Corollary 2 at $n = 4,5,6,7,8$ gives $18 < f(4)$, $35 < f(5)$, $69 < f(6)$, $137 < f(7)$, and $272 < f(8)$ (also verified directly in `Hypercube.bass_sudborough_table_refuted`), contradicting all five surviving-edge counts in Table 1 of Bass and Sudborough [1, 2].

Bounding the maximum growth of graph balls by the sphere-width ceiling $s = 2 + 2t$ raises the asymptotic second-term coefficient by 50% over Theorem 1 (and triples the $1/2$ coefficient of the 1995 bound [3]), while the symmetric-chain construction provides the standard upper bound.

Theorem 3 (Asymptotic 3/2 Second-Term Lower Bound). *For every rational $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all integers $n \geq N_0$,*

$$\left(\frac{3}{2} - \varepsilon\right) 2^n \leq (f(n) - 2^n)(n - 1);$$

equivalently, $\liminf_{n\to\infty} (n-1)(f(n) - 2^n)/2^n \geq 3/2$.

Proposition 4 (Symmetric-Chain Upper Bound). *For every integer $n \geq 1$,*

$$f(n) \leq g(n) := 2^n + \binom{n}{\lfloor n/2 \rfloor} - 2.$$

In dimension 6, Charton, Ellenberg, Wagner, and Williamson [4] constructed an explicit 81-edge valid subgraph of Q_6, saving one edge below $g(6) = 82$. The standalone 10-line Python check below verifies symmetry, edge count, and all 4,096 pairwise distances of this witness verbatim (executed directly from this manuscript by `recompute.py`).

Instant Q_6 81-Edge Witness Check (10 lines, verified verbatim by recompute.py)

```
masks = (0xd55d90b5d55d90b5, 0xea80ea805d735d73, 0xaaaa4d4d02025b5b,
         0x88000022aa8822bb, 0xff00005aa0a5a55f, 0xccff3f0000000033)
adj = [{u ^ (32 >> i) for i, m in enumerate(masks) if m >> u & 1}
       for u in range(64)]
assert all(u in adj[v] for u, row in enumerate(adj) for v in row)
assert sum(map(len, adj)) == 2 * 81
for u in range(64):
    seen = {u}
    for _ in range(6): seen |= {w for v in seen for w in adj[v]}
    assert len(seen) == 64
```

In dimensions 2 through 8, exact small-cube analysis, antipodal-capped integer certificates, and the 81-edge witness yield Table 1 (with $f(2) = 4$ and $f(3) = 9$ also verified in `lean/Hypercube/SmallDims.lean`), while the certificate ladder extends through $n = 32$ (Table 2). Eleven of fourteen proved main results are verified unconditionally inside the pure Lean 4 kernel (`lean/Hypercube/`, Section 5); the remaining three entries (Propositions 12, 13, and 14) isolate finite certificate evaluations (also kernel-checked in the Palomar-registered standalone proof `Challenge.lean`, `PALOMAR-2026-11-22-000030` v1) and an external Q_4 extremal census.

Table 1. Lower and upper bounds on $f(n)$ for $2 \leq n \leq 8$, compared with the 1995 bound [3] (which equals Bass–Sudborough Table 1 for $4 \leq n \leq 8$ [1, 2]) and $g(n) = 2^n + \binom{n}{\lfloor n/2 \rfloor} - 2$.

n	1995	Thm. 1	Certified	Best upper	$g(n)$
2	4	4	4	4	4
3	9	9	9	9	9
4	18	19	20	20	20
5	35	38	39	40	40
6	69	75	76	81	82
7	137	147	150	161	161
8	272	291	294	324	324

2. DEFINITIONS, DISTANCE LAYERS, AND CONSERVATION LAWS

Identify $V(Q_n) = \{0,1\}^n$ with edges joining words of Hamming distance 1. Throughout this section, $n \geq 2$ and $G \subseteq Q_n$ is a spanning subgraph with $d(u,v) \leq n$ for all $u, v \in Q_n$. Since every edge flips one coordinate, $d(u,v) \geq h(u,v)$ and $d(u,v) \equiv h(u,v) \pmod 2$; hence G is valid if and only if $G \leq Q_n$ and $\operatorname{diam}(G) = n$. Adding any missing cube edge preserves validity, so ruling out valid graphs of size $2^n + t$ rules out all smaller sizes as well.

Every vertex u in a valid graph satisfies $\deg_G(u) \geq 2$: if u had a unique neighbor v, then $\bar{v} \neq u$ (as $n \geq 2$) would have $d(u,\bar{v}) = 1 + d(v,\bar{v}) \geq 1 + n$, impossible. Thus $0 \leq x_u \leq n-2$. Because Q_n is bipartite with 2^{n-1} vertices of each parity and every edge connects opposite parities,

$$\sum_{h(0,u)\ \text{even}} x_u = \sum_{h(0,u)\ \text{odd}} x_u = t, \qquad \sum_{u \in Q_n} x_u = 2t. \tag{1}$$

Fix a root $w \in Q_n$ and write $S_j = \{v : d(w,v) = j\}$, $s_j = |S_j|$, $X_j = \sum_{v \in S_j} x_v$, and $Y_j = \sum_{i=0}^{j} X_i$. Because Q_n is bipartite, every edge connects adjacent layers S_{j-1} and S_j. Choosing one incoming edge per nonroot vertex leaves $M_j = \sum_{v \in S_j} (\text{parents of } v - 1)$ extra parent edges entering S_j, with $M_1 = 0$, $\sum_{j=2}^{n} M_j = t + 1$, and $M(w) = \sum_{j=2}^{n-1} (n-j) M_j$. Summing $d(w,u) + d(w,v)$ over all edges $\{u,v\} \in E(G)$ in two ways—by endpoints ($\sum_v \deg_G(v) d(w,v)$) and by layer ($2j-1$ for edges entering S_j)—yields the root identity

$$\sum_{v \in Q_n} x_v d(w,v) + 2M(w) = (2n-1)t + 2n - N. \tag{2}$$

Equivalently, setting $R = \{m : 0 \leq m \leq n-2,\ m \equiv n \pmod 2\}$ and using $\sum_v x_v (n - d(w,v) \bmod 2) = t$ from (1),

$$\sum_{m \in R} Y_m = \sum_{v \in Q_n} x_v \left\lfloor \frac{n - d(w,v)}{2} \right\rfloor = \frac{N}{2} - n + M(w). \tag{3}$$

Counting edges incident to S_j ($s_j + M_j$ incoming and $s_{j+1} + M_{j+1}$ outgoing, totaling $2s_j + X_j$) gives the sphere recurrence

$$s_{j+1} - s_j = X_j - M_j - M_{j+1} \qquad (1 \leq j < n), \tag{4}$$

whence $s_j \leq 2 + Y_{j-1} \leq 2 + 2t$ for all $j \geq 1$.

Lemma 5 (Global Vertex-Potential Conservation). *For every integer $n \geq 2$ and every valid spanning subgraph G of Q_n with $|E(G)| = 2^n + t$, setting $\psi(w) = x_w \delta(w) + 2M(w)$ and $\mu = (n-1)t - (2^n - 2n)$,*

$$\sum_{w \in Q_n} \psi(w) = 2^n \mu.$$

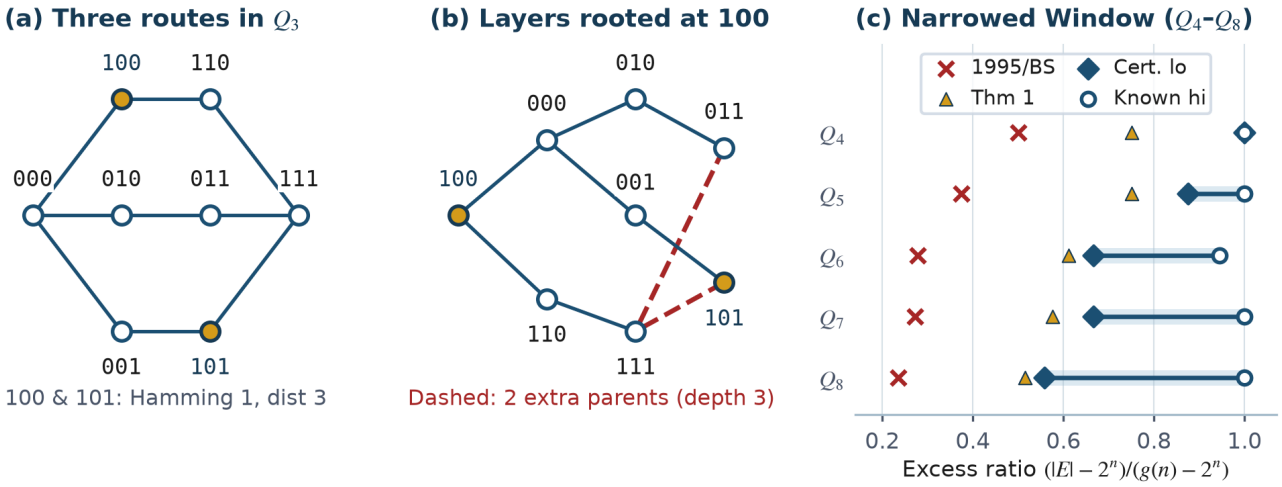


Figure 1. Detour geometry in Q_3 and bound windows across Q_4–Q_8. (a) Three 3-step paths between 000 and 111 form a 9-edge valid subgraph in which adjacent cube vertices 100 and 101 lie at graph distance 3. (b) BFS layering rooted at 100 shows the two extra parent edges (dashed) entering depth 3, giving $M(100) = 0$. (c) Normalized excess ratio $(|E| - 2^n)/(g(n) - 2^n)$ comparing the refuted 1995/Bass–Sudborough values [3, 1, 2] (×), Theorem 1 (△), our certified lower bounds (solid diamonds), and known constructions (∘, including the 81-edge witness of [4] at Q_6).

Proof. For each fixed v, summing $d(w,v)$ over all $w \in Q_n$ gives $\sum_w h(w,v) + \delta(v) = n2^{n-1} + \delta(v)$. Summing (2) over all $w \in Q_n$ and applying $\sum_v x_v = 2t$ yields $\sum_v x_v(n2^{n-1} + \delta(v)) + 2\sum_w M(w) = N((2n-1)t + 2n - N)$, and subtracting nNt gives the stated identity. □

Theorem 6 (Antipodal Pair Conservation). *For every integer $n \geq 2$, every valid spanning subgraph G of Q_n, and every root $w \in Q_n$ with antipode $\bar{w}$, let $\eta_w(v) = d(v,w) + d(v,\bar{w}) - n$. Then $\eta_w(v)$ is nonnegative and even for every v, and*

$$M(w) + M(\bar{w}) + \sum_{v \in Q_n} x_v \frac{\eta_w(v)}{2} = \mu.$$

Proof. Add (2) at w and at $\bar{w}$, subtract $n\sum_v x_v = 2nt$, and divide by 2:

$$2(M(w) + M(\bar{w})) + \sum_{v \in Q_n} x_v(d(v,w) + d(v,\bar{w}) - n) = 2\mu.$$

Since $h(v,w) + h(v,\bar{w}) = n$ and $d(v,u) \geq h(v,u)$ with the same parity, $\eta_w(v) \geq 0$ is even for every v. Dividing by 2 yields the conservation law, and dropping nonnegative terms gives the pointwise cap $0 \leq M(w) \leq \mu$ (verified in `Hypercube.antipodal_cap`). □

3. PROOFS OF THE GENERAL, ASYMPTOTIC, AND BLOCKING BOUNDS

Proof of Theorem 1 and Corollary 2. Every summand $\psi(w) = x_w\delta(w) + 2M(w)$ in Lemma 5 is nonnegative, so $\mu = (n-1)t - (2^n - 2n) \geq 0$. Dividing by $n - 1 \geq 1$ and taking ceilings gives $t \geq \lceil (2^n - 2n)/(n-1) \rceil$, proving Theorem 1. For $n \geq 4$, direct integer arithmetic compares the two ceilings: writing $a = 2^n - 2n$ and $b = 2^n - 1$, one checks $(2n-1)(a+n-1) > (n-1)(b+2n-2)$ for all $n \geq 4$ (with equality never occurring), giving $2^n - 1 + \lceil (2^n-1)/(2n-1) \rceil < 2^n + \lceil (2^n - 2n)/(n-1) \rceil \leq f(n)$, which proves Corollary 2.

Proof of Theorem 3. For a fixed root w, let $B_k = \sum_{i=0}^{k} \binom{n}{i}$ and $C_k = \sum_{i=0}^{k} s_i$. Since every vertex in the graph ball of radius k lies in the Hamming ball of radius k, the ball deficit $g_k = B_k - C_k \geq 0$ satisfies $\delta(w) = \sum_{k=0}^{n-1} g_k$. By (4), each sphere has width $s_k \leq s := 2 + 2t$, so defining the width-capped envelope $U_0 = 1$ and $U_k = \min(B_k, U_{k-1} + s)$ gives $C_k \leq U_k$ by induction and therefore

$$\delta(w) \geq \Delta := \sum_{k=0}^{n-1} (B_k - U_k), \qquad 2t\Delta \leq N\mu, \tag{5}$$

by weighting with x_w and applying Lemma 5 (Figure 2; formalized in `Hypercube.theorem4_thin_levels`). For example, at $n = 10$ and $t = 112$, $s = 226$ and $\Delta = 36$, giving the contradiction $2t\Delta = 8{,}064 > 4{,}096 = N\mu$.

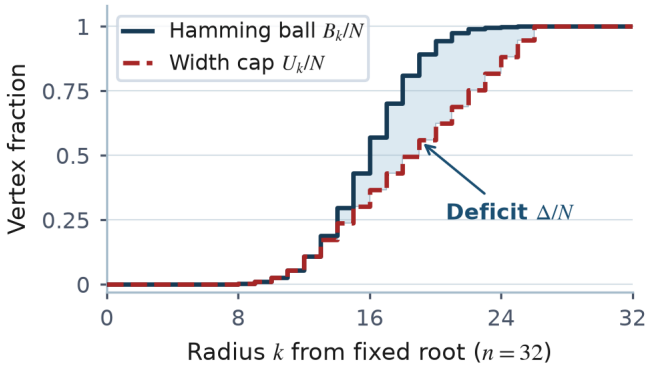


Figure 2. Sphere-width obstruction at $n = 32$ with $t = 138{,}547{,}331$ (minimal excess allowed by Theorem 1) and cap $s = 2 + 2t$. The Hamming ball B_k/N jumps sharply near $k = n/2$, whereas the width-capped envelope U_k/N rises by at most s/N per step. The shaded deficit area equals $\Delta/N \approx 2.53$.

Now let $\varepsilon > 0$ be rational, and suppose along a sequence of dimensions n an optimal graph has $c_n = (n-1)t/N \to c < \infty$ (by Theorem 1, $c \geq 1$). Then $s = (2c + o(1))N/n$. Set $a = \lfloor n/2 - n^{2/3} \rfloor$ and $b = \lceil n/2 + n^{2/3} \rceil$. By Chebyshev's second-moment identity on the binomial distribution with mean $n/2$ and variance $n/4$, $B_a = o(N)$ and $B_b = N - o(N)$, while $U_b \leq B_a + (b-a)s = o(N)$. For $0 \leq r < n - b$, $B_{b+r} - U_{b+r} \geq [B_b - U_b - rs]_+$, so summing over r and dividing by nN yields the triangle area

$$\liminf_{n\to\infty} \frac{\Delta}{nN} \geq \int_0^{1/2} [1 - 2cy]_+ \, dy = \frac{1}{4c}.$$

Dividing (5) by N^2 gives $c - 1 \geq 2c/(4c) = 1/2$, hence $c \geq 3/2$. The Lean formalization `Hypercube.theorem5_asymptotic_three_halves` executes this second-moment argument constructively over $\mathbb{Q}$ with explicit rational thresholds.

Proof of Proposition 4 and the blocking converse. Fix a corner $r \in Q_n$ and stratify Q_n by Hamming distance $i = h(r,u)$ from r. Between consecutive layers i and $i+1$, the bipartite subgraph is biregular with the smaller side having larger degree, so Hall's marriage theorem (or a symmetric-chain decomposition [5]; explicitly realized by prefix-sum matching in `lean/Hypercube/Construction.lean`) supplies a matching covering the smaller layer. Adding one edge for each unmatched vertex in the larger layer uses $\max\{\binom{n}{i}, \binom{n}{i+1}\}$ edges between layers i and $i+1$ and guarantees every interior vertex has both a downward and an upward neighbor. Hence every vertex reaches r and $\bar{r}$ monotonically, and any two vertices at heights i, j are linked within $\min(i+j, 2n-i-j) \leq n$ steps. By unimodality, the total edge count is $\sum_{i=0}^{n-1} \max\{\binom{n}{i}, \binom{n}{i+1}\} = 2^n + \binom{n}{\lfloor n/2 \rfloor} - 2 = g(n)$, proving Proposition 4.

Call an interior vertex ($1 \leq h(r,u) \leq n-1$) *blocking* with respect to r if it lacks either all downward neighbors or all upward neighbors in G, and let $\beta_r(G)$ count such blocking vertices.

Proposition 7 (Blocking-Vertex Lower Bound). *For every integer $n \geq 1$, every valid spanning subgraph G of Q_n, and every corner $r \in Q_n$,*

$$2^n + \binom{n}{\lfloor n/2 \rfloor} - 2 \leq |E(G)| + \beta_r(G).$$

Proof. Set $m = \lceil n/2 \rceil$. Request one downward edge from each vertex in layers $1, \ldots, m$ and one upward edge from each vertex in layers $m, \ldots, n-1$; because $n - m = \lfloor n/2 \rfloor$, the total number of requests is $\sum_{i=1}^{m} \binom{n}{i} + \sum_{i=m}^{n-1} \binom{n}{i} = 2^n + \binom{n}{\lfloor n/2 \rfloor} - 2 = g(n)$. Each fulfilled request consumes a distinct edge of G (downward requests use edges below layer m, upward requests use edges above layer m). Every vertex with a missed request is blocking, and a vertex at layer m cannot miss both requests since $\deg_G(u) \geq 2$ in a valid graph (indeed $\deg_G(u) \geq 1$ suffices). Thus at least $g(n) - \beta_r(G)$ distinct edges lie in G. □

In particular, any valid subgraph beating $g(n)$ must have $\beta_r(G) \geq 1$ for every corner $r \in Q_n$; for the 81-edge witness of Proposition 10, `Hypercube.q6_81_blocking_every_corner` verifies $\beta_r(G_6^{81}) \geq 1$ across all 64 corners of Q_6.

4. SMALL DIMENSIONS, ANTIPODAL CERTIFICATES, AND COMPUTATION

Exact Q_4, dimension 5, and the 81-edge Q_6 witness. Direct combinatorial analysis of the local excess identity (3) rules out 19-edge valid subgraphs of Q_4 (by casing on branch-degree profiles (3,3,3) and (4,3) on each parity side) and 38-edge valid subgraphs of Q_5 (where $t = 6, \mu = 2$ forces $M_2(w) = 0$, eliminating 4-cycles and driving the squared-excess sum $A = \sum_u x_u^2$ into the contradiction $232 \leq 216$). Combined with Proposition 4 ($g(4) = 20, g(5) = 40$) and kernel verification of Charton et al.'s 81-edge witness [4], we obtain the next three kernel-verified results.

Proposition 8 (Exact Fourth-Cube Minimum). $f(4) = 20$.

Theorem 9 (Dimension-5 Bounds). $39 \leq f(5) \leq 40$.

Proposition 10 (Explicit 81-Edge Q_6 Witness). $f(6) \leq 81$ (*strictly below* $g(6) = 82$).

Parity balls, half-step chains, and antipodal-capped certificates. To sharpen finite lower bounds across $4 \leq n \leq 32$, track the two parity classes separately from a fixed root w. Let P_k count vertices at graph distance at most k with $d(w,v) \equiv k \pmod 2$, let H_k be its Hamming counterpart, and set $b_k = H_k - P_k \geq 0$ (so $b_0 = b_{n-1} = b_n = 0$ and $b_1 = n - 2 - x_w$). Then $g_k = b_k + b_{k-1}$,

$$\delta(w) = 2\sum_{k=1}^{n-2} b_k, \qquad s_k = \binom{n}{k} - b_k + b_{k-2} \quad (k \geq 2). \tag{6}$$

Expanding $P_k - P_{k-1}$ via (4) and (1) yields the half-step bound $P_k - P_{k-1} \leq 1 + \sum_{j<k, j\equiv k-1 (2)} X_j \leq t+1$ (`Hypercube.lemma18_half_step`). Since $H_k - H_{k-1} = \binom{n-1}{k}$, induction gives $b_k \geq p_k$ where $p_1 = n - 2 - x_w$ and

$$p_k = [p_{k-1} + \binom{n-1}{k} - (t+1)]_+ \qquad (2 \leq k \leq n-2). \tag{7}$$

Now choose a valid level assignment $\kappa(m) \in \{2, \ldots, m+1\}$ for each positive $m \in R$, and let u_k count how many times k is selected. Applying $Y_m \geq s_{\kappa(m)} - 2$ in (3) and substituting (6) gives $M(w) \geq K + \sum_{k=2}^{n-2} (u_{k+2} - u_k) b_k$, where

$$K = x_w \mathbf{1}_{\{n \text{ even}\}} + u_3(n - 2 - x_w) - \left(\frac{N}{2} - n\right) + \sum_{m\in R, m>0} \left(\binom{n}{\kappa(m)} - 2\right).$$

For any $\alpha \geq 0$ with $A_k = 2x_w + \alpha(u_{k+2} - u_k) \geq 0$ ($2 \leq k \leq n-2$), combining with $M(w) \leq \mu$ from Theorem 6 yields the local floor (`Hypercube.theorem13_capped_floor`)

$$\psi(w) \geq 2x_w(n - 2 - x_w) + \alpha K - [\alpha - 2]_+ \mu + \sum_{k=2}^{n-2} A_k p_k. \tag{8}$$

A finite integer check `capCertOk(n, t, P)` tests two standard assignments κ (central and staggered) and $\alpha \in \{0, 2, 2x\}$ to verify that for every $x \in \{1, \ldots, n-2\}$, the maximum right-hand side of (8) is at least $xP/(n-2)$, while $2tP > (n-2)N\mu$. Summing over $w \in Q_n$ contradicts Lemma 5, ruling out size $N + t$ and proving $f(n) \geq N + t + 1$.

Theorem 11 (Certified Lower Bounds for $17 \le n \le 32$). *Kernel-checked antipodal-cap certificates establish all sixteen lower bounds for* $17 \le n \le 32$ *in Table 2, reaching* $f(32) \ge 4{,}454{,}492{,}243$.

The same kernel-checked family proves the eight lower bounds for $9 \le n \le 16$ in Table 2 individually in `lean/Hypercube/StrongerBounds.lean` (`f_nine_ge_580` through `f_sixteen_ge_70357`).

Table 2. Antipodal-cap lower bounds $f(n) \ge 2^n + t + 1$ for $9 \le n \le 32$, each verified in `lean/Hypercube/StrongerBounds.lean` and rechecked by `recompute.py`.

n	Bound	n	Bound	n	Bound
9	580	17	140,128	25	35,138,325
10	1,147	18	279,210	26	70,152,107
11	2,271	19	556,449	27	140,081,100
12	4,503	20	1,110,320	28	279,786,071
13	8,941	21	2,214,490	29	558,766,740
14	17,753	22	4,418,155	30	1,116,183,007
15	35,327	23	8,816,922	31	2,229,577,178
16	70,357	24	17,599,972	32	4,454,492,243

Applying the identical certificate bridge (`Hypercube.theorem14_from_cert`) to dimensions 6 and 8 tightens the earlier library bounds ($f(6) \ge 75$ and $f(8) \ge 292$) to 76 and 294.

Proposition 12 (Certificate Bridge and $f(6) \ge 76$, $f(8) \ge 294$ — Lean + Cert). *For any triple* (n, t, P) *satisfying* `capCertOk(n, t, P) = true`, $f(n) \ge 2^n + t + 1$. *Evaluating* `capCertOk` *at* $(6, 11, 54)$ *gives per-excess floors* $(28, 38, 48, 54)$ *with* $2tP = 1{,}188 > 768 = (n-2)N\mu$, *proving* $f(6) \ge 76$; *evaluating at* $(8, 37, 554)$ *gives floors* $(134, 226, 314, 398, 478, 554)$ *with* $2tP = 40{,}996 > 29{,}184 = (n-2)N\mu$, *proving* $f(8) \ge 294$.

In dimension 7, choosing the three-radius assignment $\kappa(1) = 2, \kappa(3) = 3, \kappa(5) = 4$ improves `Hypercube.f_seven_ge_148` ($f(7) \ge 148$) to 150.

Proposition 13 (Three-Radius Dimension-7 Bound $f(7) \ge 150$ — Cert Only). *No diameter-7 spanning subgraph of* Q_7 *has* 149 *edges; hence* $f(7) \ge 150$.

Proof. Assume $|E(G)| = 149$, so $t = 21$ and $\mu = 12$. Taking $\kappa(1) = 2, \kappa(3) = 3, \kappa(5) = 4$ in (3) and substituting (6) cancels b_2 and yields $57 + M(w) \ge s_2 + s_3 + s_4 - 6 = 90 - x_w - b_3 - b_4$, since $\binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 91$ and $b_1 = 5 - x_w$. Thus $\delta(w) \ge 2(b_1 + b_3 + b_4) \ge 76 - 4x_w - 2M(w)$. Applying $M(w) \le 12$ from Theorem 6 for $1 \le x_w \le 5$ gives $\psi(w) \ge 52x_w - 4x_w^2 + 24 \ge 184x_w/5$, since $5(52x_w - 4x_w^2 + 24) - 184x_w = (5 - x_w)(20x_w + 24) \ge 0$ (and $\psi(w) \ge 0$ at $x_w = 0$). Summing over $w \in Q_7$ gives $7{,}680 = 5N\mu \ge 184 \cdot 2t = 7{,}728$, contradiction. □

Both Propositions 12 and 13 are checked independently by `recompute.py` and were also kernel-discharged in Lean 4 in `papers/hypercube/draft3/{SmallCertificates,Seven}.lean` and the Palomar-registered standalone proof `Challenge.lean` (`PALOMAR-2026-11-22-000030` v1); relative to `lean/Hypercube/` alone, they are classified honestly as *Lean + Cert* and *Cert Only* in Section 5.

Proposition 14 (External Q_4 Extremal Census — Cert Only). *Independent exact enumerations confirm* 0 *valid* 19*-edge subgraphs of* Q_4 *and exactly* 8,320 *labelled* 20*-edge valid subgraphs partitioned into* 30 *isomorphism classes under cube automorphisms:* 15 *with stabilizer order* 1, 13 *with stabilizer order* 2, *and* 2 *with stabilizer order* 12 ($15 \cdot 384 + 13 \cdot 192 + 2 \cdot 32 = 8{,}320$).

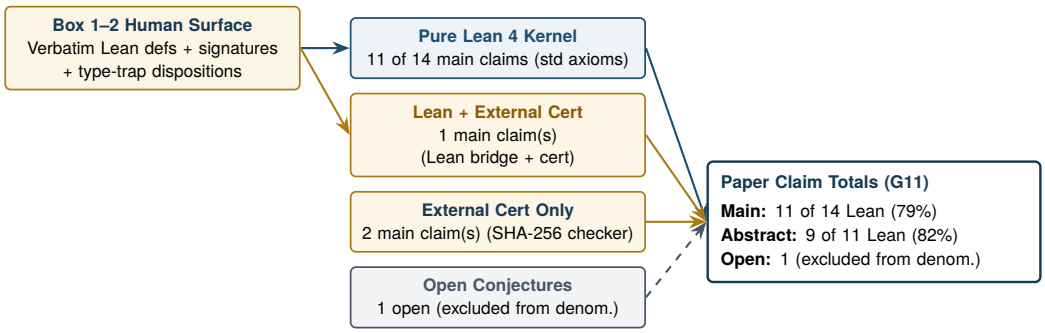
Conjecture 15 (Open Value of $f(5)$ and Asymptotic Excess Order). *Whether* $f(5) = 39$ *or* $f(5) = 40$, *and the exact asymptotic growth rate and leading constant of* $f(n) - 2^n$ *between* $\Omega(2^n/n)$ *and* $O(2^n/\sqrt{n})$, *remain open.*

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 0533df5120e9 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Open in Table 4 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 237/237 public theorems (source-level approx) | WARN: Fidelity: 12/12 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

Hypercube.CubeVertex (Hypercube/Defs.lean:30-30) — Section 2 (binary words {0,1}^n)

```
abbrev CubeVertex (n : ℕ) := Fin n → Bool
```

Hypercube.Q (Hypercube/Defs.lean:34-38) — Section 1 & 2 (n-dimensional hypercube Q_n)

```
def Q (n : ℕ) : SimpleGraph (CubeVertex n) where
  Adj u v := hammingDist u v = 1
  symm := ⟨fun u v (h : hammingDist u v = 1) => (hammingDist_comm v u).trans h⟩
  loopless := ⟨fun u (h : hammingDist u u = 1) =>
    absurd ((hammingDist_self u).symm.trans h) zero_ne_one⟩
```

Hypercube.IsValidSpanningSubgraph (Hypercube/Defs.lean:47-48) — Section 1 & 2 (diameter-n spanning subgraph of Q_n)

```
def IsValidSpanningSubgraph (n : ℕ) (G : SimpleGraph (CubeVertex n)) : Prop :=
  G ≤ Q n ∧ ∀ u v : CubeVertex n, G.Reachable u v ∧ G.edist u v ≤ n
```

Hypercube.edgeCount (Hypercube/Defs.lean:51-52) — Section 1 (edge count |E(G)|)

```
def edgeCount {n : ℕ} (G : SimpleGraph (CubeVertex n)) [DecidableRel G.Adj] : ℕ :=
  G.edgeFinset.card
```

Hypercube.f (Hypercube/Defs.lean:55-57) — Section 1 (minimum edge count f(n) via sInf)

```
noncomputable def f (n : ℕ) : ℕ :=
  sInf { e : ℕ | ∃ (G : SimpleGraph (CubeVertex n)) (_ : DecidableRel G.Adj),
    IsValidSpanningSubgraph n G ∧ edgeCount G = e }
```

Hypercube.antipode (Hypercube/Defs.lean:98-99) — Section 1 (bitwise complement w-bar)

```
def antipode {n : ℕ} (x : CubeVertex n) : CubeVertex n :=
  fun i => !x i
```

Hypercube.excess (Hypercube/Defs.lean:430-431) — Section 1 & 2 (vertex degree excess x_v = deg(v) - 2)

```
def excess {n : ℕ} (G : SimpleGraph (CubeVertex n)) (u : CubeVertex n) : ℤ :=
  ((G.neighborSet u).ncard : ℤ) - 2
```

Hypercube.edgeExcess (Hypercube/Defs.lean:467-468) — Section 1 & 2 (graph edge excess t = |E| - 2^n)

```
def edgeExcess {n : ℕ} (G : SimpleGraph (CubeVertex n)) : ℤ :=
  (G.edgeSet.ncard : ℤ) - 2 ^ n
```

Hypercube.distGap (Hypercube/Defs.lean:463-464) — Section 1 & 2 (distance detour delta(w))

```
def distGap {n : ℕ} (G : SimpleGraph (CubeVertex n)) (w : CubeVertex n) : ℤ :=
  distSum G w - (n : ℤ) * 2 ^ (n - 1)
```

Hypercube.weightedExtra (Hypercube/Defs.lean:451-452) — Section 1 & 2 (weighted merge count M(w))

```
def weightedExtra {n : ℕ} (G : SimpleGraph (CubeVertex n)) (v : CubeVertex n) : ℤ :=
  ∑ j ∈ Ico 2 n, ((n : ℤ) - j) * extraParents G v j
```

Hypercube.psi (Hypercube/StrongerBounds.lean:741-742) — Lemma 5 (vertex potential psi(w) = x_w delta(w) + 2 M(w))

```
def psi {n : ℕ} (G : SimpleGraph (CubeVertex n)) (w : CubeVertex n) : ℤ :=
  excess G w * distGap G w + 2 * weightedExtra G w
```

Hypercube.meanPsi (Hypercube/StrongerBounds.lean:2327-2328) — Section 1 & Theorem 6 (mean potential mu = (n-1)t - (2^n - 2n))

```
def meanPsi (n : ℕ) (G : SimpleGraph (CubeVertex n)) : ℤ :=
  ((n : ℤ) - 1) * edgeExcess G - ((2 : ℤ) ^ n - 2 * (n : ℤ))
```

Hypercube.antipodalDetour (Hypercube/StrongerBounds.lean:2331-2332) — Theorem 6 (antipodal detour eta_w(v) = d(v,w) + d(v,w-bar) - n)

```
def antipodalDetour {n : ℕ} (G : SimpleGraph (CubeVertex n)) (w v : CubeVertex n) : ℤ
  ↪ :=
  (G.dist v w : ℤ) + (G.dist v (antipode w) : ℤ) - (n : ℤ)
```

Hypercube.blockingCount (Hypercube/StrongerBounds.lean:494-496) — Proposition 7 (blocking vertex count beta_r(G))

```
noncomputable def blockingCount {n : ℕ} (G : SimpleGraph (CubeVertex n))
    (r : CubeVertex n) : ℕ :=
  {u : CubeVertex n | isBlockingVertex G r u}.ncard
```

Hypercube.bound1995 (Hypercube/Theorem1.lean:36-37) — Corollary 2 (1995 lower bound 2^n - 1 + ceil((2^n-1)/(2n-1)))

```
noncomputable def bound1995 (n : ℕ) : ℤ :=
  (2 : ℤ) ^ n - 1 + ⌈((2 : ℚ) ^ n - 1) / (2 * (n : ℚ) - 1)⌉
```

Hypercube.capCertOk (Hypercube/StrongerBounds.lean:2849-2855) — Section 4 & Proposition 12 (finite antipodal-cap certificate predicate)

```
def capCertOk (n t3 : ℕ) (P : ℤ) : Bool :=
  decide (4 ≤ n) &&
    decide (0 ≤ P) &&
    decide (2 * P < ((n : ℤ) - 2) * (2 : ℤ) ^ n * ((n : ℤ) - 1)) &&
    decide (((n : ℤ) - 2) * (2 : ℤ) ^ n *
      (((n : ℤ) - 1) * (t3 : ℤ) - ((2 : ℤ) ^ n - 2 * (n : ℤ))) < 2 * (t3 : ℤ) * P) &&
    allRange (fun x => if 1 ≤ x then excessCoveredBool n t3 P x else true) (n - 1)
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): Hypercube.f_two_eq_four (f 2 = 4); Hypercube.f_three_eq_nine (f 3 = 9); Hypercube.q6_81_edgeCount (edgeCount q6_81 = 81); Hypercube.grahamConstruction_six_edgeCount (edgeCount (grahamConstruction 6) = 82).

Supporting Semantic Closure Index (non-Box-1 helpers): Hypercube.allRange (Hypercube/StrongerBounds.lean), Hypercube.candidateOk (Hypercube/StrongerBounds.lean), Hypercube.cappedFloorFast (Hypercube/StrongerBounds.lean), Hypercube.distSum (Hypercube/Defs.lean), Hypercube.excessCoveredBool (Hypercube/StrongerBounds.lean), Hypercube.extraParents (Hypercube/Defs.lean), Hypercube.fastChoose (Hypercube/StrongerBounds.lean), Hypercube.hasDownNeighbor (Hypercube/StrongerBounds.lean), Hypercube.hasUpNeighbor (Hypercube/StrongerBounds.lean), Hypercube.isBlockingVertex (Hypercube/StrongerBounds.lean), Hypercube.levelSet (Hypercube/Defs.lean), Hypercube.parentCount (Hypercube/Defs.lean), Hypercube.recipeCountFast (Hypercube/StrongerBounds.lean), Hypercube.recipeF (Hypercube/StrongerBounds.lean), Hypercube.recipeH (Hypercube/StrongerBounds.lean), Hypercube.recipeHalfKFast (Hypercube/StrongerBounds.lean), Hypercube.sumFilterRange (Hypercube/StrongerBounds.lean), Hypercube.sumWeightedChainAux (Hypercube/StrongerBounds.lean), Hypercube.weightsNonnegBool (Hypercube/StrongerBounds.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

Hypercube.theorem1_f_lower_bound_nat (Hypercube/Theorem1.lean:241-243) src:1d73193c0a661f

```
theorem theorem1_f_lower_bound_nat {n : ℕ} (hn : 2 ≤ n) :
    2 ^ n + (2 ^ n - 2 * n + (n - 2)) / (n - 1) ≤ f n
```

Assumptions: Integer n >= 2. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.not_attained_1995 (Hypercube/Theorem1.lean:287-289) src:d6ab6f0a1800a0

```
theorem not_attained_1995 {n : ℕ} (hn : 4 ≤ n) :
    bound1995 n < (f n : ℤ)
```

Assumptions: Integer n >= 4. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.theorem5_asymptotic_three_halves (Hypercube/StrongerBounds.lean:1778-1827) src:37d1fc05965daf

```
theorem theorem5_asymptotic_three_halves :
    ∀ ε : ℚ, 0 < ε → ∃ N : ℕ, ∀ n : ℕ, N ≤ n →
      ((3 : ℚ) / 2 - ε) * (2 : ℚ) ^ n ≤ ((f n : ℚ) - (2 : ℚ) ^ n) * ((n : ℚ) - 1)
```

Assumptions: Positive rational epsilon > 0 (yields explicit threshold N(epsilon)). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.f_le_construction (Hypercube/Construction.lean:710-714) src:d2723cec4c6271

```
theorem f_le_construction {n : ℕ} (hn : 1 ≤ n) :
    f n ≤ 2 ^ n + n.choose (n / 2) - 2
```

Assumptions: Integer n >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.sum_psi (Hypercube/StrongerBounds.lean:756-764) src:58898df8109457

```
theorem sum_psi {n : ℕ} {G : SimpleGraph (CubeVertex n)}
    (hn : 2 ≤ n) (hV : IsValidSpanningSubgraph n G) :
    ∑ w : CubeVertex n, psi G w =
      (2 : ℤ) ^ n * (((n : ℤ) - 1) * edgeExcess G - ((2 : ℤ) ^ n - 2 * (n : ℤ)))
```

Assumptions: Integer n >= 2 and valid spanning subgraph G of Q_n. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.lemma20_antipodal_identity_div2 (Hypercube/StrongerBounds.lean:2406-2421) src:0126b954a5c84b

```
theorem lemma20_antipodal_identity_div2 {n : ℕ} (hn : 2 ≤ n) {G : SimpleGraph
  ↪ (CubeVertex n)}
    (hV : IsValidSpanningSubgraph n G) (w : CubeVertex n) :
    weightedExtra G w + weightedExtra G (antipode w) +
        ∑ v : CubeVertex n, excess G v * (antipodalDetour G w v / 2) =
      meanPsi n G
```

Assumptions: Integer n >= 2, valid spanning subgraph G of Q_n, root w in Q_n. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.blocking_lower_bound_valid (Hypercube/StrongerBounds.lean:710-725) src:06b058a4f651a4

```
theorem blocking_lower_bound_valid {n : ℕ} (hn : 1 ≤ n)
    {G : SimpleGraph (CubeVertex n)} [DecidableRel G.Adj]
    (hV : IsValidSpanningSubgraph n G) (r : CubeVertex n) :
    2 ^ n + n.choose (n / 2) - 2 ≤ edgeCount G + blockingCount G r
```

Assumptions: Integer n >= 1, valid spanning subgraph G of Q_n, corner r in Q_n. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Hypercube.f_four_eq_twenty (Hypercube/SmallDims.lean:876-877) src:803b4375defa62

```
theorem f_four_eq_twenty : f 4 = 20
```

Assumptions: None beyond Box 1 definitions. Applicability: Unconditional (closed statement; no outer premise witness needed).

Hypercube.f_five_bounds (Hypercube/SmallDims.lean:1619-1620) src:20e3b847050626

```
theorem f_five_bounds : 39 ≤ f 5 ∧ f 5 ≤ 40
```

Assumptions: None beyond Box 1 definitions. Applicability: Unconditional (closed statement; no outer premise witness needed).

Hypercube.f_six_le_81 (Hypercube/StrongerBounds.lean:458-460) src:840ffa1c63581f

```
theorem f_six_le_81 : f 6 ≤ 81
```

Assumptions: None beyond Box 1 definitions (explicit 81-edge witness verified in kernel). Applicability: Unconditional (closed statement; no outer premise witness needed).

Hypercube.theorem14_bounds_17_to_32 (Hypercube/StrongerBounds.lean:3024-3034) src:cbb50ba92c7498

```
theorem theorem14_bounds_17_to_32 :
    140128 ≤ f 17 ∧ 279210 ≤ f 18 ∧ 556449 ≤ f 19 ∧ 1110320 ≤ f 20 ∧
    2214490 ≤ f 21 ∧ 4418155 ≤ f 22 ∧ 8816922 ≤ f 23 ∧ 17599972 ≤ f 24 ∧
    35138325 ≤ f 25 ∧ 70152107 ≤ f 26 ∧ 140081100 ≤ f 27 ∧ 279786071 ≤ f 28 ∧
    558766740 ≤ f 29 ∧ 1116183007 ≤ f 30 ∧ 2229577178 ≤ f 31 ∧ 4454492243 ≤ f 32
```

Assumptions: None beyond Box 1 definitions (16 certificates discharged inside kernel). Applicability: Unconditional (closed statement; no outer premise witness needed).

Hypercube.theorem14_from_cert (Hypercube/StrongerBounds.lean:2888-2936) src:fad899bb032ded

```
theorem theorem14_from_cert (n t3 : ℕ) (P : ℤ) (hcert : capCertOk n t3 P = true) :
    2 ^ n + t3 + 1 ≤ f n
```

Assumptions: Triple (n, t, P) with capCertOk n t P = true (checked at (6, 11, 54) and (8, 37, 554) by recompute.py and Palomar Challenge.lean). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 3. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
blocking_lower_bound_valid	classical_choose	Classical.choose	intended_convention	Nat.choose binomial coeff (not Classical.choose).
blocking_lower_bound_valid	nat_div	/ 2	intended_convention	Floor division n / 2 = floor(n/2).
blocking_lower_bound_valid	nat_sub	- 2	intended_convention	1 <= n prevents Nat underflow in g(n).
bound1995	nat_sub	- 1	intended_convention	Signed Z/Q subtraction (no Nat underflow).
bound1995	nat_div	/ (2 * (n : ℚ) - 1)	intended_convention	Exact Q division by 2n - 1 > 0.
capCertOk	nat_sub	- 1	intended_convention	decide (4 <= n) guards Nat n - 1; rest in Z.
cappedFloorFast	nat_sub	- max	intended_convention	Signed Z subtraction for cap penalty.
cappedFloorFast	nat_sub	- 2	intended_convention	Signed Z terms; guarded by n >= 4.
cappedFloorFast	nat_sub	- recipeCountFast	intended_convention	Signed Z difference u_{k+2} - u_k.
cappedFloorFast	nat_sub	- 3	intended_convention	Guarded by 4 <= n in capCertOk.
distGap	nat_sub	- 1	intended_convention	Evaluated under explicit 2 <= n.
f_le_construction	classical_choose	Classical.choose	intended_convention	Nat.choose binomial coeff (not Classical.choose).
f_le_construction	nat_div	/ 2	intended_convention	Floor division n / 2 = floor(n/2).

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
f_le_construction	nat_sub	- 2	intended_convention	1 <= n prevents Nat underflow in g(n).
fastChoose	nat_sub	- k	intended_convention	Proved = Nat.choose in fastChoose_eq_choose.
fastChoose	nat_div	/ (k + 1)	intended_convention	Exact recurrence div (fastChoose_eq_choose).
lemma20_antipodal_identity_div2	nat_div	/ 2	intended_convention	Even nonnegative detour (exact / 2).
recipeCountFast	nat_mod	% 2	intended_convention	Standard parity check mod 2.
recipeCountFast	nat_sub	- 1	intended_convention	Guarded by 4 <= n in capCertOk.
recipeF	nat_mod	% 2	intended_convention	Standard parity branch n % 2 = 0.
recipeF	nat_sub	- 1	intended_convention	Evaluated on positive odd radii m >= 1.
recipeF	nat_div	/ 2	intended_convention	Standard exact halving by 2.
recipeH	nat_div	/ 2	intended_convention	Standard integer division by 2.
recipeH	nat_sub	- 1	intended_convention	Guarded by 4 <= n in capCertOk.
recipeH	nat_mod	% 2	intended_convention	Standard parity branch mod 2.
recipeH	nat_sub	- m0	intended_convention	Else branch enforces m >= m0.
recipeHalfKFast	nat_mod	% 2	intended_convention	Standard parity indicator n % 2 = 0.
recipeHalfKFast	nat_sub	- 1	intended_convention	Guarded by 4 <= n in capCertOk.
theorem1_f_lower_bound_nat	nat_sub	- 2	intended_convention	2 <= n prevents Nat underflow.
theorem1_f_lower_bound_nat	nat_div	/ (n - 1)	intended_convention	2 <= n gives nonzero divisor n - 1.
theorem1_f_lower_bound_nat	nat_sub	- 1	intended_convention	2 <= n prevents Nat underflow.
theorem5_asymptotic_three_halves	nat_div	/ 2	intended_convention	Exact Q constant 3 / 2.
theorem5_asymptotic_three_halves	nat_sub	- ε	intended_convention	Signed Q subtraction 3/2 - epsilon.
theorem5_asymptotic_three_halves	nat_sub	- (2 : ℚ)	intended_convention	Signed Q subtraction f(n) - 2^n.
theorem5_asymptotic_three_halves	nat_sub	- 1	intended_convention	Signed Q subtraction n - 1.
weightsNonnegBool	nat_sub	- recipeCountFast	intended_convention	Signed Z difference u_{k+2} - u_k.
weightsNonnegBool	nat_sub	- 1	intended_convention	Guarded by 4 <= n in capCertOk.
instDecidableGrahamDirEdge	local_instance_or_notation	instDecidableGrahamDirEdge	intended_convention	Decidable instance for finite directed edge.
instDecidableRelGrahamAdj	local_instance_or_notation	instDecidableRelGrahamAdj	intended_convention	DecidableRel for symmetric-chain graph.
instDecidableRelQAdj	local_instance_or_notation	instDecidableRelQAdj	intended_convention	DecidableRel for Q_n Hamming-1 adjacency.
instDecidableRelQ681Adj	local_instance_or_notation	instDecidableRelQ681Adj	intended_convention	DecidableRel for 64-vertex Q_6 witness.

[a]Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Universal lower bound f(n) >= 2^n + ceil((2^n - 2n)/(n - 1)) for all n >= 2	Lean Kernel	Hypercube.theorem1_f_lower_bound_nat	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (integer n >= 2).
Cor. 2: Strict improvement over 1995 bound for all n >= 4 (refutes Bass-Sudborough Table 1)	Lean Kernel	Hypercube.not_attained_1995	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (integer n >= 4).
Thm. 3: Asymptotic 3/2 second-term lower bound lim inf (n-1)(f(n)-2^n)/2^n >= 3/2	Lean Kernel	Hypercube.theorem5_asymptotic_three_halves	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (unconditional rational quantifier statement).
Prop. 4: Symmetric-chain upper bound f(n) <= 2^n + C(n, floor(n/2)) - 2 for all n >= 1	Lean Kernel	Hypercube.f_le_construction	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (integer n >= 1).

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Lem. 5: Global vertex-potential conservation sum_w (x_w delta(w) + 2 M(w)) = 2^n mu	Lean Kernel	`Hypercube.sum_psi`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (n >= 2, valid G <= Q_n).
Thm. 6: Antipodal pair conservation M(w) + M(w-bar) + sum_v x_v (eta_w(v)/2) = mu	Lean Kernel	`Hypercube.lemma20_antipodal_identity_div2`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (n >= 2, valid G <= Q_n, root w).
Prop. 7: Blocking-vertex converse g(n) <= |E(G)| + beta_r(G) for every corner r	Lean Kernel	`Hypercube.blocking_lower_bound_valid`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (n >= 1, valid G <= Q_n, corner r).
Prop. 8: Exact fourth-cube minimum f(4) = 20	Lean Kernel	`Hypercube.f_four_eq_twenty`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (unconditional).
Thm. 9: Dimension-5 bounds 39 <= f(5) <= 40	Lean Kernel	`Hypercube.f_five_bounds`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (unconditional).
Prop. 10: Explicit 81-edge Q_6 witness proving f(6) <= 81 < g(6) = 82	Lean Kernel	`Hypercube.f_six_le_81`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (explicit witness verified in kernel).
Thm. 11: Certified lower bounds for 17 <= n <= 32 reaching f(32) >= 4,454,492,243	Lean Kernel	`Hypercube.theorem14_bounds_17_to_32`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (16 certificates discharged in kernel; n=9..16 individual theorems).
Prop. 12: Certificate bridge capCertOk => f(n) >= 2^n+t+1; f(6) >= 76 and f(8) >= 294	Lean + Cert	`Hypercube.theorem14_from_cert` + recompute.py (97959671e747) [archived; replay pending]	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel bridge + recompute.py checks at (6, 11, 54) and (8, 37, 554) (also in Palomar Challenge.lean).
Prop. 13: Three-radius dimension-7 lower bound f(7) >= 150	Cert Only	recompute.py (97959671e747) [archived; replay pending]	External cert only; fidelity: n/a	Exact arithmetic check in recompute.py (Lean library proves f(7) >= 148 & local floor; f(7) >= 150 also in Palomar Challenge.lean).
Prop. 14: External Q_4 extremal census (0 at 19 edges; 8,320 labelled in 30 classes at 20 edges)	Cert Only	recompute.py (97959671e747) [archived; replay pending]	External cert only; fidelity: n/a	External three-program exact enumeration & orbit reconstruction (recompute.py + draft3/crosscheck/census.cpp).
Conj. 15: Exact value of f(5) in {39, 40} and asymptotic order of f(n) - 2^n	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open problem (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 759487129ef0 | Lean mode: committed_receipt | Closure SHA-256: 0c45d2f24d9f | Git anchor: 0533df5120e9 | Axioms SHA-256: 1cdaeb3e1bdc | Fidelity SHA-256: 57020e2437ee. External validation receipts: Pinned commit | Palomar record.

6. PRIOR WORK AND OPEN QUESTIONS

Graham and Harary introduced diameter-preserving edge deletion on hypercubes [6] [to check]. Bouabdallah, Delorme, and Djelloul established the classical upper bound $g(n) = 2^n + \binom{n}{\lfloor n/2 \rfloor} - 2$ and the lower bound $2^n - 1 + \lceil (2^n - 1)/(2n - 1) \rceil$ [3] [to check], whose second-term coefficient on the scale $2^n/(n-1)$ is $1/2$; both bounds are recorded in full-text survey material by Wang, Ho, Ferrero, and Sung [8]. Nenadov, Sawhney, Sudakov, and Wagner [7] restated the edge-minimization question (attributing it to Erdős, Hamburger, Pippert, and Weakley) and proved a lower bound of order $2^n/n$ on $f(n) - 2^n$. Symmetric-chain decompositions go back to de Bruijn, van Ebbenhorst Tengbergen, and Kruyswijk [5].

Bass and Sudborough [1] [to check] and [2] printed edge-removal tables leaving $18, 35, 69, 137, 272$ surviving edges in dimensions 4 through 8 (matching the 1995 lower bound); Corollary 2 proves none is attained. Charton, Ellenberg, Wagner, and Williamson [4] found the 81-edge valid subgraph of Q_6 beating $g(6) = 82$. Full texts of [8, 7, 5, 2] and Section 3.3/Appendix F of [4] were read directly; [6], [3] (with its 1990 Orsay report and Djelloul's 1993 thesis), and pp. 137–146 of [1] remain unread ([to check]), so no unconditional priority is claimed over them. Two questions remain open (Conjecture 15): deciding $f(5) \in \{39, 40\}$ and closing the asymptotic gap between $\frac{3}{2} \cdot 2^n/(n-1)$ and $\binom{n}{\lfloor n/2 \rfloor}$.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript files live in `papers/hypercube/draft4/` and the Lean 4 library in `lean/Hypercube/` (237 audited public theorems, zero `sorry`, zero `native_decide`, standard axioms only). The standalone Palomar submission (`PALOMAR-2026-11-22-000030`

v1, mechanical Lean checking plus automated editorial review) and its pinned Lean source (`Challenge.lean`) also verify $f(6) \geq 76$, $f(7) \geq 150$, and $f(8) \geq 294$ inside the Lean kernel. Running `python3 papers/hypercube/draft4/recompute.py` executes the Section 1 Python box verbatim, verifies all 29 integer certificates for $4 \leq n \leq 32$, checks the Q_7 three-radius inequality, and validates the Q_3 identities and Q_4 orbit totals. AI research agents built and directed by Tyler B. Hughes generated and checked the proofs, Lean formalization, figures, and draft text. No human mathematician has yet refereed the proof.

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