AI-assisted research draft for author and mathematical review

Last-Digit Collatz Maps Across All Bases

Competing basins in base ten, and why it is the unique twin of Collatz

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Abstract

Fix an integer base $b \geq 2$, let $d_b(n) = n \bmod b$, and define $f_b(n) = n/2$ for even n and $f_b(n) = d_b(n)\, n + 1$ for odd n. Base 4 recovers the classical shortcut Collatz map verbatim on odd integers $n = 2y+1$, with Collatz's excluded fixed point $y = 0$ appearing as an isolated $1 \leftrightarrow 2$ cycle whose basin is strictly the powers of two. Among all other bases, base 10—the *Ouroboros map*—stands out as the sole near-critical contracting twin of Collatz, yet its barrier around $1 \leftrightarrow 2$ breaks open, creating a contest between two infinite cycles through 1 and 5 with no analogue in $3x + 1$.

In base 10, we prove exponential density-one descent ($p_k(10) \leq 1.124953\,(0.996423)^k$) and separate the competing basins unconditionally: $0.0145\, X^{0.49875} \leq N_1(X) \leq C\, X^{0.6245}$ alongside $N_5(X) > 0.02134\, X^{0.65}$ ($X \geq 3$), so $N_1(X)/N_5(X) = O(X^{-0.0255}) \to 0$. Swapping two backward digit-1 moves yields an exact $+30$ modular shift, proving that every congruence class $a \bmod m$ meets the basin of 5 infinitely often while an exact finite halving test governs the basin of 1. Written in any base $b = 2^t u$ (u odd), the endings of the density-zero basin of 1 close to a set of Hausdorff and box dimension $(\log u + t \log \lambda)/\log b$—0.626382 in binary (strictly above 0.6245), 0.887530 in decimal, and 1 (every finite ending!) in base 7. We exclude further cycles with ≤ 8 odd steps, verify attraction through 10^{15} (forcing $K \geq 34{,}905$, $m \geq 15{,}496$, $K + m \geq 53{,}185$), prove limiting non-descent vanishes among even bases precisely for $b \in \{2, 4, 6, 10\}$, and resolve density-one odd-base behavior in 905 of 999 odd bases through 1,999. Core proofs are formalized in Lean 4 and an offline archive replays every certificate; universal base-10 attraction and the exact exponent of $N_1(X)$ remain open.

MSC 2020 11B83, 11Y16, 11Y55, 37P99, 60J10. **Keywords** Generalized Collatz maps, 2-adic Cantor sets, stopping times, Markov chains, inverse trees, natural density.

1 One rule, nineteen bases

For any integer base $b \geq 2$, let $d_b(n) = n \bmod b \in \{0, 1, \ldots, b-1\}$ denote the final base-b digit of a positive integer n. Consider the iteration

$$f_b(n) = \begin{cases} n/2, & n \text{ even}, \\ d_b(n)\, n + 1, & n \text{ odd}. \end{cases} \tag{1}$$

An even integer halves as in the classical $3x+1$ problem; an odd integer reads its own last digit as its multiplier. Unlike fixed-multiplier generalizations of Collatz, the rule carries no external parameters beyond the base b.

Hold sixty starting integers fixed in $[10^9, 10^{10})$ and watch their *orbit values* $f_b^k(n)$ unfold as b steps from 2 through 20 (Figure 1). Three sharply distinct regimes appear at a glance. In bases 2 and 6, every curve collapses almost immediately. In bases 4 and 10, the curves wander irregularly for hundreds of steps before landing in a cycle (median entry times 279 and 353.5). In every other even base—8, 12, 14, 16, 18, 20—trajectories climb away, with base 14 hovering so close to the threshold that 6 of the 60 starts still drop into a cycle while all 54 crossings above 10^{60} happen after step 1,500. Odd bases behave altogether differently: orbits either drop straight to 1 on a green triangle (an odd multiple of b, whose last digit is 0, sending $0 \cdot n + 1 = 1$ in a single step) or shoot nearly vertically past 10^{60}.

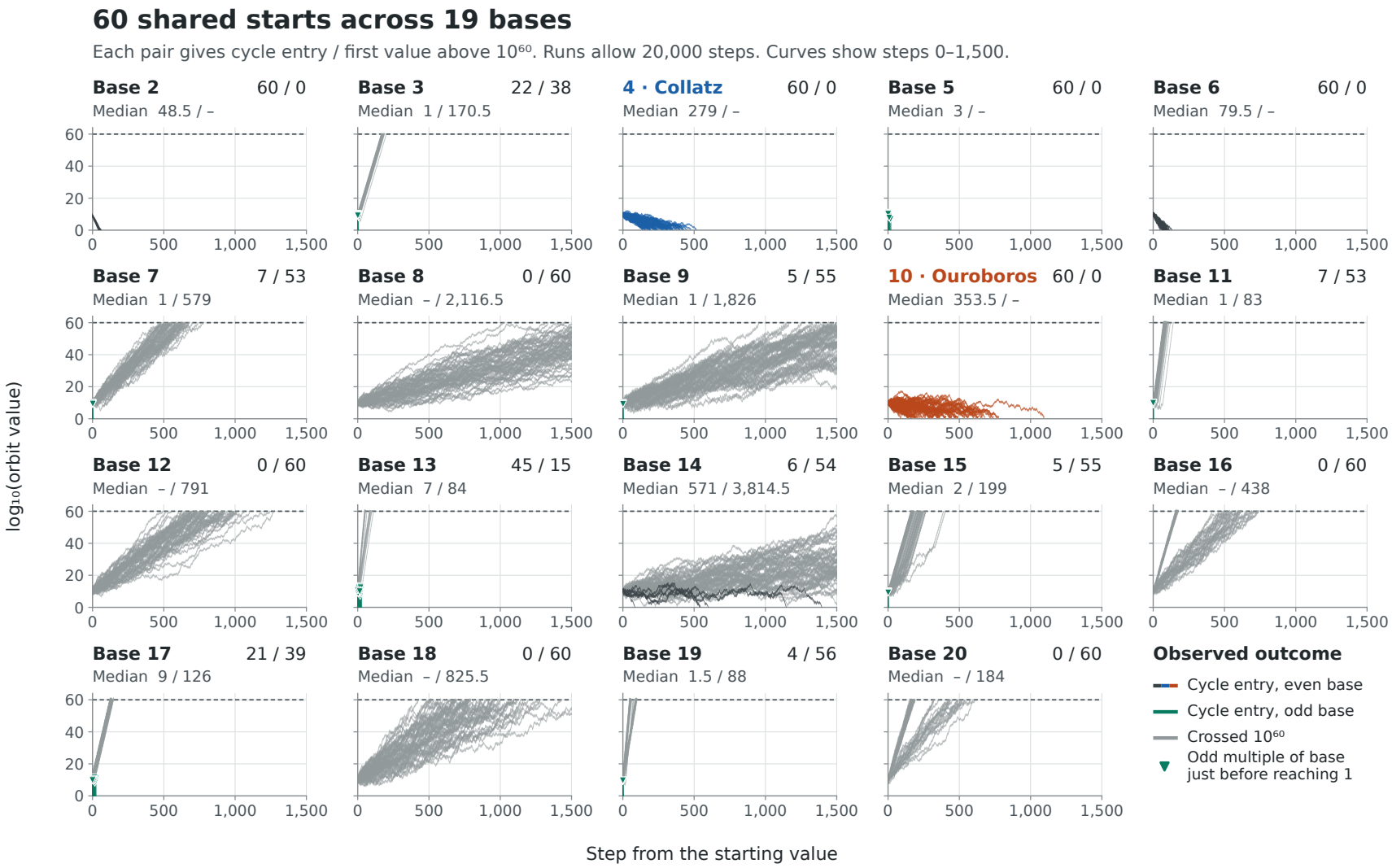

Figure 1: Sixty shared starts in $[10^9, 10^{10})$ iterated across bases $b = 2$ through 20. At step k, the *orbit value* is the k-th iterate $f_b^k(n)$, where $f_b(n) = n/2$ for even n and $f_b(n) = (n \bmod b)\, n + 1$ for odd n. Horizontal distance counts ordinary steps k; vertical height plots $\log_{10} f_b^k(n)$. Curves are drawn through step 1,500, while panel summaries report the full run until cycle entry or first crossing above 10^{60} (limit 20,000 steps). In base 14, six starts enter a cycle within the window and all 54 crossings above 10^{60} occur beyond step 1,500. A threshold crossing records a finite excursion, not a proof of divergence.

Now fix the starting range $1 \le n \le 20{,}000$ and look inward toward the cycles themselves (Figure 2). Across bases 2 through 11, two panels wear bright solid rings: *only bases* 4 *and* 10 *send every tested start into exactly two loops with zero escapes.* Bases 2, 5, and 6 send all 20,000 starts into the single loop $1 \to 2 \to 1$; bases $3, 7, 8, 9, 11$ lose large fractions past 2^{256}, with bases 7 and 8 already hosting three and four cycles. Yet the two brightly ringed panels—labeled *base* 4 *(Collatz)* and *base* 10 *(Ouroboros)*—pose an immediate visual puzzle. Compare the amber loop of 1 at the heart of each panel:

- In base 4, the loop of 1 attracts just 15 of the 20,000 starts along a single unbranched amber spike, while the remaining 19,985 starts spiral into the blue loop of 3.
- In base 10, the loop of 1 attracts 167 starts arranged in a *branching* amber tree, while 19,833 starts enter the blue loop of 5.

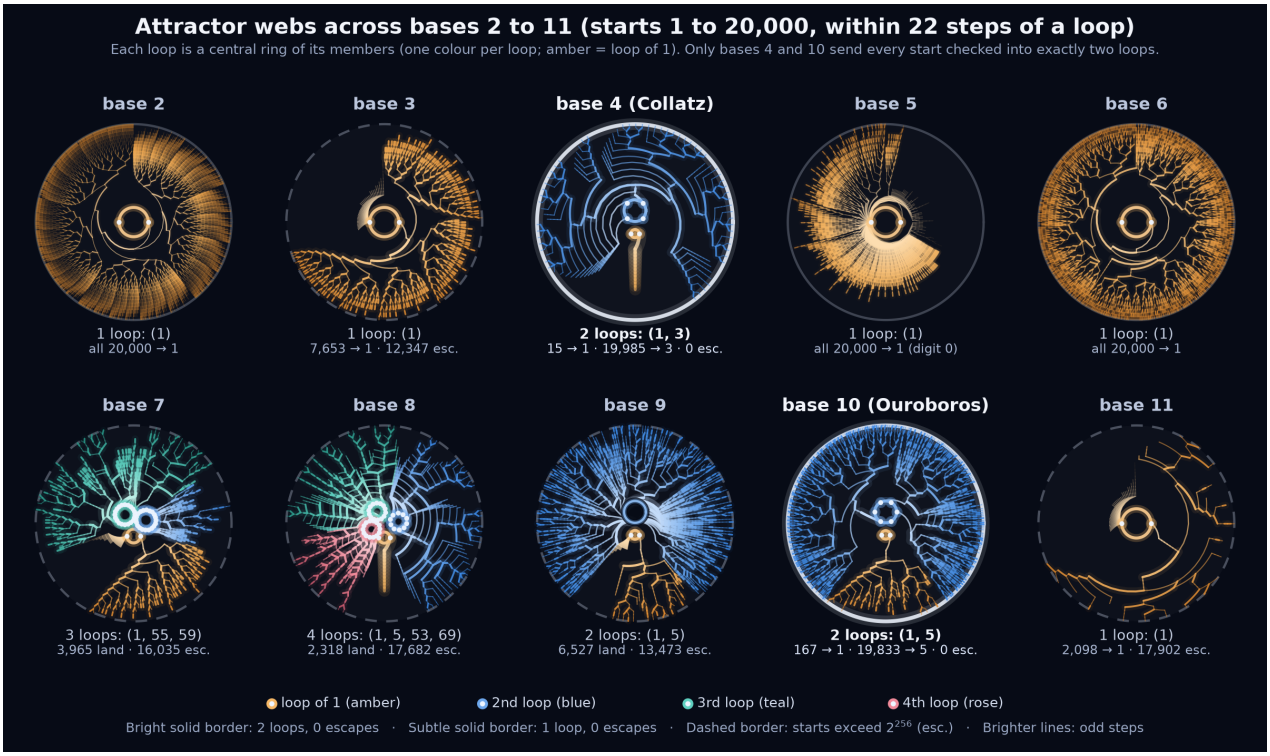

Figure 2: Attractor webs across bases 2 through 11, starting from all $1 \le n \le 20{,}000$ and tracing up to 22 ordinary steps near each observed cycle. Central rings depict the cycles (amber marks the loop of 1; blue, teal, and rose mark further cycles), and concentric outer rings count backward steps. Only bases 4 and 10 carry bright solid borders (two loops, zero escapes above 2^{256}). Odd-base panels sample shallow spokes so deeper branches stay visible. Neither experiment settles starts beyond its finite range.

1.1 Why base ten is the last contracting even base

Figures 1 and 2 raise an immediate question: why does the even-base behavior flip upward at base 8, reverse back downward one final time at base 10, and then climb from base 12 onward? Part of the answer can be seen by hand (Figure 3). Whenever $4 \mid b$, an odd integer n and its last base-b digit $d_b(n)$ agree modulo 4, so $d_b(n)\, n + 1 \equiv 2 \pmod 4$ is never divisible by 4: every odd step gets trapped at a single halving (such as $19 \to 58 \to 29$ in base 8), capping its halving budget at $-\log 2$. Whenever $b \equiv 2 \pmod 4$, by contrast, an odd step can purchase cascades of halvings—in base 10, $17 \to 120 \to 60 \to 30 \to 15$ drops below its start even after multiplying by 7—doubling the average halving budget to $-2 \log 2$. Section 6 proves that this extra halving budget pulls base 10 back below zero one last time before growing digits overtake it permanently at base 14.

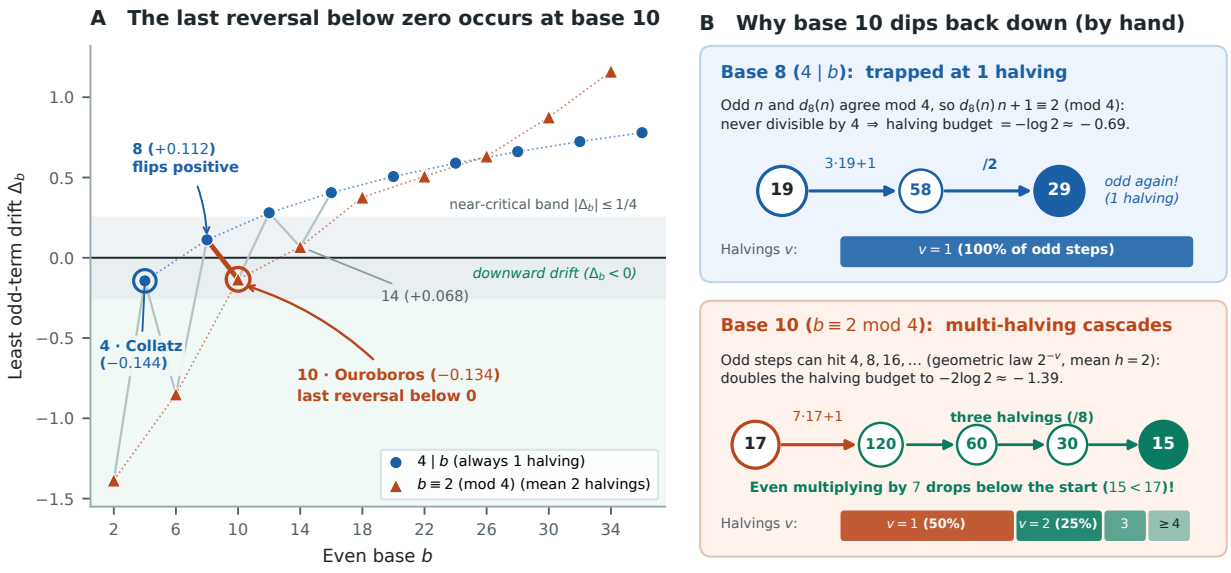

Figure 3: Why base 10 is the last even base with downward drift. **A:** Least odd-term drift Δ_b across even bases $b \le 36$. Drift is negative at $2, 4, 6$, turns positive at 8 ($+0.112$), reverses back below zero one last time at 10 (-0.134), and remains positive from 12 onward. **B:** The hand-checkable mechanism: in base 8 ($4 \mid b$), every odd step satisfies $d_8(n)n + 1 \equiv 2 \pmod 4$ and gets stuck at 1 halving ($19 \to 58 \to 29$), whereas in base 10 ($b \equiv 2 \pmod 4$) odd steps can buy cascades of halvings ($17 \to 120 \to 60 \to 30 \to 15 < 17$), doubling the mean halving budget from $-\log 2$ to $-2 \log 2$.

1.2 Zooming in on bases four and ten: why base four is Collatz

That exact same modulo-4 congruence solves the visual puzzle in Figure 2. Calling base 4 "Collatz" looks paradoxical at first sight: the classical Collatz map $C(y) = y/2$ (even y), $C(y) = 3y + 1$ (odd y) is famous for having *one* positive cycle, $1 \to 4 \to 2 \to 1$, whereas Figure 2 displays *two* cycles in base 4, namely $1 \to 2 \to 1$ and $3 \to 10 \to 5 \to 6 \to 3$. Figure 4 dissolves this paradox completely.

Proposition 1 (Base four is ordinary Collatz on $\mathbb{Z}_{\ge 0}$)**.** *Write every odd positive integer as $n = 2y+1$ with $y \ge 0$. Then:*

1. *If $y = 2k$ is even ($n = 4k + 1$, base-4 digit 1), then $f_4(n) = 2y + 2 \equiv 2 \pmod 4$ and $f_4^2(n) = 2(y/2) + 1 = 2C(y) + 1$.*
2. *If $y = 2k + 1$ is odd ($n = 4k + 3$, base-4 digit 3), then $f_4(n) = 2(3y + 1) + 2 \equiv 2 \pmod 4$ and $f_4^2(n) = 2((3y + 1)/2) + 1 = 2C^2(y) + 1$.*

Consequently, every even integer halves directly onto its odd part $2y+1$; no odd $n \ge 3$ can ever reach 1; the basin of $1 \leftrightarrow 2$ consists precisely of the powers of two $\{2^a : a \ge 0\}$; and universal attraction to $\{1, 3\}$ under f_4 is equivalent to the classical Collatz conjecture.

Proof. Direct substitution verifies both branches: for $n = 4k + 1$, $1 \cdot n + 1 = 4k + 2 = 2(2k + 1)$; for $n = 4k + 3$, $3n + 1 = 12k + 10 = 2(6k + 5)$. In both cases the intermediate even integer is congruent to 2 (mod 4), so f_4 performs exactly one halving and lands back on an odd integer $2y' + 1$, where $y' = T_C(y)$ is the shortcut Collatz iterate $y/2$ (even y) or $(3y + 1)/2$ (odd y). Thus $f_4^{2k}(2y + 1) = 2T_C^k(y) + 1$ for every $k \ge 0$. Because T_C maps $\mathbb{Z}_{\ge 1}$ into itself, no $y > 0$ ever reaches $y = 0$, so no odd $n \ge 3$ reaches $n = 1$. Any even integer $2^a m$ (m odd) halves a times to m, so it reaches 1 if and only if $m = 1$. Finally, $2y + 1$ reaches 3 under f_4 if and only if y reaches 1 under C. □

The same change of variables $n = 2y + 1$ appears in Yolcu, Aaronson, and Heule's analysis of Farkas's map [7]. Here it exposes the exact wall separating the two basins (Figure 4). On $\mathbb{Z}_{\geq 0}$, ordinary Collatz *already* has two loops: the trivial fixed point $0 \to 0$ (omitted on positive integers) and the 3-step cycle $1 \to 4 \to 2 \to 1$. Shifting by $n = 2y + 1$ brings $y = 0$ into $\mathbb{N}$ as $n = 1$, turning $0 \to 0$ into $1 \to 2 \to 1$ and turning the 15 powers of two up to 20,000 $(1, 2, 4, \ldots, 16{,}384)$ into the unbranched amber spike of Figure 2. Meanwhile, on odd y, Base 4 and ordinary Collatz march in lockstep ($3x+1$, then $/2$); on even y, Collatz divides by 2 in a single step while Base 4 pauses for one $+1$ move because $2y + 1$ is odd. That extra pause explains why Collatz's 3-step loop $4 \to 2 \to 1 \to 4$ becomes Base 4's 4-step loop $5 \to 6 \to 3 \to 10 \to 5$.

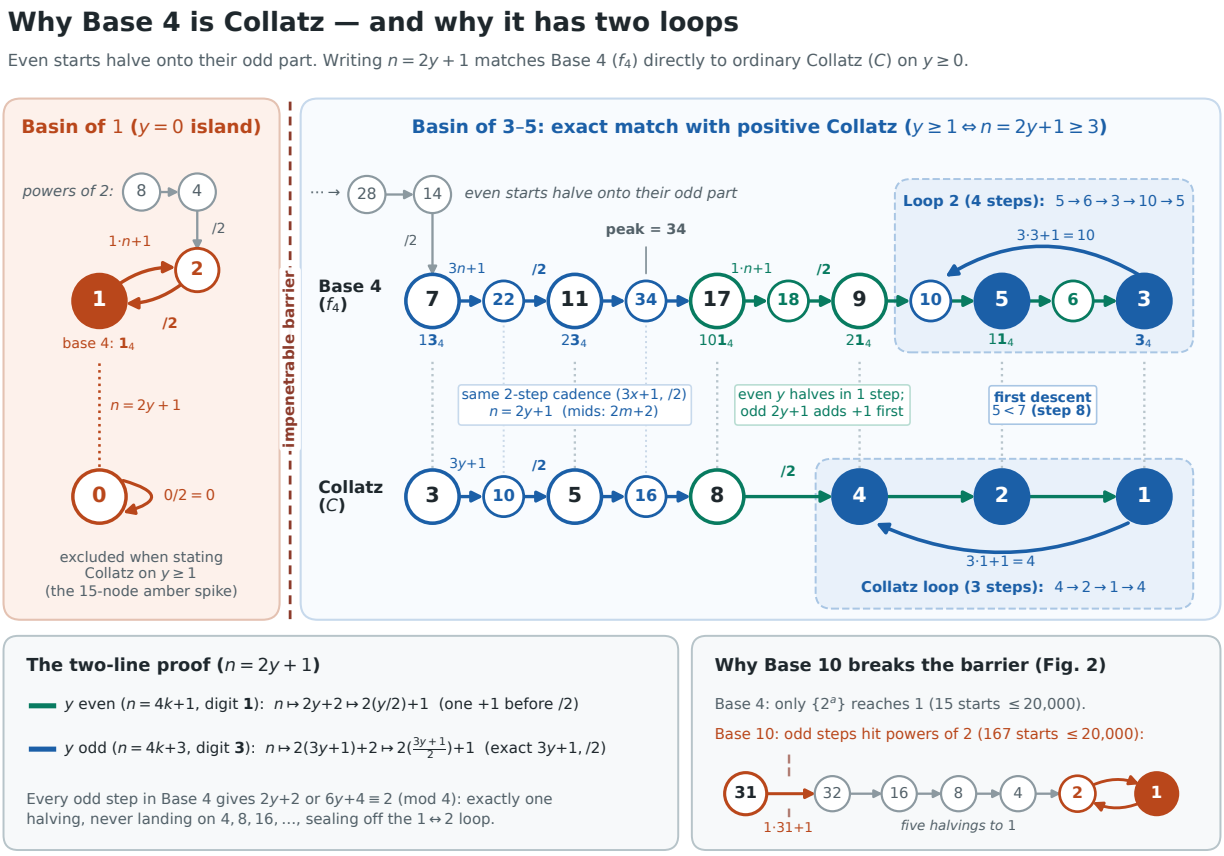

Figure 4: Why Base 4 is ordinary Collatz C on $\mathbb{Z}_{\geq 0}$, and why Base 10 breaks the barrier separating the two loops. Every even start halves onto its odd part. On odd integers, $n = 2y + 1$ aligns f_4 directly with uncompressed Collatz: odd-y steps share the exact two-step cadence ($3x + 1$, then $/2$), while even-y halvings gain one $+1$ pause before $/2$. The $1 \leftrightarrow 2$ loop corresponds to Collatz's excluded fixed point $0 \to 0$, explaining the 15-node amber spike of Figure 2. On the example orbit of $n = 7$ ($y = 3$), the peak is 34 and *first descent* occurs at step 8 ($5 < 7$). In Base 10, odd steps can land on higher powers of 2 (bottom right, $31 \to 32 \to \cdots \to 1$), breaking the barrier and sprouting a branching 1-basin.

In Base 10—our *Ouroboros map* $f = f_{10}$, found during a screen of 16 digit-driven Collatz rules—that wall collapses. Because 31 ends in 1, its odd step gives $1 \cdot 31 + 1 = 32 = 2^5$, vaulting straight onto the powers-of-two highway to 1. Meanwhile 7 follows $7, 50, 25, 126, 63, 190, \ldots$, climbs to a peak of 2,576, and settles after 40 steps into a second loop:

$$1 \longrightarrow 2 \longrightarrow 1, \qquad 5 \longrightarrow 26 \longrightarrow 13 \longrightarrow 40 \longrightarrow 20 \longrightarrow 10 \longrightarrow 5. \tag{2}$$

Santi independently recorded this six-cycle as "cycle A" in a different Collatz-type map that alter-

nates multipliers 3 and 5 on odd steps without reading a digit [8]: his multiplier-5 step at 5 and multiplier-3 step at 13 match the two odd moves of (2), whereas his other two listed cycles send 7 to 22 or 36 instead of $f(7) = 50$. The following eight lines verify our four opening decimal starts (counting ordinary steps to the first positive visit to 1 or 5):

```
def step(n):
    return n // 2 if n % 2 == 0 else (n % 10) * n + 1
for start in (1, 5, 7, 31):
    n, path = start, [start]
    while len(path) == 1 or n not in (1, 5):
        n = step(n)
        path.append(n)
    print(start, n, len(path) - 1, max(path))
```

Its four output lines end in $(1, 2, 2)$, $(5, 6, 40)$, $(5, 40, 2576)$, and $(1, 6, 32)$, respectively.

1.3 Main results and roadmap

Because Figure 2 singles out base 10 as the sole twin of Collatz base 4 among small bases, we follow the zoom straight into base 10 first (Sections 2–5), pull back across all bases $b \geq 2$ in Section 6 to prove rigorously that no larger base produces a third twin, and close in Section 7 by holding the decimal rule fixed while varying the base in which starting integers are written.

Write $N_i(X)$ for the count of *odd* integers $n \leq X$ whose f_{10}-orbit visits $i \in \{1, 5\}$, and write $d(S) = \lim_{X\to\infty} |S \cap [1, X]|/X$ for natural density (with $\underline{d}, \overline{d}$ for lower and upper densities). For any even base b, fold the guaranteed first halving after an odd step into the *shortcut map* $T_b(n) = n/2$ (even n), $T_b(n) = (d_b(n)\, n + 1)/2$ (odd n), and put

$$D_k(b) = \{n \geq 1 : T_b^j(n) \geq n \text{ for } 1 \leq j \leq k\}, \qquad p_k(b) = d(D_k(b)), \qquad p_\infty(b) = \lim_{k\to\infty} p_k(b).$$

Theorem 2 (Decimal Ouroboros: descent, competing basins, and cycles)**.** *For the base-*10 *map* $f = f_{10}$*:*

1. Exponential density-one descent: *For every* $k \geq 1$*,* $p_k(10) \leq 1.124953\,(0.996423)^k$*. Hence almost every positive integer (in natural density) has an* f*-iterate strictly smaller than itself.*
2. Thin basin of 1 and 2-adic Cantor closure: *Every odd start reaching* 1 *lies in* $\{1, 15, 17\} \pmod{30}$*. Embedded in the* 2*-adic integers* $\mathbb{Z}_2$*, the odd basin* B_1 *of* 1 *closes to a nowhere-dense, self-similar Cantor set* $\overline{B_1}$ *with exact Hausdorff and box-counting dimension* $\dim_H(\overline{B_1}) = \dim_B(\overline{B_1}) = \beta = -\log_2 a \simeq 0.626382$ *(*$a + 2a^4 = 1$*), immediately yielding the elementary bounds* $a_1^9 X^\gamma \leq N_1(X) \leq X^\beta$ *(*$\gamma \simeq 0.464958$*). Certified integer weight vectors break strictly below* β*, giving*

$$0.0806\,(X/31)^{399/800} \leq N_1(X) \leq C\, X^{0.6245}, \qquad N_1(X) \geq 0.0145\, X^{0.49875}$$

   for a constant $C > 0$ *and all real* $X \geq 1$*, and hence strict* 2*-adic/Archimedean separation* $\limsup_{X\to\infty}(\log N_1(X)/\log X) \leq 0.6245 < \dim_H(\overline{B_1})$*.*
3. Dominant basin of 5 and separation: *For every integer* $X \geq 3$*,*

$$N_5(X) \geq \frac{98{,}959}{2{,}269{,}527 \cdot 3^{13/20}}\, X^{13/20} > 0.02134\, X^{0.65}.$$

   Consequently $N_1(X)/N_5(X) = O(X^{-0.0255})$ *(already* $O(X^{-0.0236})$ *from the elementary upper bound), and the same ratio vanishes when even starts are included.*

4. Cycle exclusions: *The only primitive odd cycles with at most eight odd members are* (1) *and* $(5, 13)$ *(with no size cutoff). Every positive integer through* 10^{15} *reaches* $\{1, 5\}$*, and any cycle whose least odd member is at least* 10^{15} *must have halving count* K*, odd-step count* m*, and ordinary length* $K + m$ *satisfying*

$$K \geq 34{,}905, \qquad m \geq 15{,}496, \qquad K + m \geq 53{,}185.$$

5. Congruence completions and endings in every base: *Every congruence class* $a \bmod m$ *(*$m \geq 1$*) contains infinitely many starts reaching* 5*, while an exact finite halving test decides whether it contains infinitely many or zero starts reaching* 1*. Writing the density-zero basin of* 1 *in any base* $b = 2^t u$ *(*u *odd), its reversed endings close to a subset* $\mathcal{E}_b \subset [0, 1]$ *of Hausdorff and box dimension* $(\log u + t \log \lambda)/\log b$ *(*$\lambda = 1/a$*):* 0.62638214 *in binary,* 0.88752982 *in decimal, and* 1 *(*$\mathcal{E}_7 = [0, 1]$*) in base* 7*.*

Theorem 3 (All even bases: why only $2, 4, 6, 10$ contract)**.** *For every even base* $b \geq 2$*, every finite-horizon density* $p_k(b)$ *exists, and*

$$p_\infty(b) = 0 \quad \Longleftrightarrow \quad b \in \{2, 4, 6, 10\}.$$

In bases $2, 4, 6, 10$*, the set of integers eventually descending below their starting value has natural density one. In every other even base, there exists* $\eta_b > 0$ *such that* $p_k(b) \geq \eta_b$ *for all* $k \geq 1$*.*

Theorem 4 (All odd bases: trapdoors and certified divergence)**.** *Let* $b \geq 3$ *be odd. Write* Z_b *for starts reaching a multiple of* b*,* R_b *for starts reaching* 1*, and* E_b *for starts tending to infinity. Computable finite-cylinder lower bounds converge to masses* τ_b *for* Z_b *and* ρ_b *for* E_b*. Whenever* $\tau_b + \rho_b = 1$*, all three natural densities exist and satisfy*

$$d(Z_b) = d(R_b) = \tau_b, \qquad d(E_b) = \rho_b.$$

If any prime $p \equiv 3 \pmod 4$ *divides* b*, then* $Z_b = b\mathbb{N}$ *exactly. Across the* 999 *odd bases* $3 \leq b \leq 1{,}999$*, the certificate establishes* $\tau_b + \rho_b = 1$ *in* 905 *bases (*27 *trap-only with* $\tau_b = 1$*, and* 878 *split with both* $\tau_b, \rho_b > 0$*).*

2 Base ten: competing corals and forward descent

Stay inside base 10, where Figure 4 just broke open the wall around $1 \leftrightarrow 2$. In classical Collatz only one nontrivial cycle exists, so there is no question of competing basins; in base 10, breaking that barrier creates a genuine contest between two infinite attracting trees—a phenomenon with no analogue in $3x + 1$. Trace the first 30,000 positive integers backward from their destinations 1 and 5, letting orbits that share a suffix fuse into a single strand (Figure 5). Before we prove a single inequality, the asymmetry leaps off the page: 29,813 trajectories build the towering blue coral of 5, while just 187 build the fragile amber sprig of 1.

How can we tell mathematically that typical decimal orbits fall below their starting value? Abbreviate $d(n) = n \bmod 10$ and $T = T_{10}$. Write $r = n \bmod 5$. An odd integer in class r has a unique last digit $\delta(r)$, given in order $r = 0, 1, 2, 3, 4$ by $(5, 1, 7, 3, 9)$. Because halving multiplies by 3 mod 5, one shortcut step sends residue r to

$$E(r) = 3r \bmod 5 \text{ (even } n), \qquad O(r) = 3(r^2 + 1) \bmod 5 \text{ (odd } n). \tag{3}$$

Notice a remarkable structural quirk right away: $O(r) = 0$ precisely for $r \in \{2, 3\}$ (digits 7 and 3), and $E(0) = 0$. Thus *every odd term ending in* 3 *or* 7 *lands next on an odd term ending in* 5*, and no other odd digit ever sends flow into* 5 (Figure 7).

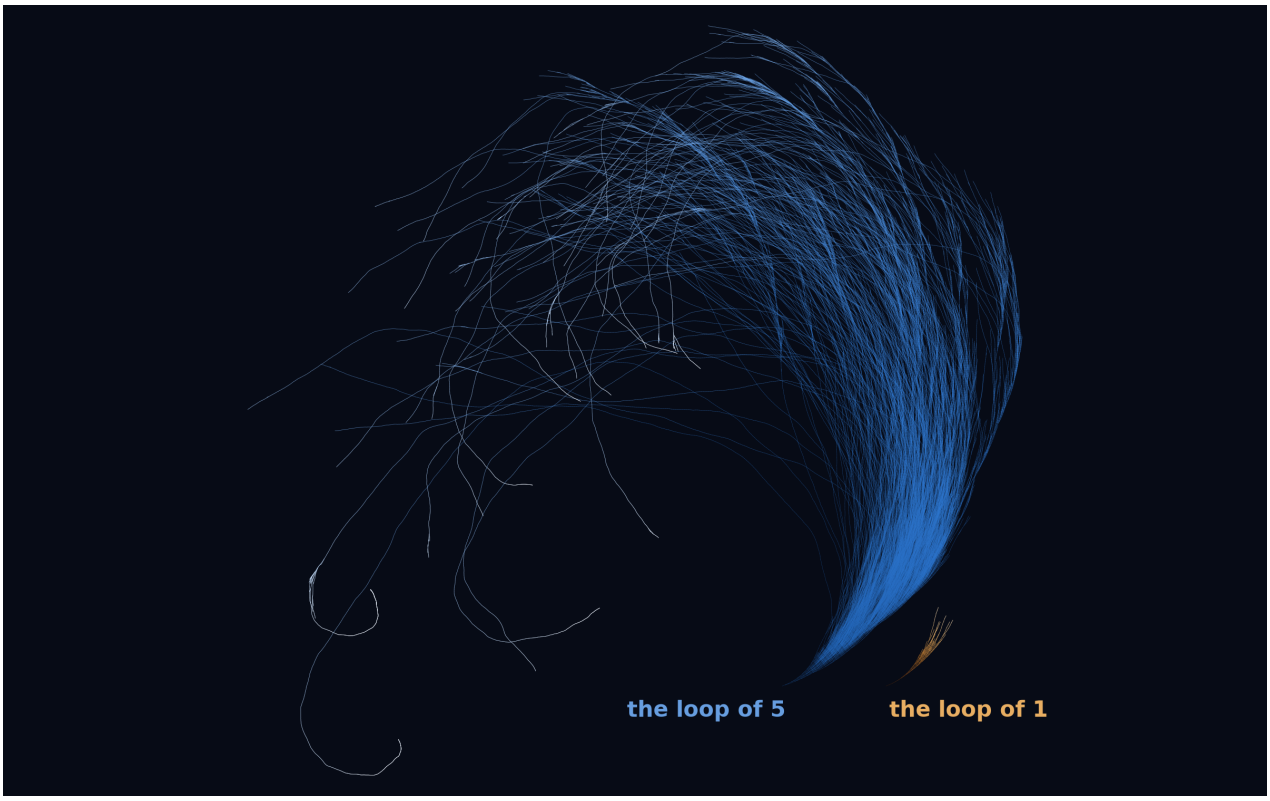

Figure 5: The two orbit corals of base 10. Every integer $1 \le n \le 30{,}000$ traces its path to first arrival at 1 or 5: 29,813 starts form the blue coral of 5, and 187 form the amber coral of 1. Reading outward from the roots, each halving bends left and each odd move bends right; shared segments represent identical iterates.

Lemma 5 (Base-ten parity coding)**.** *Fix $k \ge 0$. A starting residue $r_0 = n \bmod 5$ and a string of k shortcut parities specify precisely one residue class modulo $5 \cdot 2^k$. On every such class there exist an odd integer A_j and an integer $B_j \ge 0$ such that $T^j(n) = (A_j n + B_j)/2^j$ for $0 \le j \le k$, where A_j is the product of the odd digits multiplied along the first j steps.*

Proof. Start at $(A_0, B_0) = (1, 0)$. An even step leaves (A_j, B_j) unchanged; an odd step replaces it by $(dA_j,\, dB_j + 2^j)$, where $d = \delta(r_j)$ is determined by the current residue modulo 5, and advances the residue via E or O. Suppose the word of length j specifies $n_0 \bmod (5 \cdot 2^j)$. Its two lifts n_0 and $n_0 + 5 \cdot 2^j$ have j-th iterates differing by $5A_j$, which is odd; hence one lift is even and the other is odd. Induction gives both claims. □

For a uniformly chosen residue modulo $5 \cdot 2^k$, each successive parity bit is therefore an independent fair coin—an exact finite-counting identity generalizing Terras's parity theorem for $3x+1$ [1]. Direct substitution in (3) gives the unique stationary distribution $\pi = (5, 3, 1, 4, 2)/15$, which also governs the embedded odd-to-odd chain (Figure 7). Since an odd step is followed on average by two halvings, the stationary logarithmic change per odd step is

$$\Delta_{10} = \sum_{r=0}^{4} \pi_r \log \delta(r) - 2 \log 2 = \frac{1}{15} \log\left(\frac{3^8 \cdot 5^5 \cdot 7}{2^{30}}\right) \simeq -0.134161.$$

Proof of Theorem 2, Part 1. On each residue class modulo $5 \cdot 2^k$, set $H_j = A_j/2^j$. Because A_j is odd, $H_j \ne 1$ for every $j \ge 1$: if $H_j < 1$, then $T^j(n) < n$ holds for all $n > B_j/(2^j - A_j)$, while if $H_j > 1$, then $T^j(n) > n$ throughout the class. Discarding the finitely many exceptions across finitely

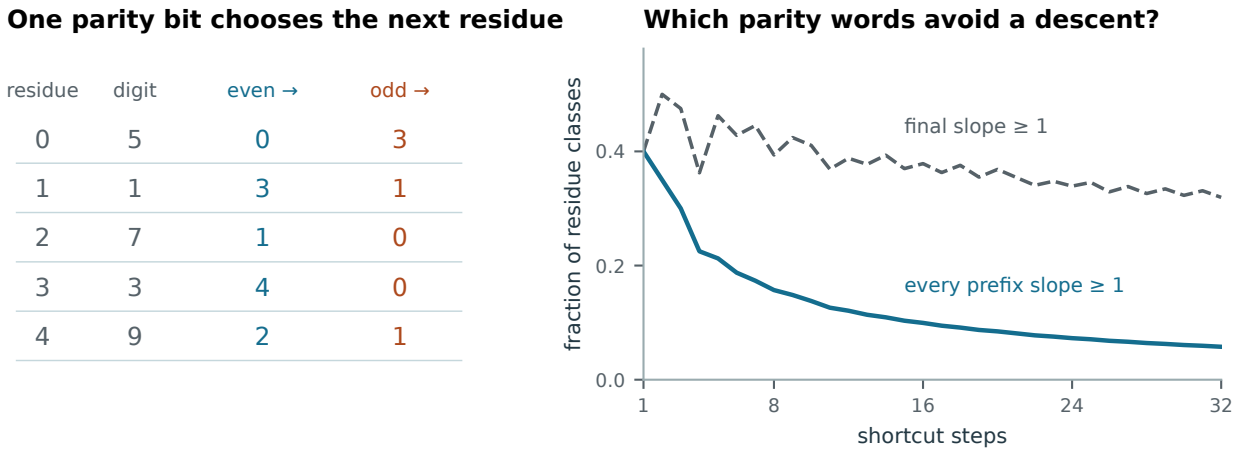

Figure 6: Base-10 finite parity coding. Left: the deterministic transition table (3) modulo 5. Right: exact fraction of the $5 \cdot 2^k$ residue classes avoiding contraction through $k \leq 32$ shortcut steps, comparing the final-slope test $A_k \geq 2^k$ against the true first-descent condition $A_j \geq 2^j$ for all $1 \leq j \leq k$.

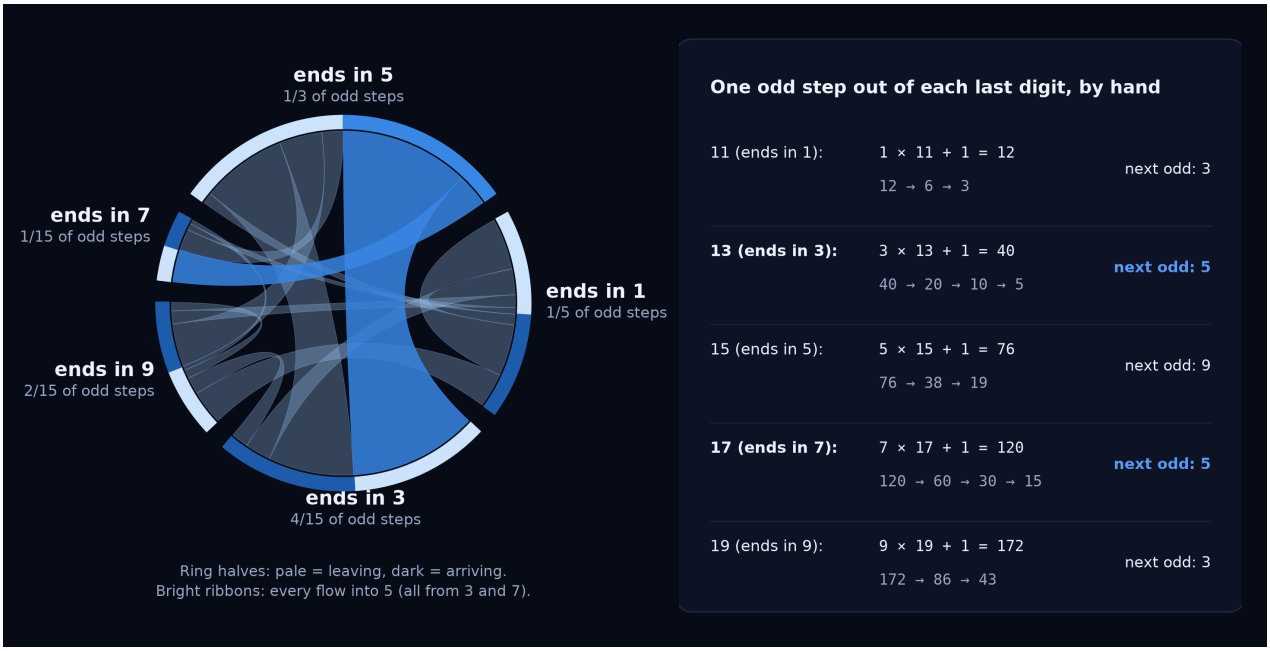

Figure 7: Stationary odd-to-odd digit flow in base 10. Pale arcs emit flow and dark arcs receive it; ribbon width measures joint frequency. The bright blue ribbons highlight the forced funnel: digits 3 (4/15) and 7 (1/15) account for all transitions into digit 5 (1/3). Right: one hand-worked odd step out of each last digit, including all intervening halvings.

many classes proves that $p_k(10) = d(D_k(10))$ exists and equals the fraction of classes with $H_j > 1$ for all $1 \leq j \leq k$ (bounded above by the fraction with $H_k \geq 1$).

Tilt the five-state transition matrix by $t = 11/100$:

$$(Mu)_r = \tfrac{1}{2}(1/2)^t u_{E(r)} + \tfrac{1}{2}(\delta(r)/2)^t u_{O(r)},$$

so that $\mathbb{E}[H_k^t] = \mathbf{1}^\top M^k \mathbf{1}/5$. Take the positive vector $u = (1{,}000{,}000, 837{,}748, 965{,}453, 964{,}025,$

944,908) and denominator $D = 10^{12}$. The five integer numerators

$$(h_1, h_3, h_5, h_7, h_9) = (926{,}588{,}061{,}891,\ 1{,}045{,}610{,}747{,}322,\ 1{,}106{,}046{,}538{,}591,\\ 1{,}147{,}750{,}484{,}796,\ 1{,}179{,}922{,}211{,}263)$$

each satisfy $h_d^{100}\, 2^{11} > D^{100}\, d^{11}$, certifying $(d/2)^{11/100} < h_d/D$. Substituting directly into all five rows confirms

$$10^6 (h_1 u_{E(r)} + h_{\delta(r)} u_{O(r)}) < 2D \cdot 996{,}423\, u_r, \qquad \frac{\sum_r u_r}{5 \min_r u_r} = \frac{2{,}356{,}067}{2{,}094{,}370} < 1.124953.$$

Since M is nonnegative, $\mathbf{1} \le u / \min u$ and Markov's inequality yield $p_k(10) \le \mathbb{E}[H_k^t]$, giving $p_k(10) \le 1.124953\,(0.996423)^k$ (Figure 6). Any integer whose ordinary f-orbit never drops below itself lies in every $D_k(10)$ (as $d(n)n + 1 > n$); letting $k \to \infty$ proves natural-density-one descent. □

Density-one first descent shows that typical decimal orbits turn downward, but it cannot tell us *which* of the two corals in Figure 5 captures them. For that, we reverse time and climb both inverse trees.

3 Backward toward one: the thin amber basin

Strip all powers of two after an odd step to define the odd-to-odd decimal map

$$F(n) = \frac{d(n)\, n + 1}{2^v}, \qquad v = v_2(d(n)\, n + 1), \qquad n \text{ odd}.$$

For example, $F(991) = 31$ and $F(31) = 1$ use only multiplier 1, whereas $F(195) = 61$ and $F(61) = 31$ bring multiplier 5 into the same tree. Plotting the unbounded ordinary-step predecessor trees of 1 and 5 through depth 26 (Figure 8) immediately exposes a stark arithmetic selection rule: among all 261 odd predecessors of 1 within 26 steps (excluding 1), *zero* end in 3 and *zero* end in 9.

3.1 The modulo-30 residue law and binary runs

Proposition 6 (Three residue classes for the basin of one)**.** *Every odd integer reaching* 1 *under* f_{10} *is congruent to* $1, 15,$ *or* 17 *modulo* 30 *(equivalently* $r \in \{1, 0, 2\} \pmod{15}$*). Label these three classes* $A \equiv 1$*,* $B \equiv 0$*, and* $C \equiv 2 \pmod{15}$*. Every odd-to-odd move* $m \xrightarrow{v} o$ *inside the basin of* 1 *has* $m = (2^v o - 1)/d$ *governed by the exhaustive table below:*

move $m \to o$	$A \to A$	$A \to C$	$B \to A$	$B \to C$	$C \to B$
digit d	1	1	5	5	7
condition	$v \equiv 1 \bmod 4$	$v \equiv 0 \bmod 4$	$2^v o \equiv 1 \bmod 25$	$2^v o \equiv 1 \bmod 25$	$2^v o \equiv 1 \bmod 7$

In particular, $v \equiv 0 \pmod 4$ *on* $B \to A$*,* $v \equiv 3 \pmod 4$ *on* $B \to C$*, and every* $v \ge 1$ *is modulo*-15 *admissible on* $C \to B$*, with no residue admitting two distinct moves at the same* v*.*

Proof. For an odd residue r mod 15, the target residue of F is $(\delta(r \bmod 5)\, r + 1)\, 2^{-v} \bmod 15$. Because 2 has multiplicative order 4 modulo 15, hitting 1 or 2 (mod 15) requires the numerator to lie in $\{1, 2, 4, 8\} \pmod{15}$, while hitting 0 (mod 15) requires it to vanish. In order $r = 0, 1, \ldots, 14$, the numerators $\delta(r \bmod 5)\, r + 1 \bmod 15$ are

$$(1, 2, 0, 10, 7,\ 11, 7, 5, 10, 7,\ 6, 12, 10, 10, 7).$$

Only $r \in \{0, 1, 2\}$ land in $\{0, 1, 2, 4, 8\}$. Backward induction from $1 \in A$ therefore confines the entire odd basin of 1 to A, B, C. Checking the powers of 2 modulo 15—together with exact divisibility $d \mid 2^v o - 1$ and $m \equiv d \pmod 5$ (which strengthens $5 \mid 2^v o - 1$ to $25 \mid 2^v o - 1$ for $d = 5$)—yields the exact table. □

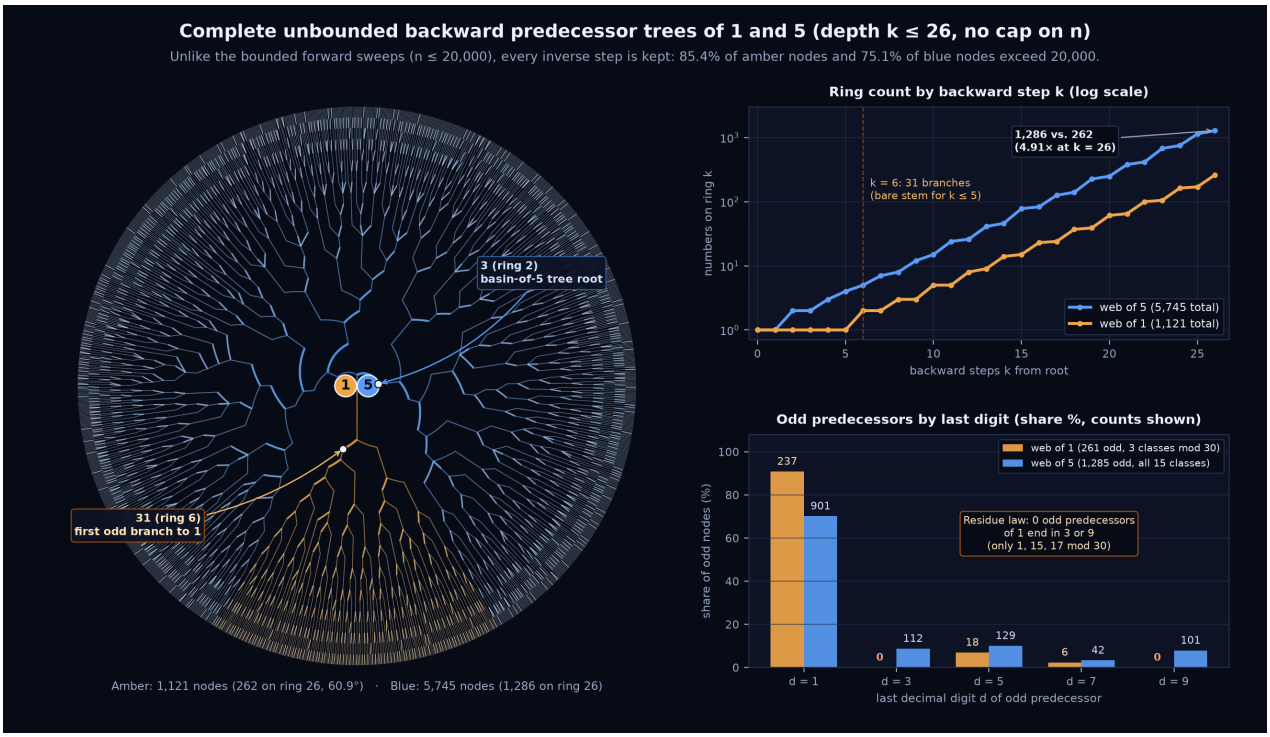

Figure 8: Complete unbounded backward predecessor trees of 1 and 5 through 26 ordinary steps (no cap on n, omitting periodic repeats). At depth 26, the outer ring holds 262 nodes for 1 (1,121 total) versus 1,286 for 5 (5,745 total). The bare stem $1, 2, 4, 8, 16, 32$ branches for the first time at 31 (ring 6), and the lower-right histogram shows zero odd predecessors of 1 ending in 3 or 9.

Inside class A sits a charming binary characterization of paths that use *only* digit 1. Write $u = (n-1)/2$. An odd n reaches 1 via digit-1 moves alone if and only if *every maximal run of* 1*-bits in the binary expansion of* u *has length divisible by* 4. Indeed, if $u = 2^{r+1}c + 2^r - 1$ ends in r ones, then $n+1 = 2^{r+1}(2c+1)$, so $F(n) = 2c+1$ deletes the trailing block 01^r from u; reducing modulo 5 when both n and $F(n)$ end in 1 forces $2^r \equiv 1 \pmod 5$, i.e., $4 \mid r$ (for instance, $(991-1)/2 = (111101111)_2$).

3.2 Exact 2-adic Cantor closure and elementary quartic bounds

Rather than tracking odd-to-odd jumps all at once, watch the shortcut map $T = T_{10}$ act one halving at a time on the three admissible sheets $n = 30k + r$ ($r \in \{1, 15, 17\}$). Writing $k = 2k' + s$ ($s \in \{0, 1\}$) tests a single binary digit of k and advances $T(n)$ mod 30:

- from 1 or 2 (mod 30): $s = 0 \mapsto 1$, $s = 1 \mapsto 16$;
- from 0 or 17 (mod 30): $s = 0 \mapsto 0$, $s = 1 \mapsto 15$;
- from 15 or 16 (mod 30): $s = 0 \mapsto 8$, $s = 1 \mapsto 23$ (forbidden);
- from 8 (mod 30): $s = 0 \mapsto 4$, $s = 1 \mapsto 19$ (forbidden);
- from 4 (mod 30): $s = 0 \mapsto 2$, $s = 1 \mapsto 17$.

Only eight residues $\{0, 1, 2, 4, 8, 15, 16, 17\}$ (mod 30) are reachable while staying in $\{1, 15, 17\}$ (mod 30), and the first forbidden odd residue hit is always 19 or 23 (mod 30). Grouping bisimilar residues into four states $S_0 = \{0, 1, 2, 17\}$, $S_1 = \{15, 16\}$, $S_2 = \{8\}$, and $S_3 = \{4\}$ collapses the entire modulo-30 survival dynamics to the four-state automaton of Figure 9A:

$$S_0 \xrightarrow{0} S_0, \qquad S_0 \xrightarrow{1} S_1 \xrightarrow{0} S_2 \xrightarrow{0} S_3 \xrightarrow{0,1} S_0.$$

Consequently, a binary itinerary starting in S_0 survives every halving test if and only if it belongs to the infinite prefix-code Cantor set $\Sigma = \{0,\ 1000,\ 1001\}^\omega \subset \{0, 1\}^{\mathbb{N}}$.

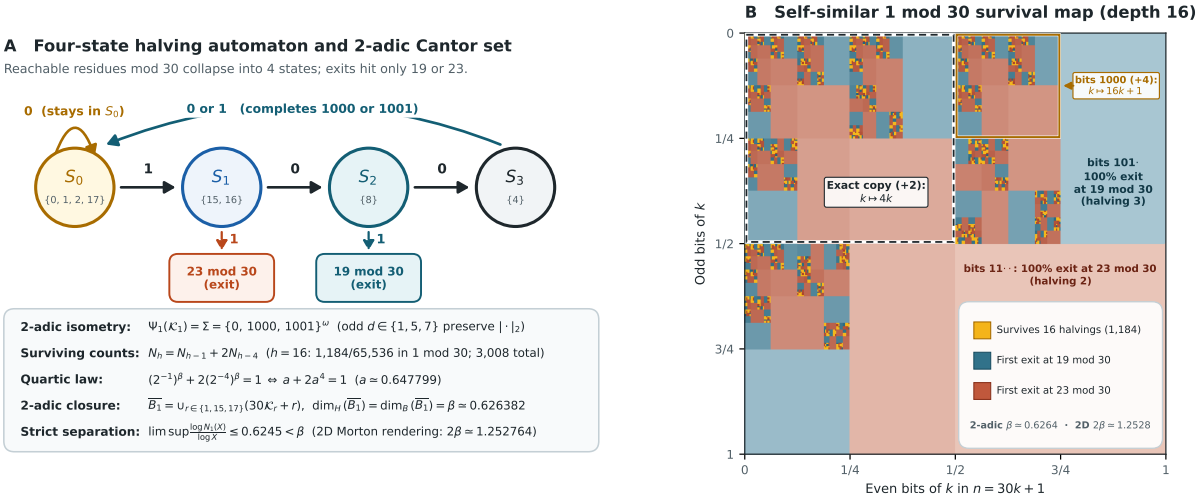

Figure 9: (A) Exact four-state collapse of the eight reachable residues modulo 30 under single halving steps T_{10}, yielding the prefix-code Cantor set $\Sigma = \{0, 1000, 1001\}^\omega$ and the recurrence $N_h = N_{h-1} + 2N_{h-4}$. **(B)** The 1 mod 30 survival landscape at depth $h = 16$ (256×256 classes), interleaving even and odd binary digits of k in $n = 30k + 1$ into $[0, 1]^2$. Just 1,184 of 65,536 classes survive (gold; 3,008 of 983,040 across all 15 odd classes modulo 30). The upper-left quarter ($k \mapsto 4k$, $4n - 3 \to 2n - 1 \to n$) and the 1000 sub-square ($k \mapsto 16k + 1$) reproduce the entire square identically at depths +2 and +4.

Let $B_1 = \{n \in \mathbb{Z}_{\ge 1} \text{ odd} : f_{10}^j(n) = 1 \text{ for some } j \ge 0\}$ be the odd basin of 1, embedded in the 2-adic integers $(\mathbb{Z}_2, |\cdot|_2)$ with the standard ultrametric $|x - y|_2 = 2^{-v_2(x-y)}$. For each $r \in \{1, 15, 17\}$, let $K_r = \{k \in \mathbb{Z}_{\ge 0} : 30k + r \in B_1\}$, and let $\mathcal{K}_r \subset \mathbb{Z}_2$ be the compact set of 2-adic coordinates k whose congruence class $k \bmod 2^h$ survives the first h halvings without hitting a forbidden odd residue modulo 30 for every $h \ge 0$. Remarkably, $\mathcal{K}_r$ is not merely a necessary outer shell: it is the *exact* 2-adic closure of the true basin of 1.

Theorem 7 (Exact 2-adic Cantor closure and elementary quartic bounds)**.** *Let $a_1, a \in (0, 1)$ be the unique positive roots of $a_1 + a_1^4 = 1$ and $a + 2a^4 = 1$, and put $\gamma = -\log_2 a_1 \simeq 0.464958$ and $\beta = -\log_2 a \simeq 0.62638214$ ($0.626382 < \beta < 0.626383$).*

1. Exact 2-adic closure: *For each $r \in \{1, 15, 17\}$, the 2-adic closure of K_r equals $\mathcal{K}_r$, and hence the 2-adic closure of the odd basin B_1 in $\mathbb{Z}_2$ is*

$$\overline{B_1} = (30\mathcal{K}_1 + 1) \cup (30\mathcal{K}_{15} + 15) \cup (30\mathcal{K}_{17} + 17).$$

   Every nonempty 2-adic ball meeting $\overline{B_1}$ contains infinitely many positive integers reaching 1 and infinitely many exiting $\{1, 15, 17\} \pmod{30}$ at 19 or 23 $\pmod{30}$, and $\overline{B_1}$ is nowhere dense in $\mathbb{Z}_2$.

2. Self-similar isometry and Hausdorff dimension: *Equip $\{0, 1\}^{\mathbb{N}}$ with $d(x, y) = 2^{-\inf\{j : x_j \ne y_j\}}$. The halving-itinerary map $\Psi_r : \mathcal{K}_r \to \{0, 1\}^{\mathbb{N}}$ is a surjective 2-adic isometry onto $\Sigma = \{0, 1000, 1001\}^\omega$ for $r \in \{1, 17\}$ and onto $000\Sigma \sqcup 001\Sigma$ for $r = 15$. Consequently $\overline{B_1}$ has positive finite β-dimensional Hausdorff measure ($\frac{1}{2} \le \mathcal{H}^\beta(\overline{B_1}) \le a^{-2} + a < 3.031$) and exact 2-adic Hausdorff and box-counting dimension*

$$\dim_H(\overline{B_1}) = \dim_B(\overline{B_1}) = \beta = -\log_2 a \simeq 0.62638214.$$

3. Elementary integer bounds: *For every real $X \ge 1$,*

$$a_1^9 X^\gamma \le N_1(X) \le X^\beta.$$

   In particular, the full basin of 1 (including even starts) has natural density zero.

Proof. Write $n_0 = 30(2^h k' + k_0) + r$ with $r \in \{1, 15, 17\}$. Induction on $j \le h$ shows that along any surviving path, $n_j = T^j(n_0) = 30 \cdot 2^{h-j} D_j k' + b_j$, where $D_j = \prod_{i<j,\ n_i \text{ odd}} d(n_i)$ is an odd product of digits in $\{1, 5, 7\}$ and $b_j = T^j(30k_0 + r)$ is independent of k'. Because $2 \mid 2^{h-j}$ for $j < h$, both parity and last digit of n_j depend only on $b_j \bmod 30$, so survival through h halvings depends solely on $k \bmod 2^h$. Thus $\mathcal{K}_r^{(h)}$ is a clopen union of $N_h(r)$ balls of radius 2^{-h}, and $\mathcal{K}_r = \bigcap_{h \ge 0} \mathcal{K}_r^{(h)}$ is compact.

Part 2 (isometry and dimension). For $k \in \mathcal{K}_r$ and $j \ge 0$, choose any nonnegative integer lift $\tilde{k} \equiv k \pmod{2^{j+1}}$ and write $T^j(30\tilde{k} + r) = 30q_j + r_j$ $(0 \le r_j < 30)$; the depth-$(j+1)$ formula above changes $T^j(30\tilde{k} + r)$ by a multiple of 60, so the j-th itinerary bit $s_j(k) = q_j \bmod 2 \in \{0, 1\}$ is well defined. Take distinct $k, k' \in \mathcal{K}_r$ with $v_2(k - k') = m$, and fix integer lifts $\tilde{k}, \tilde{k}' \equiv k, k' \pmod{2^{m+1}}$. Along $0 \le j \le m$, both lifts follow the same residues modulo 30 and odd multipliers D_j, giving $\tilde{n}_j - \tilde{n}'_j = 30D_j(\tilde{k} - \tilde{k}')/2^j$ in $\mathbb{Z}$. For $j < m$ this is divisible by 60, so $s_j(k) = s_j(k')$; at $j = m$, both D_m and $(\tilde{k} - \tilde{k}')/2^m$ are odd, so $\tilde{n}_m - \tilde{n}'_m \equiv 30 \pmod{60}$ and $s_m(k) \ne s_m(k')$. Hence $d(\Psi_r(k), \Psi_r(k')) = |k - k'|_2$ identically. Conversely, each transition bit s_j bisects a class modulo 2^j into two classes modulo 2^{j+1}, so by compactness of $\mathbb{Z}_2$ every infinite word in the automaton's language is realized uniquely: $\Psi_r(\mathcal{K}_r) = \Sigma$ for $r \in \{1, 17\}$ and $000\Sigma \sqcup 001\Sigma$ for $r = 15$ (starting from state S_1).

Assign the Bernoulli product measure μ on Σ with word weights $p(0) = a$ and $p(1000) = p(1001) = a^4$ (summing to $a + 2a^4 = 1$). Every length-h valid prefix w ends in state S_0, S_1, S_2, or S_3, giving cylinder mass $\mu(C_w) = a^h$, $2a^{h+3}$, $2a^{h+2}$, or $2a^{h+1}$ respectively. Since $2a^3 < 2a^2 < 1 < 2a$, every ambient length-h cylinder $C_w \subset \{0, 1\}^{\mathbb{N}}$ (of diameter 2^{-h}) meeting Σ satisfies

$$2a^3\,(2^{-h})^\beta \le \mu(C_w) \le 2a\,(2^{-h})^\beta \qquad (a = 2^{-\beta}). \tag{4}$$

Summing (4) over the $N_h(1)$ disjoint length-h cylinders covering Σ yields $(2a)^{-1}a^{-h} \le N_h(1) \le (2a^3)^{-1}a^{-h}$ for all $h \ge 0$ (with $N_h(1) = N_h(17)$ obeying $N_h = N_{h-1} + 2N_{h-4}$ for $h \ge 4$, and $N_h(15) = 2N_{h-3}(1)$ for $h \ge 3$). In the ultrametric space $\{0, 1\}^{\mathbb{N}}$, any subset $U \subseteq \Sigma$ of positive diameter 2^{-h} lies in a single cylinder C_w of length h, so $\mu(U) \le 2a\,(\operatorname{diam} U)^\beta$ (and $\mu(\{x\}) = 0$ for singletons). For any countable cover $\{U_i\}$ of Σ, $1 = \mu(\Sigma) \le 2a \sum (\operatorname{diam} U_i)^\beta$, while covering by all $N_h(1)$ length-h cylinders gives $\mathcal{H}^\beta(\Sigma) \le (2a^3)^{-1}$. Thus $(2a)^{-1} \le \mathcal{H}^\beta(\Sigma) \le (2a^3)^{-1}$. For each $r \in \{1, 15, 17\}$, the affine map $k \mapsto 30k + r$ contracts 2-adic distances by $|30|_2 = 1/2$, scaling $\mathcal{H}^\beta$ by $(1/2)^\beta = a$. Monotonicity and finite subadditivity on $\overline{B_1} = \bigcup_r (30\mathcal{K}_r + r)$ (once Part 1 is established) therefore give $\frac{1}{2} \le \mathcal{H}^\beta(\overline{B_1}) \le 2 \cdot a(2a^3)^{-1} + 2a^4(2a^3)^{-1} = a^{-2} + a < 3.031$, and the same upper and lower cylinder counts establish $\dim_H(\overline{B_1}) = \dim_B(\overline{B_1}) = \beta$.

Part 1 (exact 2-adic closure via the $+30$ inverse commutator). Every $k \in K_r$ passes all modulo-30 halving tests by Proposition 6, so $\overline{K_r} \subseteq \mathcal{K}_r$. Conversely, take any $k \in \mathcal{K}_r$ and any 2-adic ball $k_0 + 2^h\mathbb{Z}_2$ around k. Inside the eight-state automaton, every state reaches 1 (mod 30) in at most 5 halvings and an exit (19 or 23 (mod 30)) in at most 3 halvings. Following the exit path of length $d \le 3$ specifies an open sub-ball $k_{\text{exit}} + 2^{h+d}\mathbb{Z}_2 \subseteq (k_0 + 2^h\mathbb{Z}_2) \setminus \mathcal{K}_r$ whose positive integer elements all leave $\{1, 15, 17\}$ (mod 30) at 19 or 23 (mod 30). Following the path to 1 (mod 30) yields a sub-cylinder $k_1 + 2^{h'}\mathbb{Z}_2 \subseteq k_0 + 2^h\mathbb{Z}_2$ $(h' \le h + 5)$ on which $T^{h'}(30(2^{h'}k'' + k_1) + r) = 30D'k'' + b$, with $D' = 5^{e_5}7^{e_7}$ odd and $b \equiv 1 \pmod{30}$.

To hit $b \bmod 30D'$ with an arbitrarily large member of B_1, observe that for any integer $u \equiv 1 \pmod{10}$, both backward digit-1 moves $L(u) = 2u - 1$ and $H(u) = 32u - 1$ still end in 1 and return to u under f_{10}. Composing them in opposite orders shifts the start by exactly 30:

$$P(u) = H(L(u)) = 64u - 33, \qquad Q(u) = L(H(u)) = 64u - 3 = P(u) + 30. \tag{5}$$

(For example, at $u = 31$ the odd paths are $1{,}951 \to 61 \to 31$ and $1{,}981 \to 991 \to 31$; cf. [6] for modular affine translations in the $3x + 1$ graph.) For any odd modulus $q \ge 1$, P permutes $\mathbb{Z}/q\mathbb{Z}$;

choosing $s \geq 1$ with $P^s \equiv \mathrm{id} \pmod q$, the legal inverse composite $Q \circ P^{s-1}$ adds 30 (mod q) while preserving u mod 30 and returning to u, and iterating P^s makes representatives arbitrarily large since $P(x) > x$ for all $x \geq 1$. Applying this at $u = 1$ with $q = 15D'$ reaches every residue congruent to 1 (mod 15) modulo $15D'$; oddness upgrades this to $o \equiv b \pmod{30D'}$ with $o \in B_1$ and $o \geq b$. Setting $k'' = (o-b)/(30D') \in \mathbb{Z}_{\geq 0}$ places $2^{h'}k'' + k_1 \in K_r \cap (k_0 + 2^h\mathbb{Z}_2)$. Hence $\overline{K_r} = \mathcal{K}_r$, and because $B_1 = \bigcup_r (30K_r + r)$ is a finite union of affine images of K_r, $\overline{B_1} = \bigcup_r (30\mathcal{K}_r + r)$. Any nonempty 2-adic ball $x + 2^m\mathbb{Z}_2$ meeting $\overline{B_1}$ contains some $30k + r$ ($k \in \mathcal{K}_r$) and therefore contains $30(k + 2^{\max(m-1,0)}\mathbb{Z}_2) + r$, yielding both infinitely many integers in B_1 and infinitely many exiting at 19 or 23 (mod 30). Because each exit sub-ball lies outside $\mathcal{K}_r$, every compact sheet $\mathcal{K}_r$ (and hence its homeomorphic copy $30\mathcal{K}_r + r$ inside $1 + 2\mathbb{Z}_2$) has empty interior in $\mathbb{Z}_2$; the finite union $\overline{B_1}$ is therefore closed and nowhere dense in $\mathbb{Z}_2$ (as also follows directly from Part 2 since $\beta < 1$).

Part 3 (elementary integer bounds). For the lower bound, build all-digit-1 chains backward from 1 via $o \mapsto 2^v o - 1$ with $v \equiv 1 \pmod 4$ and initial $v \geq 5$ (entering $31, 511, \ldots$). Each step stays in class $A \equiv 1 \pmod{15}$ of Proposition 6, strictly increases the vertex, and is uniquely invertible via $v_2(o+1)$. Any chain of total halving count $m > 0$ ends strictly below 2^m. Let $U(m)$ count ordered compositions of m into parts $\equiv 1 \pmod 4$ (with $U(0) = 1$). Then $U(0) = \cdots = U(4) = 1$, and for $m \geq 5$ either stripping a leading 1 or subtracting 4 from a larger first part gives $U(m) = U(m-1) + U(m-4)$, so $U(m) \geq a_1^{4-m}$ by induction. Requiring the first part to be at least 5 leaves $U(m-4)$ distinct odd starts below 2^m. Taking $m = \lfloor \log_2 X \rfloor$ gives $N_1(X) \geq a_1^9 X^\gamma$ for $X \geq 32$, and $n = 1$ suffices for $X < 32$.

For the upper bound, if $1 \leq X < 31$ then $15 \to 76 \to 38 \to 19$ and $17 \to 120 \to 60 \to 30 \to 15$ (exiting at 19) both exit, leaving $N_1(X) = 1 \leq X^\beta$. For $X \geq 31$, choose the unique integer $h \geq 1$ with $30 \cdot 2^{h-1} < X \leq 30 \cdot 2^h$. Each congruence class modulo $30 \cdot 2^h$ meets $[1, X]$ at most once, while Part 2 gives $N_h(1) = N_h(17) \leq (2a^3)^{-1}a^{-h}$ and $N_h(15) \leq a^{-h}$ for all $h \geq 0$ (for $h \in \{0,1,2\}$, $N_h(15) = 1 \leq a^{-h}$; for $h \geq 3$, $N_h(15) = 2N_{h-3}(1) \leq 2(2a^3)^{-1}a^{-(h-3)} = a^{-h}$). Using $2a^4 = 1 - a$ (so $a^{-3} + 1 = (1+a)/(1-a)$) and $a^{-h} = (2^h)^\beta < (X/15)^\beta$,

$$N_1(X) \leq N_h(1) + N_h(15) + N_h(17) \leq \frac{1+a}{1-a}\, a^{-h} < \frac{1+a}{(1-a)\,15^\beta}\, X^\beta < 0.858\, X^\beta < X^\beta.$$

Summing over $X/2^j$ includes all even multiples and preserves natural density zero. (The pure-kernel Lean 4 library proves the looser prefactor $N_1(X) \leq 2.44\, X^\beta$ via the odd-step 3×3 transfer matrix.) □

Remark (planar self-similarity and embedding dimension). In Figure 9B, the bit-interleaved Morton map $\Phi(\sum_{j\geq 0} b_j 2^j) = (\sum b_{2d} 2^{-(d+1)}, \sum b_{2d+1} 2^{-(d+1)})$ sends each 2-adic ball of radius 2^{-2m} to a dyadic square of side 2^{-m}, taking the square root of the metric (Hölder exponent 1/2, pulling back side $\varepsilon = 2^{-m}$ to 2-adic radius $\varepsilon^2 = 2^{-2m}$) and therefore doubling the box-counting dimension to $\dim_B(\Phi(\mathcal{K}_1)) = 2\beta \simeq 1.2527643$ (a coordinate effect of the planar embedding, whereas $\beta \simeq 0.6263821$ is intrinsic to $\mathbb{Z}_2$). Because digit-1 moves introduce no odd multiplier 5 or 7, the odd-depth shift $k \mapsto 2k$ ($2n-1 \to n$, +1 halving) swaps even and odd bits into a transposed, half-width affine image $\Phi(2k) = (y/2, x)$ on $[0, 1/2] \times [0, 1]$, whereas the even-depth shifts—the upper-left quarter $k \mapsto 4k$ ($\Phi(4k) = (x/2, y/2)$, $4n-3 \to 2n-1 \to n$, +2 halvings; e.g., $31 \mapsto 121$ and $91 \mapsto 361$) and the `1000` sub-square $k \mapsto 16k + 1$ ($\Phi(16k+1) = (1/2 + x/4, y/4)$, $480k + 31 \to \cdots \to 30k + 2$, which shares the exact orbit of $30k + 1$ after one micro-step, +4 halvings)—are untwisted Euclidean copies of the entire 1 mod 30 square, and `1001` ($k \mapsto 16k + 9$, +4 halvings) is an untwisted copy of the 17 mod 30 square.

3.3 Sharpening $N_1(X)$ to $[0.49875,\ 0.6245]$ and 2-adic/Archimedean separation

Theorem 7 shows that modulo-30 halving obstructions alone can never push the exponent of $N_1(X)$ below the 2-adic Hausdorff dimension $\beta \simeq 0.626382$, because every surviving 2-adic cylinder contains infinitely many integers reaching 1. To break strictly below β—and simultaneously raise the lower bound by restoring $d = 5$ and $d = 7$ branches—we must exploit the prime-to-2 congruences $2^v o \equiv 1 \pmod{25}$ and $2^v o \equiv 1 \pmod{7}$ of Proposition 6. Working modulo $Q_1 = 3 \cdot 5^4 \cdot 7^3 = 643{,}125$ assigns weights to all 128,625 residues in the classes A, B, C of Proposition 6. Knowing $o \bmod Q_1$ determines each reverse move $m = (2^v o - 1)/d$ modulo Q_1/d; writing $w_{\min}(m)$ and $w_{\max}(m)$ for the least and greatest weights among its d completions modulo Q_1 lets us certify sharper bounds in exact integer arithmetic.

Sharpened lower bound. Set $\gamma' = 399/800 = 0.49875$. The bundled integer vector verifies across all 128,625 residues (2,038,400 truncated moves with $v \le 64$) that

$$\sum_{\text{moves at } o,\ v \le 64} (d/2^v)^{\gamma'} w_{\min}(m) \ge w(o), \qquad \frac{w(31)}{\max w} \ge 0.2017. \tag{6}$$

Let $N(o, X)$ count distinct odd descendants of o in $[1, X]$, including o. For any basin member $o \ne 1$ and $X \ge o$, strong induction on $N(o, X)$ proves $N(o, X) \ge \frac{2}{5}\frac{w(o)}{\max w}(X/o)^{\gamma'}$. Admit children with $2^v o/d \le X$ (so $m < 2^v o/d \le X$, yielding disjoint subtrees excluding o); for discarded moves $Xd/(o2^v) < 1$, and valuations are spaced by $4, 20, 3$ for $d = 1, 5, 7$ respectively. Because $\gamma' \ge 2/5$, the discarded sum $\sum (Xd/(o2^v))^{\gamma'}$ stays below $(1 - 2^{-8/5})^{-1} + (1 - 2^{-8})^{-1} < 5/2$ on A, C and $(1 - 2^{-6/5})^{-1} < 5/2$ on B. Multiplying by $2/5$ keeps the loss strictly below 1, which the root o absorbs, and (6) closes the induction. Evaluating at $o = 31$ (with $n = 1$ for $X < 31$) yields $N_1(X) \ge 0.0806\,(X/31)^{399/800}$, and $31^{\gamma'} < 5.544$ gives $N_1(X) \ge 0.0145\,X^{0.49875}$ for all real $X \ge 1$.

A contracting obstruction—and the depth-based upper bound. Why can we not simply reverse (6) to sharpen the upper bound? Remarkably, the move graph admits *contracting double moves*: from a suitable odd multiple o of 15, applying $(d, v) = (7, 1)$ and then $(5, 3)$ produces $o' = (16o - 15)/35$, shrinking o by more than a factor of 2! Because $19o' + 15 = 16(19o + 15)/35$, solving $19(o/15) + 1 \equiv 0 \pmod{35^r}$ chains r successive shrinking double moves. Taking $X < o$ equal to the final descendant and letting $r \to \infty$ proves that *no finite residue vector can bound descendant counts at every node for all $X > 0$ by $w(o \bmod Q)(X/o)^c$ with $c > 0$.*

Organizing reverse paths by *halving depth* V and *digit product* D sidesteps this obstruction completely. Any positive vector ($\min w \ge 1$) obeying $\sum_{\text{moves at } o} d^s a^v w_{\max}(m) \le w(o)$ satisfies $\sum_{V=L} D^s \le w(o)\, a^{-L}$ by induction on L. Two verified 128,625-entry vectors at $(a_H, s_H) = (0.72135, 0)$ and $(a_P, s_P) = (0.7116, 9/25)$ (each testing 3,057,600 moves through $v = 96$ plus a geometric tail $7^s(\max w)a^{97}/(1 - a)$) together with a three-state forward vector at $(a_F, s_F) = (0.6331, 7/100)$ yield $N_1(X) \le C\,X^{0.6245}$:

Set $K = \lceil \log_2 X \rceil$, $\delta = 0.505$, $\beta' = 0.6245$, and $\theta_i = \log_2(1/a_i)$. Follow each odd $n \le X$ reaching 1 while its cumulative halvings stay below K:

1. If it hits 1 with $V < K$, the depth vector $(a_H, 0)$ gives $H(L) \le 2{,}953\,(0.72135)^{-L}$, totaling $O(2^{\theta_H K})$ starts.
2. Otherwise, let D be the digit product of the last prefix with $V < K$. If $D \ge 2^{\delta K}$, the forward vector (a_F, s_F) and the modulo-$15 \cdot 2^K$ coding injection yield $O(2^{(\theta_F - s_F\delta)K})$ starts.
3. If $D < 2^{\delta K}$, the crossing step enters an odd node $o \le Y = 7\sqrt{K+1}\,2^{\delta K}$ (multiplying $(1 + 1/(d_i n_i))$ over at most K distinct odd terms ≥ 15 stays below $\sqrt{K+1}$). Bounding

$o \le \sqrt{K+1}\,XdD/2^{v+V}$, raising to s_P, and summing by parts via $N_1(t) \le Ct^{\beta'}$ for $t \le Y$ contributes $O\big(C(K+1)^{\beta'/2}\,2^{(\theta_P+\delta(\beta'-s_P))K}\big)$ starts.

Exact integer comparisons verify that θ_H, $\theta_F - s_F\delta$, and $\theta_P + \delta(\beta' - s_P)$ all lie strictly below $\beta' = 0.6245$. For large X, $Y < X - 1$ and the third term drops below $CX^{\beta'}/2$; choosing C large enough to cover the first two terms and small X closes the induction, proving Part 2 of Theorem 2 and the following strict separation.

Corollary 8 (Strict 2-adic/Archimedean separation)**.** *The Archimedean growth exponent of the odd basin of* 1 *lies strictly below its* 2*-adic Hausdorff dimension:*

$$\limsup_{X\to\infty} \frac{\log N_1(X)}{\log X} \;\le\; 0.6245 \;<\; 0.626382\ldots \;=\; \dim_H(\overline{B_1}).$$

In contrast with the classical $3x+1$ map (where the integer basin of 1 is conjectured to be dense in all of $\mathbb{Z}_2$, giving dimension 1 in both topologies), base 10 exhibits a three-way split: 2-adic closure dimension $\beta \simeq 0.626382$, proved Archimedean ceiling ≤ 0.6245, and empirical growth near 0.49945 (Section 5).

4 Backward toward five: why the blue coral dominates

To prove that the blue coral of Figure 5 unconditionally dwarfs the amber coral, we need a lower bound on $N_5(X)$ with exponent strictly above 0.626383 (and 0.6245). Following Krasikov and Lagarias's difference-inequality framework for $3x+1$ [4], root an inverse tree at $o = 3$: since $3 \to 10 \to 5$ enters the six-cycle without ever returning to 3, the odd predecessors of 3 form a genuine tree whose members all reach 5.

An odd child of a vertex o under F^{-1} has the form $m = (2^v o - 1)/d$ ($d \in \{1,3,5,7,9\}$, $v \ge 1$), subject to the single congruence $2^v o - 1 \equiv d^2 \pmod{5d}$ enforcing integrality, oddness, and $d(m) = d$ all at once (for instance, $5\cdot 35+1 = 16\cdot 11$ gives $F(35) = 11$). Restrict to $1 \le v \le 16$, put $Q = 14{,}175$, and assign a nonnegative integer weight w_t to each of the $25{,}335 = \sum_{d\in\{1,3,5,7,9\}} Q/d$ residue types $t = (M, r)$ with $M = Q/d$. Because $5d \mid Q$ for all five digits, $R = o \bmod Q$ determines the complete list $\mathcal{M}(R)$ of admissible pairs (d, v) together with each child's type

$$t'(R, d, v) = \Big(\frac{Q}{d},\; \frac{2^v R - 1}{d} \bmod \frac{Q}{d}\Big).$$

Set $s = 13/20 = 0.65$, and test two finite inequalities at every residue $0 \le R < Q$:

$$\sum_{(d,v)\in\mathcal{M}(R)} (d/2^v)^s\, w_{t'} \ge w_{(M,\, R \bmod M)} \qquad \text{for all five moduli } M, \tag{7}$$

$$\sum_{(d,v)\in\mathcal{M}(R)} w_{t'} \le L. \tag{8}$$

The first inequality drives exponential branching across all available children; the second caps the weight lost when children exceed the cutoff X. Charging each discarded child its *own* weight—rather than worst-casing all 80 pairs by $\max w$—raises the resulting prefactor more than 35-fold using the exact same weights.

Lemma 9 (Charging individual child weights)**.** *Suppose* (7) *and* (8) *hold with* $L > 0$*. For any odd vertex* o *in the inverse tree of* 3 *and any integer* $X \ge o$*, let* $H(o, X)$ *count the odd* $n \le X$ *whose forward orbit visits* o*. Then for every type* t *of* o*,*

$$L\,H(o, X) \ge (X/o)^s\, w_t.$$

Proof. Set $q = X/o$ and $R = o \bmod Q$. Admit a pair $(d, v) \in \mathcal{M}(R)$ whenever $2^v o/d \le X$, so its child $m = (2^v o - 1)/d$ satisfies $m < 2^v o/d \le X$. Since F is a function and o is nonperiodic, admitted subtrees are pairwise disjoint, exclude o, and satisfy $H(m, X) < H(o, X)$. Strong induction on $H(o, X)$ gives

$$L\,H(o, X) \ge L + \sum_{\text{admitted}} L\,H(m, X) \ge L + q^s \sum_{\text{admitted}} (d/2^v)^s\, w_{t'}.$$

Every discarded pair satisfies $q\,d/2^v < 1$, hence $(q\,d/2^v)^s w_{t'} \le w_{t'}$. Summing over all discarded pairs loses at most $\sum_{\mathcal{M}(R)} w_{t'} \le L$, which the leading term L (from the vertex o itself) absorbs completely. Restoring the discarded terms and applying (7) closes the induction, including when no children are admitted. □

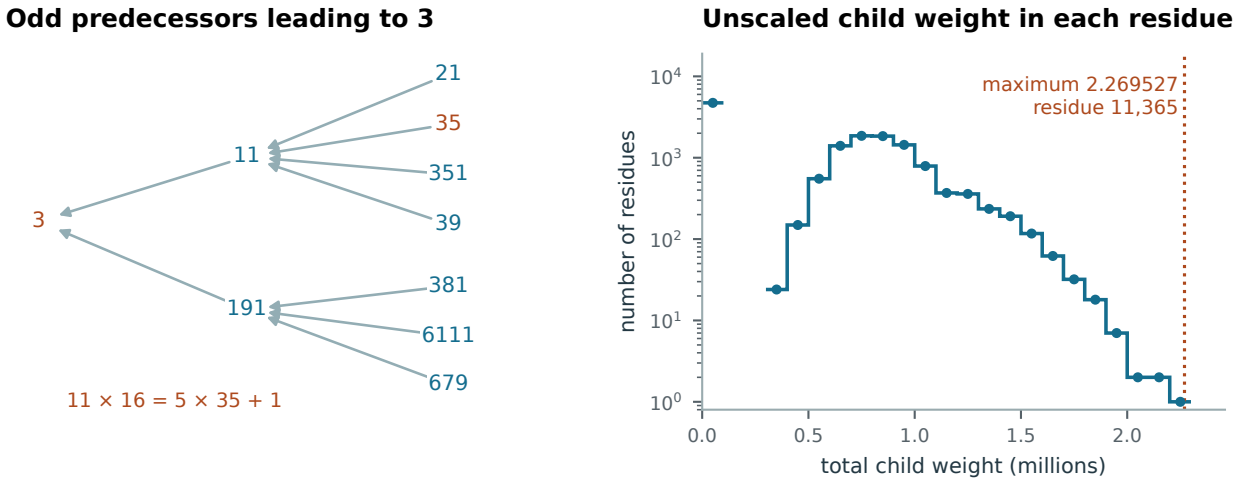

Figure 10: The basin-of-5 certificate. Left: odd predecessors of 3 with $v \le 6$ through two generations (arrows point forward). Right: distribution of the unscaled child-weight sum $\sum_{\mathcal{M}(R)} w_{t'}$ across all 14,175 residues modulo Q ($v \le 16$), peaking at $L = 2{,}269{,}527$ at residue $R = 11{,}365$.

Proof of Theorem 2, Part 3. The supplied 25,335-entry integer vector has root weight $w_{(Q,3)} = 98{,}959$ and maximum entry 10^6. For each (d, v), let $c_{d,v}$ be the greatest integer with $c_{d,v}^{20}\, 2^{13v} \le (10^6)^{20} d^{13}$. Across all 14,175 residues there are 81,072 admissible moves; every one of the $70{,}875 = 5Q$ integer comparisons $\sum c_{d,v} w_{t'} \ge 10^6 w_{(M,\,R \bmod M)}$ holds, proving (7). The largest raw row sum in (8) is $L = 2{,}269{,}527$ at $R = 11{,}365$ (Figure 10; the pure-kernel Lean 4 proof checks the same vector under the conservative budget $80 \cdot 10^6$, yielding 0.000605). Lemma 9 at $o = 3$ gives, for every integer $X \ge 3$,

$$N_5(X) \ge H(3, X) \ge \frac{98{,}959}{2{,}269{,}527 \cdot 3^{13/20}}\, X^{13/20} > 0.02134\, X^{0.65},$$

verified by exact 20th-power integer comparison. Combining with Part 2 gives $N_1(X)/N_5(X) = O(X^{-0.0255})$ (and already $O(X^{-0.0236})$ from Theorem 7 alone). Including even starts multiplies the numerator by at most $\sum_{j \ge 0} 2^{-j\beta'}$ without shrinking the denominator, so the all-start ratio also tends to zero. □

5 Excursions, cycles, and the quadrillion frontier

Suppose distinct odd integers $x_1, \dots, x_m$ form a cycle under F, with last digits $d_i = d(x_i)$ and halving counts $v_i \geq 1$ satisfying $2^{v_i} x_{i+1} = d_i x_i + 1$ (indices cyclic). Put $K = \sum_{i=1}^{m} v_i$ and $D = \prod_{i=1}^{m} d_i = 3^a 5^e 7^c$; the ordinary period is $K + m$. Multiplying around the cycle cancels the x_i and leaves

$$\frac{2^K}{D} = \prod_{i=1}^{m} \left(1 + \frac{1}{d_i x_i}\right), \qquad n_5 = n_3 + n_7, \tag{9}$$

where n_d counts digit d (recalling from Figure 7 that 5 enters precisely after 3 or 7). Hence (a, e, c) fixes four digit counts: $n_3 = e - c$, $n_5 = e$, $n_7 = c$, and $n_9 = (a - e + c)/2$. Furthermore, the modulo-5 recurrence $d_{i+1} \equiv (d_i^2 + 1)2^{-v_i} \pmod 5$ enforces a minimum halving budget. Assign incoming digit weights $w(1), w(3), w(5), w(7), w(9) = 1, 2, 0, 4, 3$ and the potential $\phi(d) = \mathbf{1}_{\{d=5\}}$. Checking all 25 pairs (d_i, d_{i+1}) gives $v_i \geq w(d_{i+1}) + 4\,\mathbf{1}_{\{(d_i, d_{i+1}) = (5,1)\}} + \phi(d_{i+1}) - \phi(d_i)$. Summing around the cycle telescopes ϕ to zero and yields

$$K \geq n_1 + 2n_3 + 4n_7 + 3n_9 + 4\,t_{51} \quad \Longrightarrow \quad 2n_1 + 3a + e + 7c \leq 2K, \tag{10}$$

where t_{51} counts $5 \to 1$ transitions.

5.1 Short cycles with no size hypothesis

Proposition 10 (No further cycles through eight odd members)**.** *The only primitive odd cycles of f_{10} with at most eight odd members are* (1) *and* $(5, 13)$*.*

Proof. For $m = 1$, $n_5 = n_3 + n_7$ leaves digit 1 or 9: $2^K = 9 + 1/x$ lies in $(9, 10)$, while $2^K = 1 + 1/x$ forces $x = 1$. An all-1 cycle has $F(x) \leq (x + 1)/2 < x$ for $x > 1$, leaving only (1). For $m = 2$, the remaining multisets are $\{1, 9\}$, $\{9, 9\}$, $\{5, 7\}$, and $\{3, 5\}$. Outside (1), the least odd elements of digits $1, 9$ are $11, 9$, confining 2^K for the first three to $(9, 10)$, $(81, 84)$, and $(35, 38)$—no powers of two. For $\{3, 5\}$, $2^K \in (15, 18)$ forces $K = 4$, and solving $5x + 1 = 2y$, $3y + 1 = 8x$ with the required digits leaves only $(5, 13)$.

For $m = 3$, exclude (1), $(5, 13)$, and 3 (which reaches 5): the least remaining odd integers of digits $1, 3, 5, 7, 9$ are $11, 13, 15, 7, 9$. The identity $n_5 = n_3 + n_7$ admits eight digit multisets:

digits	111	119	135	157	199	359	579	999
D	1	9	15	35	81	135	315	729
U	1	10	17	39	90	142	329	756

where $U = \lfloor \prod (d_i + 1/x_{\min}(d_i)) \rfloor$. Beyond 111, only 135 encloses a power of two ($K = 4$), yet its sole cyclic order is $3 \to 5 \to 1 \to 3$, where (10) demands $K \geq 1 + 2 + 4 = 7$, impossible.

For $4 \leq m \leq 8$, (9) gives the cutoff-free ceiling $2^K = \prod (d_i + 1/x_i) \leq 10^m$. Putting $L = \lfloor \log_2 10^m \rfloor$, there are exactly $5\binom{L}{m}$ choices for $(d_1, v_1, \dots, v_m)$ with $\sum v_i \leq L$. The modulo-5 recurrence determines $d_2, \dots, d_m$ uniquely, and closing the affine map $(Ax_1 + B)/2^K = x_1$ solves uniquely for $x_1 = B/(2^K - A)$, which is then tested for positivity, integrality, and exact digits and halvings. Exhausting all 7,811,375 choices at $m = 8$ (confirmed independently in Python and C++) produces only repetitions of (1) and $(5, 13)$. □

5.2 Excursions, shared peaks, and the 10^{15} sweep

Direct iteration of the first two million decimal starts (Figure 11) shows a dense glowing wave of blue orbits reaching 5 (median 160 steps; 7,762 starts exceed the 1,085-step display cap, topped by 3,337 steps from 1,761,259) riding cleanly above a thin amber ribbon of fast returns to 1. Comparing peak

heights against classical $3x+1$ through 10^7 (Figure 12) shows why base 10 feels even wilder than Collatz: its first-excursion tail decays as a heavy power law rather than plunging rapidly, and orbits coalesce into huge horizontal shared-peak bands—the densest decimal band (peak 7,823,727,640, first reached from 197) collects 252,655 starts, over 36 times the largest shared band in $3x+1$ (7,000 starts, peak 6,810,136, first reached from 4,255).

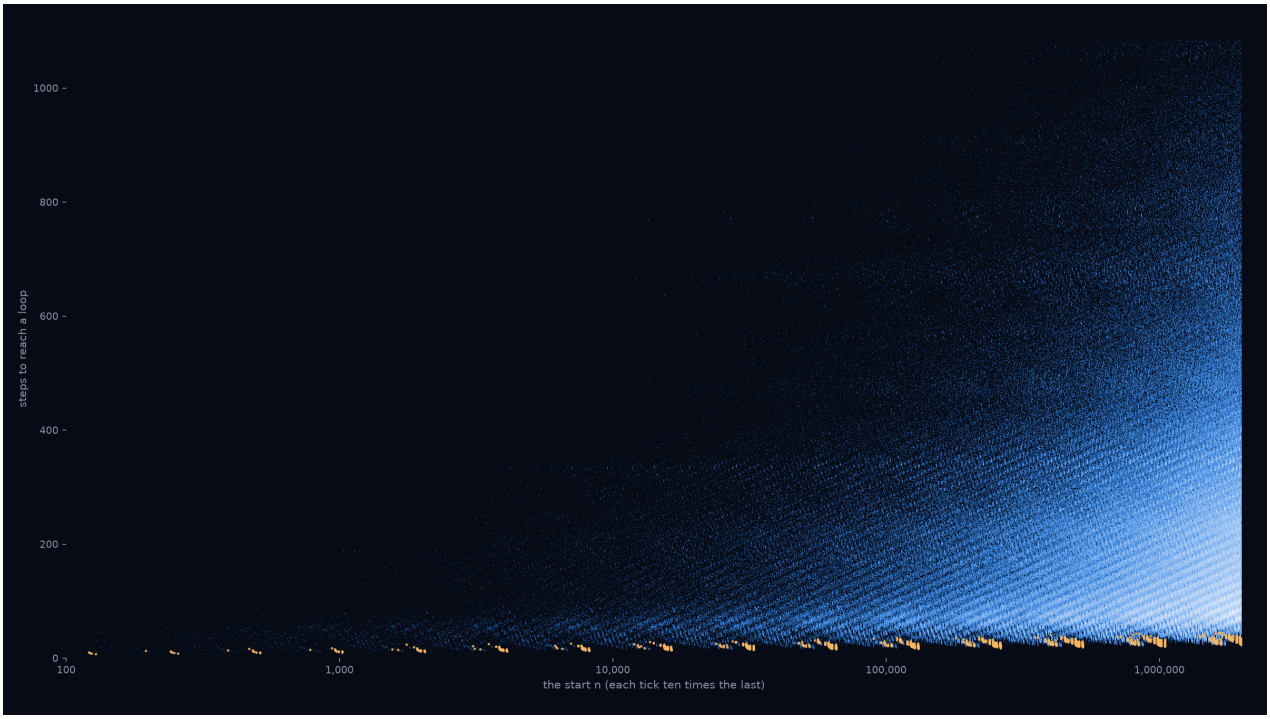

Figure 11: First-hitting times to 1 (amber) or 5 (blue density) across $1 \le n \le 2{,}000{,}000$, displayed from $n = 100$ with vertical axis clipped at 1,085 steps. In the full census, 7,762 starts exceed 1,085 steps, with maximum 3,337 at 1,761,259.

To verify far beyond 10^7, an archived exhaustive sweep tiles $[7, 10^{15})$ with 10,000 contiguous windows of width 10^{11}, covering all 499,999,999,999,997 odd starts (via modular early-descent filters and 1,009,968,528 exact arbitrary-precision continuations, zero fallbacks). Every odd start in $[7, 10^{15})$ descends below itself; 10^{15} is even and starts below 7 enter $\{1, 5\}$ directly, so strong induction proves that every positive integer $n \le 10^{15}$ reaches $\{1, 5\}$. The longest recorded first descent starts at 889,676,852,592,557 and runs 11,074 ordinary steps before dipping to 821,329,135,081,529; the widest peak reaches 268 bits, tied by 321,901,022,712,499 and 724,277,301,103,123.

5.3 From 10^{15} to a 53,185-step cycle floor

Any further cycle must have every odd member $x_i \ge X_0 = 10^{15}$. Taking logarithms in (9) forces a razor-thin linear form in logarithms, paralleling Simons's cycle obstructions for generalized Syracuse maps [5]:

$$0 < \Lambda := K \log 2 - \log D < \frac{1}{X_0} \sum_{i=1}^{m} \frac{1}{d_i} \le \frac{K}{X_0}, \qquad D = 3^a 5^e 7^c. \tag{11}$$

First assume $K \le L = 64{,}000$. Put $S = 2^{58}$ and $\lambda_p = \lfloor S \log_2 p \rfloor$ for $p \in \{3, 5, 7\}$. Then $F = KS - a\lambda_3 - e\lambda_5 - c\lambda_7$ equals $S\Lambda/\log 2$ plus three nonnegative floor errors summing to at most $a + e + c \le K$ (since $2^K > D \ge 2^{a+e+c}$), confining every solution to the thin four-dimensional box

$$1 \le F \le \left\lceil \frac{3LS}{2X_0} \right\rceil + L, \qquad 0 \le u_p \le \left\lfloor \frac{LS}{\lambda_p} \right\rfloor \quad (u_3 = a,\ u_5 = e,\ u_7 = c).$$

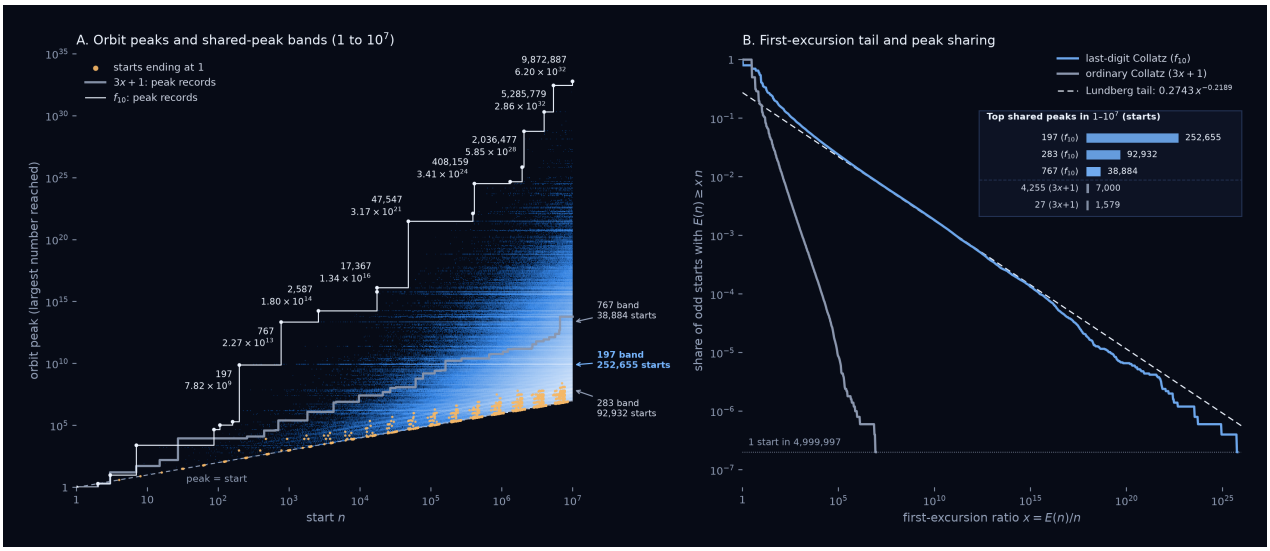

Figure 12: Excursions in f_{10} versus classical $3x+1$ through 10^7. **A:** Orbit peaks and record staircases; horizontal bands collect starts sharing the same maximum (led by the 197 band with 252,655 starts). **B:** Fraction of odd starts whose peak before first descent exceeds xn ($7 \le n \le 10^7$ for f_{10}, $3 \le n \le 10^7$ for $3x+1$), with inset comparing top shared peaks including the classical orbit of 27.

Reducing the lattice generated by $(0,0,0,S)$ and $(e_p, -\lambda_p)$ over $\mathbb{Z}^4$, enumerating all box points in exact rational arithmetic, and testing (11) with interval logarithms leaves 298 raw candidates, 44 with nonnegative integer digit counts (n_3, n_5, n_7, n_9), 32 meeting the refined gap (11) with $n_1 \le K - (n_3 + n_5 + n_7 + n_9)$, and just 19 surviving the halving budget (10) (Figure 13).

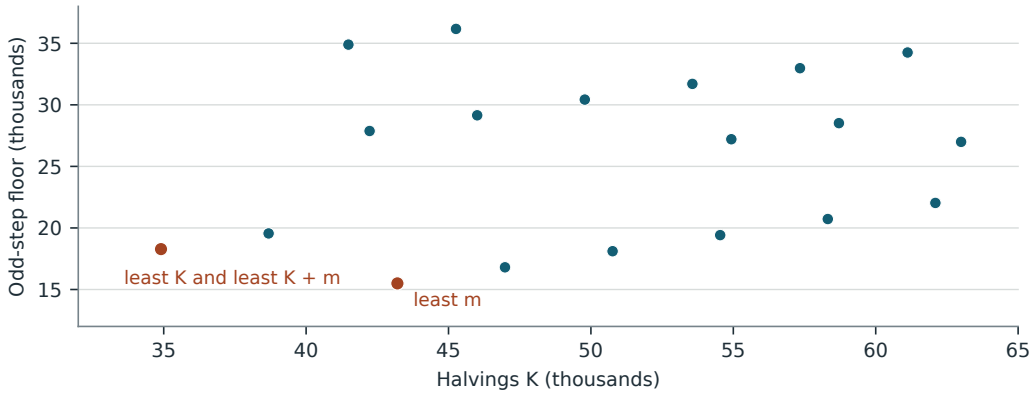

Figure 13: All 19 surviving exponent triples for $K \le 64{,}000$ plotted by halving count K and required odd-step floor m. Two distinct triples realize the three separate minima: $(8{,}842, 7{,}001, 1{,}651)$ gives $K = 34{,}905$, $m \ge 18{,}280$, $K + m \ge 53{,}185$, while $(13{,}041, 4{,}567, 4{,}250)$ gives $K = 43{,}205$, $m \ge 15{,}496$.

Taking the three separate minima over these 19 survivors yields $K \ge 34{,}905$, $m \ge 15{,}496$, and $K + m \ge 53{,}185$. In the remaining tail $K > L = 64{,}000$, if $9^m \le 2^L$, then $3m \le L$ gives $m/X_0 \le 1/6$, and (11) would force $2^K < 9^m e^{m/X_0} < 2^{L+1}$, contradicting $K \ge L+1$. Direct integer arithmetic confirms $9^{20189} < 2^{64000} < 9^{20190}$, forcing $m \ge 20{,}190$ and $K + m \ge 84{,}191$. This completes the proof of Theorem 2.

5.4 Empirical scaling of the thin basin

Direct iteration gives $N_1(X) = 11, 34, 101, 349$ at $X = 10^3, 10^4, 10^5, 10^6$ (and 668 versus 299,332 across all starts through 300,000). Enumerating the inverse tree of 1 with every odd iterate capped at 2^{72} (Figure 14) finds 383,455, 12,091,490, and 381,309,367 odd starts below 2^{40}, 2^{50}, and 2^{60}—growing nearly 32-fold per ten doublings. Because an orbit can climb above 2^{72} before landing at 1, every capped count is strictly a *lower bound* on $N_1(X)$: raising the ceiling from 2^{60} to 2^{72} recovers 10,355,985 extra starts below 2^{60} (on top of the 370,953,382 seen under the 2^{60} ceiling; Figure 14D).

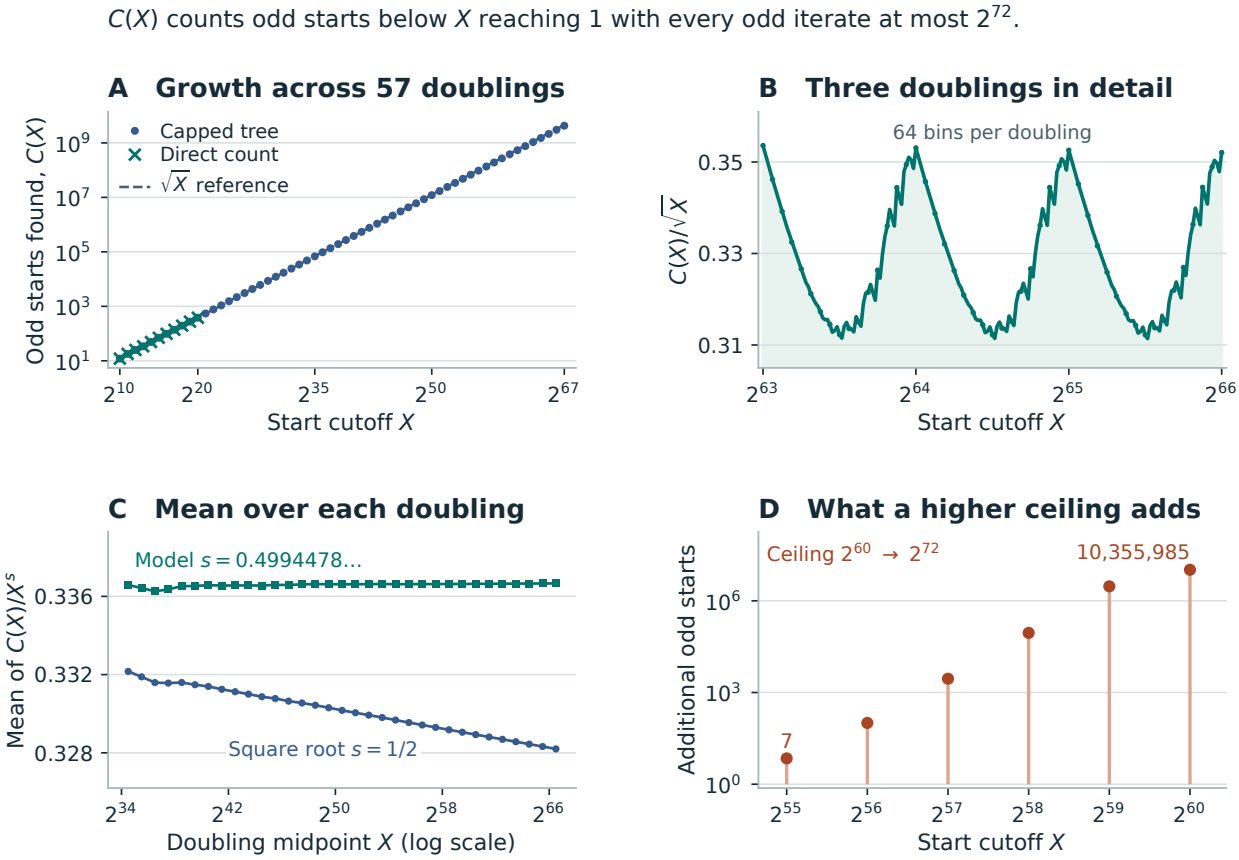

Figure 14: Odd starts below X reaching 1 with every odd iterate at most 2^{72}. **A:** Exact agreement with direct forward iteration through 2^{20}; above that, the curve is a lower bound on $N_1(X)$. **B:** Dividing by $\sqrt{X}$ exposes log-periodic ripples within each doubling. **C:** Geometric means over 64 bins per octave drift slightly below $\sqrt{X}$ while tracking the branching-model exponent $\alpha \simeq 0.4994478$. **D:** Additional starts below X recovered by raising the odd-iterate ceiling from 2^{60} to 2^{72}.

Dividing by $\sqrt{X}$ reveals sharp log-periodic ripples across each doubling (Figure 14B), and averaging over 64 bins per octave shows a slight downward tilt below $s = 1/2$ (Figure 14C). A branching model averaging the inverse moves of Proposition 6 uniformly modulo 25 and 7 predicts $\alpha \simeq 0.4994477848\ldots$, the unique root in $(0, 1)$ (with $a = 2^{-\alpha}$) of

$$1 = a + \left(1 + 35^{\alpha-1}\right) a^4. \tag{12}$$

(The 3×3 branching determinant factors off $(1 - a)(1 - a^4)$, and on $[0, 1]$ the negative derivative of $2^{-\alpha}$ exceeds $(\log 2)/2$ in magnitude while the positive derivative of $(35/16)^\alpha/35$ stays below $\log(35/16)/16$, guaranteeing a unique crossing.) Dividing the capped counts by $X^{0.4994478}$ flattens the curve across 30 doublings, though our proved window $[0.49875, 0.6245]$ still straddles $1/2$.

6 Pulling back across all bases: why base ten is unique

We zoomed into base 10 on the visual evidence of Figures 1 and 2 that bases 4 and 10 are the only two nontrivial contracting bases through $b = 20$. Could some larger base—say $b = 22$, 100, or 10^6—harbor a third Collatz twin? Pulling back across all $b \geq 2$ now settles this completely, proving Theorems 3 and 4 and explaining both the sawtooth reversal at base 10 (Figure 3) and the odd-base density spectrum (Figure 15).

6.1 Even bases: generalizing the coding and bounding the least drift

The base-10 coding of Lemma 5 generalizes cleanly to any even base $b \geq 2$. Along any prescribed branch sequence of T_b, $T_b^j(n) = (A_j n + B_j)/2^j$ with $A_0 = 1, B_0 = 0$; an even step leaves (A_j, B_j) unchanged, and an odd step of digit r sends it to $(rA_j,\, rB_j + 2^j)$. Since b is even, A_j stays odd, $B_j \geq 0$, and $A_j \neq 2^j$ for $j \geq 1$. Shifting n by $b\,2^{k-1}$ shifts $T_b^j(n)$ by $A_j b\,2^{k-1-j}$ (a multiple of b for all $j < k$), so the residue modulo $b\,2^{k-1}$ determines all k branches, and discarding finitely many exceptions $n \leq B_j/(2^j - A_j)$ proves that $p_k(b) = d(D_k(b))$ exists and equals the fraction of classes modulo $b\,2^{k-1}$ maintaining $A_j > 2^j$ for all $1 \leq j \leq k$.

Even bases split cleanly into two algebraic families:

- **Doubly even bases ($4 \mid b$).** Just as in Base 4, every odd digit r obeys $r^2 + 1 \equiv 2 \pmod 4$, so every odd step takes *always one halving* and T_b maps odd integers to odd integers. Lifting modulo $b\,2^{j-1}$ to $b\,2^j$ shifts the j-th image by $A_j b/2 \equiv b/2 \pmod b$, sending an odd digit r with probability $1/2$ each to

$$c = (r^2 + 1)/2 \pmod b, \qquad c + b/2 \pmod b. \tag{13}$$

  Even starts fail at $j = 1$, so $p_k(b)$ is half the survival probability of (13) started uniformly on odd digits.
- **Singly even bases ($b = 2M$, M odd).** Just as in Base 10 ($M = 5$), write $y = n \bmod M$ and let $\delta(y) \in [1, 2M)$ be its odd representative. A residue modulo $M2^k = b2^{k-1}$ corresponds bijectively to y_0 and k independent fair parity bits, evolving by $H(y) = y/2 \pmod M$ (even bit) and $Q(y) = (y^2 + 1)/2 \pmod M$ (odd bit). At successive odd iterates, the intervening halving count $v \geq 1$ has geometric law 2^{-v} (mean $h = 2$, versus $h = 1$ for $4 \mid b$).

On any closed communicating class of the odd-term chain with stationary law π, define the odd-term logarithmic drift $\Delta(\pi) = \sum_r \pi(r) \log r - h \log 2$, and let Δ_b be the minimum of $\Delta(\pi)$ across all closed classes. Why are there only four contracting even bases? Because the halving budget stays capped at $\log 2$ or $2 \log 2$ while the odd digits grow with b.

Lemma 11 (Drift growth and exact classification)**.** *If $4 \mid b$, then $\Delta_b = \frac{1}{2}\log(b/2+1) - \log 2$, attained uniquely on $\{1,\, b/2+1\}$. If $b = 2M$ with M odd, then $\Delta_b \geq \frac{1}{2}\log((M+1)/64)$. Hence $\Delta_b \to +\infty$, and across all even $b \geq 2$:*

$$\Delta_b < 0 \iff b \in \{2, 4, 6, 10\}, \qquad |\Delta_b| \leq \tfrac{1}{4} \iff b \in \{4, 8, 10, 14\}.$$

Proof. For $4 \mid b$, the two targets (13) are odd residues in $[1, b)$ differing by $b/2$; their product is at least $1 \cdot (b/2+1)$, with equality exclusively on the closed pair $\{1,\, b/2+1\}$. Thus $\Delta_b = \frac{1}{2}\log(b/2+1) - \log 2$, which is negative only at $b = 4$ and lies in $[-1/4, 1/4]$ only at $b \in \{4, 8\}$.

For $b = 2M$ (M odd), fix a starting digit r, put $z = (r^2+1) \bmod M$, and set $y_v = z2^{-v} \bmod M \in [0, M)$. If $z = 0$, then $\delta(y_v) = M$ for all v, giving row mean $\log M$. If y_1 is even and positive, $\delta(y_1) = y_1 + M \geq M + 2$, so $v = 1$ alone contributes $\frac{1}{2}\log(M+2)$. If y_1 is odd, then for each $v \geq 2$ the even integer $2^{v-1}y_v$ satisfies $2^{v-1}y_v = y_1 + tM$ with $t \geq 1$ ($t \leq 0$ is negative or odd), so $\delta(y_v) \geq (M+1)/2^{v-1}$; using $\sum_{v \geq 2}(v-1)2^{-v} = 1$, the row average is at least $\frac{1}{2}\log(M+1) - \log 2$.

Averaging in stationarity and subtracting $2\log 2$ gives $\Delta_b \geq \frac{1}{2}\log((M+1)/64)$, which is positive for $M \geq 65$ and exceeds $1/4$ for $M \geq 105$.

For the remaining odd $3 \leq M \leq 103$, form the exact odd-term matrix P by summing 2^{-v} over periods of $2 \bmod M$, and set $g(r) = \sum_s P(r,s)\log s - 2\log 2$. Every stationary law obeys $\Delta(\pi) \geq \min_r \ell^{-1}\sum_{i=0}^{\ell-1}(P^i g)(r)$, which is strictly positive at $\ell = 8$ for every odd $7 \leq M \leq 63$ by directed rational logarithm bounds. Exact rational stationarity solves through $b = 208$ confirm that only $b \in \{2,4,6,10\}$ are negative and only $b \in \{4,8,10,14\}$ lie in $[-1/4, 1/4]$ (Table 1 and Figure 3). □

b	Δ_b	$p_{12}(b)$	b	Δ_b	$p_{12}(b)$
2	−1.386294	0	8	+0.111572	1,407/8,192
4	−0.143841	113/4,096	10	−0.134161	495/4,096
6	−0.849815	7/6,144	14	+0.067947	5,669/28,672

Table 1: Odd-term drifts Δ_b (rounded to six places) and exact horizon-12 non-descent densities $p_{12}(b)$. Bases 4 and 10 sit almost side by side just below zero, while base 14 sits barely above zero.

Proof of Theorem 3. Write $S_j = \log A_j - j\log 2$ in the finite-state shortcut coding. For $b = 2M$ (M odd), the shortcut kernel $K = \frac{1}{2}(H+Q)$ and odd-term kernel $P = \frac{1}{2}Q(I - \frac{1}{2}H)^{-1}$ satisfy $\pi P = \pi \iff \pi K = \pi$, so they share the same extreme stationary laws and the shortcut drift per step is $\Delta(\pi)/2$ (for $4 \mid b$, it is $\Delta(\pi)$ directly). By the finite-state ergodic theorem, S_j/j converges almost surely to the drift of the eventual closed class. For $b \in \{2,4,6,10\}$ (each having a unique closed class), $S_j \to -\infty$ almost surely, so $p_k(b) \to 0$ and integers avoiding descent have natural density zero. For every other even base, $S_j \to +\infty$ almost surely; each path has a finite last minimum, so some state of positive initial mass has positive probability $\eta_b > 0$ of keeping $S_j > 0$ for all $j \geq 1$, whence $p_\infty(b) \geq \eta_b > 0$ by continuity of probability. □

Tilting the shortcut matrix $(M_t^{\pm} w)(z) = \mathbb{E}_z[(m/2)^{\pm t} w(z')]$ quantifies both sides via Markov's inequality ($M_t^{+} w \leq \lambda w$) and optional stopping of the supermartingale $(2^j/A_j)^t w(z_j)$ ($M_t^{-} w \leq w$), yielding certified bounds such as

$$p_k(6) \leq 4.595731\,(0.702852)^k, \qquad p_\infty(8) \geq 0.0861, \qquad p_\infty(14) \geq 0.0298.$$

Moreover, for every $b = 4k$ with $k \geq 4$ and $k \neq 6$, the single explicit vector $t = 1/2$, $w(1) = 1$, $w(r) = 2/5$ ($r \geq 3$) satisfies $M_{1/2}^{-} w \leq w$ universally (at $r = 1$ the ratio is $7\sqrt{2}/10 < 1$; at $r \geq 3$ the pair contains 1 only if $r^2 \equiv 1 \pmod b$, forcing $r \geq 7$ where $(7/10)\sqrt{2/r} \leq 2/5$), giving

$$p_\infty(b) \geq \frac{1}{2b} \sum_{\substack{3 \leq r < b \\ r \text{ odd}}} (1 - \sqrt{2/r}).$$

Notice the subtle limit-order caveat: $p_k(b) \geq p_\infty(b) > 0$ holds at every *finite* horizon k, whereas proving that an individual integer diverges to infinity requires ruling out an empty infinite intersection (like $\bigcap_k \{n : n \geq k\} = \emptyset$). In odd bases, however, exact divergence proofs succeed systematically.

6.2 Odd bases: zero-digit trapdoors and certified escapes

Drop $n = 33$ into two odd bases. In base 5, $33 \to 100 \to 50 \to 25$, where the zero digit sends $f_5(25) = 0 \cdot 25 + 1 = 1$. In base 17, ten steps return 33 to the exact same residue modulo 34 at 370,871: every $n \equiv 33 \pmod{34}$ obeys

$$f_{17}^{10}(n) = 11{,}200\,n + 1{,}271 \equiv 33 \pmod{34}, \tag{14}$$

so every integer in $34\mathbb{Z}_{\geq 0} + 33$ diverges to infinity! The following nine lines check (14) symbolically for all $q \geq 0$ at once (a zero in the tuple prescribes a halving):

```
a, c = 34, 33
for d in (16, 2, 5, 0, 0, 0, 0, 8, 14, 10):
    if d:
        assert a % 34 == 0 and c % 2 == 1 and c % 17 == d
        a, c = d*a, d*c + 1
    else:
        assert a % 2 == c % 2 == 0
        a, c = a//2, c//2
assert (a, c) == (34*11200, 33*11200 + 1271)
```

Yet base 17 also hides a period-8 cycle through 9 alongside both its trapdoor and this escaping progression.

For any odd $b \geq 3$, every multiple of b halves to an odd multiple of b and jumps straight to 1. Since halving preserves divisibility by odd b, a nonmultiple enters $b\mathbb{N}$ only from an odd digit r satisfying $r^2 \equiv -1 \pmod{b}$. Any prime divisor $p \equiv 3 \pmod 4$ eliminates all solutions, forcing $Z_b = b\mathbb{N}$ immediately. If every prime divisor of b is $1 \pmod 4$ (a *rooted* base), Hensel lifting and the Chinese remainder theorem yield $2^{\omega(b)}$ roots, giving one-step capture density $1/b + 2^{\omega(b)}/(2b)$. In base 5, every odd $n > 1$ either hits $5\mathbb{N}$ in 0, 1, or 2 steps (digits $0, 2, 3, 4$) or drops to $(n+1)/2 < n$ (digit 1), proving by induction that *every positive integer reaches* 1 *under* f_5.

Because odd b is coprime to 2, residue modulo b and low binary bits live in independent coordinates. Define the cylinder

$$\mathcal{C}(x, c, j) = \{n \geq 1 : n \equiv x \pmod{b},\ n \equiv c \pmod{2^j}\}, \qquad 0 \leq x < b, \quad 0 \leq c < 2^j,$$

of density $1/(b\,2^j)$. At $j = 0$, revealing one parity bit splits the cylinder in half. For $j \geq 1$ and $x \neq 0$, the next move requires *no* new bit, and an even digit x actually creates $v_2(x)$ fresh known bits:

parity of c	new residue mod b	new low bits	bits known
even	$x/2 \bmod b$	$c/2$	$j - 1$
odd	$x^2 + 1 \bmod b$	$xc + 1 \bmod 2^{j+v_2(x)}$	$j + v_2(x)$

Because specifying e output bits specifies exactly e input bits, any path using s parity splits from x pulls back to a single starting cylinder of density $1/(b\,2^s)$. Follow a *deterministic stretch* from $(x, 1, 1)$ while $j \geq 1$ and $x \neq 0$. If it enters a subset of an earlier cylinder in the stretch after h halvings and odd multiplications by $d_1, \ldots, d_\ell$, then $h \leq \sum v_2(d_i)$ and $\ell \geq 1$, so the return map $(An + B)/2^h$ has $A = \prod d_i \geq 2^h$ and $B > 0$: every integer in that cylinder diverges to infinity (as in (14), where $h = 4$).

Proof of Theorem 4. Reduce to the b zero-bit states $x \in \mathbb{Z}/b\mathbb{Z}$, giving $x = 0$ its initial share $1/b$ as an absorbing trap. From each nonzero x, move with probability $1/2$ to $x/2 \bmod b$ and with probability $1/2$ along the deterministic stretch from $(x, 1, 1)$, absorbing into trap (0), certified escape, another zero-bit state, or no absorbed mass if unresolved. Let τ_b, ρ_b be the finite-path absorption probabilities into trap and escape. Every finite union of disjoint first-hit cylinders has natural density equal to its path probability; taking suprema gives $\underline{d}(Z_b) \geq \tau_b$ and $\underline{d}(E_b) \geq \rho_b$ without commuting limits. When $\tau_b + \rho_b = 1$, the inclusions $Z_b \subseteq R_b$ and $E_b \cap R_b = \emptyset$ squeeze $\overline{d}(R_b) \leq 1 - \rho_b = \tau_b \leq \underline{d}(Z_b)$, establishing $d(Z_b) = d(R_b) = \tau_b$ and $d(E_b) = \rho_b$, where τ_b, ρ_b are the exact rational solutions of the finite absorption system. □

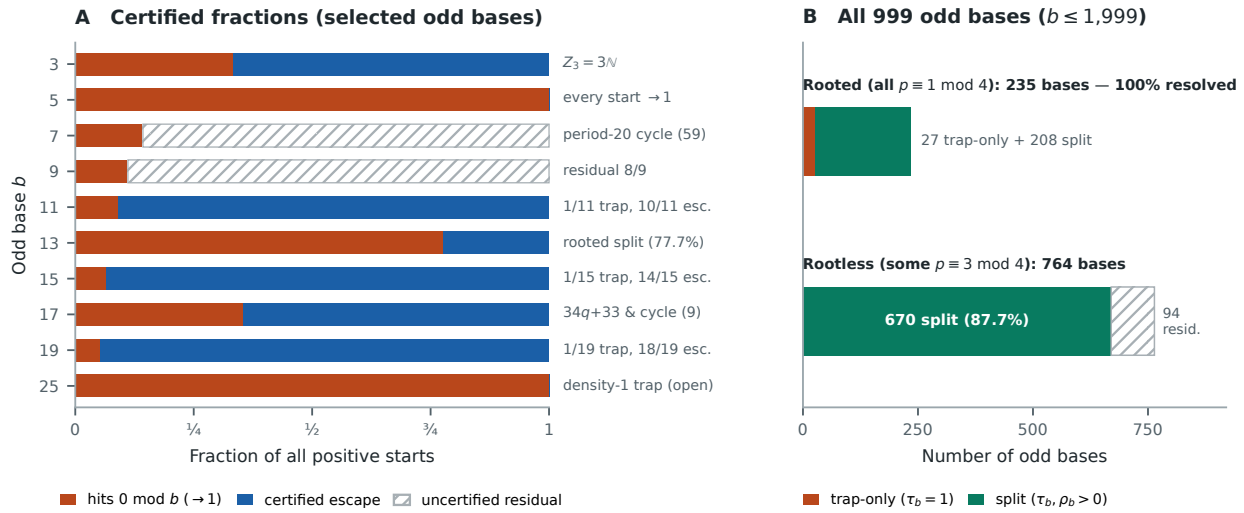

Figure 15: Certified odd-base natural densities. **A:** Selected odd bases from 3 through 25 (amber: zero-digit trapdoor into 1; blue: certified cylinder escape; hatched: uncertified residual mass, such as 6/7 at base 7 and 8/9 at base 9). In base 17, trapping and divergence already sum to 1 despite a period-8 cycle through 9. **B:** Complete survey of all 999 odd bases $3 \leq b \leq 1{,}999$: all 235 rooted bases resolve completely (27 trap-only, 208 split), while 670 of 764 rootless bases split with full absorption and 94 retain residual mass.

Surveying all 999 odd bases $3 \leq b \leq 1{,}999$ with a 2,000-step stretch budget (6,000 at $b = 1{,}983$, where the longest resolved stretch takes 2,640 steps) establishes $\tau_b + \rho_b = 1$ in 905 bases (Table 2 and Figure 15): all 235 rooted bases (27 trap-only, 208 split) and 670 of 764 rootless bases, leaving residual graph mass in 94 rootless bases (such as 6/7 in base 7, which hosts a period-20 cycle through 59).

b	τ_b	ρ_b	Behavior beyond the cylinder fractions
3	1/3	2/3	$Z_3 = 3\mathbb{N}$ exactly
5	1	0	Every positive integer reaches 1
7	1/7	0	Residual 6/7; period-20 cycle through 59
13	13,210/16,991	3,781/16,991	Complete absorption ($\tau_{13} + \rho_{13} = 1$)
17	148,789/417,911	269,122/417,911	Complete absorption, plus period-8 cycle through 9
19	1/19	18/19	Stretch enters earlier escape cylinder
25	1	0	Density-one trapping (universal capture open)

Table 2: Exact absorption probabilities for selected odd bases. Whenever $\tau_b + \rho_b = 1$, Theorem 4 promotes these fractions to exact natural densities.

7 Every ending in a set of density zero

Section 6 held the starting integers fixed and varied the base of the iteration rule. To close, hold the *rule* fixed in base 10 and vary only the base in which we write the starting integers.

The two numbers 131,071 and 431,071 share their last five decimal digits, and both leave remainder 1 on division by 30. Yet $131{,}071 + 1 = 2^{17}$, so the first drops straight to 1 by seventeen halvings, whereas the second follows odd terms $431{,}071 \to 13{,}471 \to 421 \to 211 \to 53 \to 5$. How much can trailing digits ever reveal about which loop captures a start?

Write $\mathcal{B}_i = \{n \in \mathbb{Z}_{\geq 1} : f_{10}^{j}(n) = i \text{ for some } j \geq 0\}$ for the full (even and odd) basin of $i \in \{1, 5\}$. By Theorem 7, $\mathcal{B}_1$ has natural density zero. Yet if we write starting integers in base 7 and prescribe *any* finite string of trailing base-7 digits—even a thousand zeros—infinitely many *odd* integers with that ending reach 1, and infinitely many reach 5.

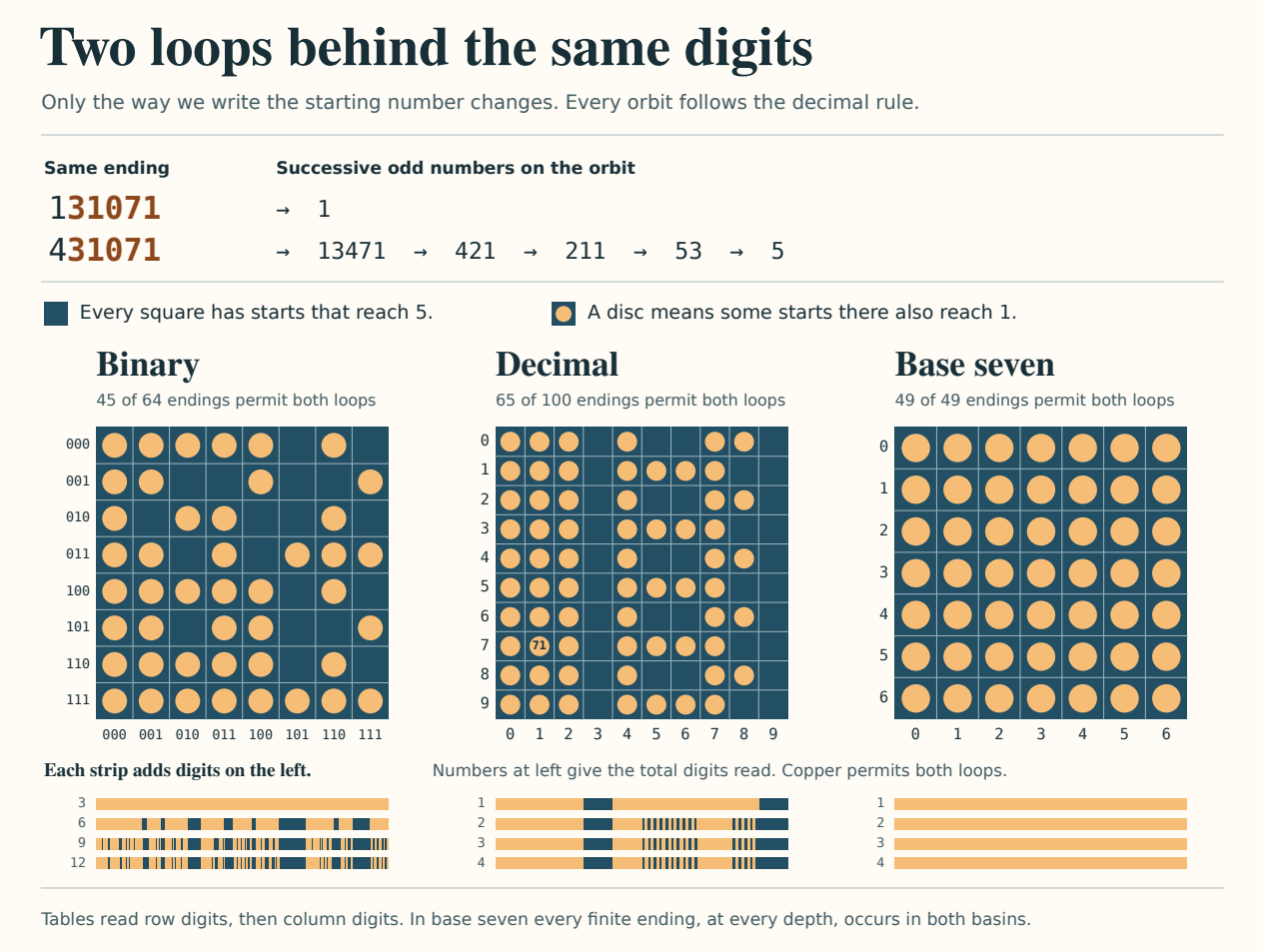

Figure 16: Exact tables of trailing endings for the decimal rule f_{10} when starting integers are written in binary (45/64 endings permit both loops), decimal (65/100; row 7, column 1 permits both loops, whereas 73 admits only 5), and base 7 (49/49: every ending permits both loops). Every cell contains infinitely many starts reaching 5; copper discs mark endings that also contain infinitely many starts reaching 1. Each strip below orders endings from the units digit outward, subdividing as digits are prepended on the left.

Figure 16 displays this contrast across binary, decimal, and base 7. The mechanism behind it is the exact +30 commutator (5) already introduced in Section 3: swapping the two legal backward digit-1 moves $L(u) = 2u - 1$ and $H(u) = 32u - 1$ yields $Q(u) = P(u) + 30$, translating by 30 modulo any odd modulus q while preserving the eventual target u. Extending that argument from $u = 1$ to

all residues modulo 30 characterizes both basins in every congruence class $a \bmod m$.

Theorem 12 (Completions of every congruence class)**.** *Every congruence class $a \bmod m$ ($m \geq 1$) contains infinitely many members of $\mathcal{B}_5$. For $\mathcal{B}_1$, put $h = \max(0, v_2(m) - 1)$, $M = 30 \cdot 2^h$, and $p = \gcd(m, M)$, and recall the shortcut map $T = T_{10}$ (extended to $x = 0$ via the even branch $T(0) = 0$). Then $a \bmod m$ contains infinitely many members of $\mathcal{B}_1$ if and only if some $0 \leq x < M$ satisfies*

$$x \equiv a \pmod{p}, \qquad T^h(x) \bmod 30 \in S = \{0, 1, 2, 4, 8, 15, 16, 17\}.$$

If no such x exists, the congruence class contains no member of $\mathcal{B}_1$.

Proof. From any residue $r \bmod 30$, the two possible one-halving successors of $r + 30s$ ($s \in \{0, 1\}$) are $T(r)$ and $T(r) + 15 \pmod{30}$. Backward breadth-first search on $\mathbb{Z}/30\mathbb{Z}$ partitions the states reaching 1 mod 30 and those reaching $\{1, 11, 21\} \pmod{30}$ (last decimal digit 1) into exact distance layers:

Distance	Target 1 mod 30	Target ending in 1 ($1, 11, 21 \bmod 30$)
0	1	1, 11, 21
1	2	2, 9, 12, 19, 22, 29
2	4	4, 8, 14, 18, 24, 28
3	8	5, 6, 15, 16, 25, 26
4	15, 16	0, 3, 7, 10, 13, 17, 20, 23, 27
5	0, 17	

Thus $S = \{0, 1, 2, 4, 8, 15, 16, 17\}$ consists precisely of all residues modulo 30 that can reach 1 mod 30 (so S is closed backward: no residue outside S can ever enter S), while *every* residue modulo 30 reaches a residue ending in 1 within at most four halvings.

By the Chinese remainder theorem, each $x \bmod M$ compatible with $a \bmod p$ refines $a \bmod m$ into arithmetic progressions of step $\operatorname{lcm}(M, m) = 30 \cdot 2^h D$ (D odd). Because the first h shortcut branches are fixed along $x + 30 \cdot 2^h Dj$, each step halves the spacing and (on an odd branch) multiplies by the odd digit in $\{1, 3, 5, 7, 9\}$, sending the progression onto a full tail $b + 30D'j$ ($j \geq 0$, D' odd) in residue $T^h(x) \bmod 30$. Splitting j by parity and applying T realizes either graph successor as a whole progression tail of the same form.

To reach 1, route any live state in S to 1 mod 30 and pick seed $u = 1$. To reach 5, route any state in $\mathbb{Z}/30\mathbb{Z}$ to 1, 11, or 21 (mod 30) and pick the corresponding odd seed 211 ($211 \to 53 \to 5$), 11 ($11 \to 3 \to 5$), or 21 ($21 \to 11 \to 3 \to 5$). For the resulting progression tail $b + 30Ej$ (E odd), apply the commutator translation $Q \circ P^{s-1}$ of (5) modulo the odd integer $q = 15E$. Because the chosen seed $u \in \{1, 11, 21, 211\}$ agrees with $b \bmod 30$, repeated translation hits $b \bmod 15E$, and oddness upgrades this to $b \bmod 30E$. Iterating P^s yields arbitrarily large representatives $\geq b$ that lift back to positive integers in $a \bmod m$ reaching the desired loop. Necessity for $\mathcal{B}_1$ follows from $T(1) = 1$ and the backward closure of S. □

Notice what Theorem 12 rules out: no modular test modulo any $m \geq 1$ can ever certify membership in $\mathcal{B}_1$ (every class meets $\mathcal{B}_5$ infinitely often), and since $p = 2^{v_2(m)} \gcd(m, 15)$, odd primes beyond 3 and 5 create no obstruction at all. If m is odd and $15 \nmid m$, both basins appear infinitely often in every class modulo m; even requiring odd starts ($m = 2 \cdot 7^k$) gives $h = 0$, $p = 2$, so $x = 1$ still succeeds for every base-7 ending! By contrast, if $15 \mid m$ is odd, the classes meeting $\mathcal{B}_1$ are precisely $\{0, 1, 2, 4, 8\} \pmod{15}$ (neither modulus 3 nor modulus 5 alone excludes any class, yet together they leave only five of fifteen).

To measure how much room the allowable endings leave in an arbitrary written base $b \geq 2$, write each $n \in \mathcal{B}_1$ in base b as $n = \sum_{j \geq 0} d_j b^j$ (padded on the left with zeros), map it to the reversed

expansion $\sum_{j\geq 0} d_j b^{-j-1} \in [0,1]$, and take the closure $\mathcal{E}_b \subset [0,1]$. Thus a base-$b$ prefix interval of length b^{-k} prescribes the last k base-b digits of n, and $\mathcal{E}_b$ represents all infinite trailing digit strings whose every finite ending occurs in $\mathcal{B}_1$.

Corollary 13 (Exact Hausdorff and box dimension in every written base)**.** *Write an integer base $b \geq 2$ as $b = 2^t u$ with u odd, and let $\lambda = 1/a \simeq 1.543689$ be the unique positive root of $\lambda^4 = \lambda^3 + 2$. Then the Hausdorff and box-counting dimensions of $\mathcal{E}_b \subset [0,1]$ both equal*

$$\dim_H(\mathcal{E}_b) = \dim_B(\mathcal{E}_b) = \frac{\log u + t \log \lambda}{\log b}.$$

To eight decimal places, $\dim \mathcal{E}_2 = 0.62638214$ (recovering β from Theorem 7 and strictly exceeding the integer ceiling 0.6245 of Corollary 8), $\dim \mathcal{E}_{10} = 0.88752982$, and $\dim \mathcal{E}_7 = 1$ with $\mathcal{E}_7 = [0,1]$. The analogous closure for $\mathcal{B}_5$ equals $[0,1]$ in every base $b \geq 2$.

Proof. Under the four-state collapse $S_0 = \{0,1,2,17\}$, $S_1 = \{15,16\}$, $S_2 = \{8\}$, and $S_3 = \{4\}$ of Section 3, Theorem 7 shows that the number $N_r(h)$ of live length-h halving paths starting from any $r \in S$ satisfies $c_1\lambda^h \leq N_r(h) \leq c_2\lambda^h$ uniformly for constants $0 < c_1 < c_2$ (with $N_1(h) = N_1(h-1) + 2N_1(h-4)$ for $h \geq 4$).

At digit depth $k \geq 1$, refine each occupied suffix modulo b^k to modulus $\ell_k = \operatorname{lcm}(30, b^k) = 30 \cdot 2^{h_k} D_k$, where $h_k = \max(0, tk-1)$ and $D_k = u^k / \gcd(u^k, 15)$. Since $\gcd(u^k, 15) = \gcd(u, 15)$ is constant for all $k \geq 1$, we have $D_{k+j}/D_k = u^j$ and $h_{k+j} - h_k = tj$ (for $t \geq 1$, or 0 for $t = 0$). From any live refined class modulo ℓ_k ending at state $r \in S$ after h_k halvings, the number of live refined extensions j digits later is exactly $(D_{k+j}/D_k)\, N_r(h_{k+j} - h_k) = u^j N_r(tj)$, because the odd-modulus factors choose freely among u^j lifts while the low-bit paths count live halvings. Since each occupied suffix modulo b^k has between 1 and 30 refined lifts modulo ℓ_k, the number of occupied depth-$(k+j)$ extensions of *every* occupied depth-k suffix lies between fixed positive constants times θ^j, where $\theta = u\lambda^t > 1$.

In particular, there are $\asymp \theta^k$ occupied level-k intervals of length b^{-k}, giving $\dim_B(\mathcal{E}_b) = \log_b \theta = (\log u + t \log \lambda)/\log b$ and $\dim_H(\mathcal{E}_b) \leq \log_b \theta$. For the Hausdorff lower bound, put uniform mass on the occupied words at depth K and take a weak-$*$ limit μ on the compact space $\{0, \ldots, b-1\}^{\mathbb{N}}$ as $K \to \infty$. By the uniform extension bounds, every length-k cylinder carries μ-mass $O(\theta^{-k})$ (hence every singleton has μ-mass 0). Pushing μ forward to $\mathcal{E}_b \subset [0,1]$ via base-b expansion, any interval $I \subset [0,1]$ with $b^{-(k+1)} \leq |I| < b^{-k}$ meets at most three level-k b-adic intervals, hence carries mass $O(\theta^{-k}) = O(|I|^{\log_b \theta})$. The mass distribution principle gives $\dim_H(\mathcal{E}_b) \geq \log_b \theta$. Finally, Theorem 12 makes every depth-k ending occupied for $\mathcal{B}_1$ in base 7 (since $\gcd(7^k, 30) = 1$) and for $\mathcal{B}_5$ in every base $b \geq 2$, giving $[0,1]$ identically. □

Concretely, in base 7 the ending 314 occurs on both odd starts $(256314)_7 = 47{,}835 \to 1$ and $(1314)_7 = 501 \to 5$; prepend another digit on the left, or a thousand more, and infinitely many odd starts in each basin still carry that exact ending—even though inside every such progression $\mathcal{B}_1$ has relative natural density zero ($O(7^k X^{-0.3755}) \to 0$). Membership in $\mathcal{B}_1$ versus $\mathcal{B}_5$ is therefore a genuinely global property of the decimal dynamics, invisible to any finite 7-adic suffix and one-sidedly invisible to all modular arithmetic.

8 Open problems, verification, and prior work

Four crisp open problems tie together the entire base-b picture:

1. *The Ouroboros conjecture (base 10):* Does every positive integer reach 1 or 5 under f_{10}?
2. *The thin-basin exponent (base 10):* Does $\lim_{X\to\infty} \log N_1(X)/\log X$ exist, and does it equal the branching-model root $\alpha \simeq 0.4994477848\ldots$ of (12)? Settling whether $\alpha < 1/2$ requires sharper residue equidistribution along the inverse tree than $w_{\min}$ and $w_{\max}$ completions provide.
3. *Individual divergence in even bases (base 8):* Theorem 3 proves $p_\infty(8) \geq 0.0861$, yet an explicit integer orbit diverging under f_8—analogous to $34q + 33$ in base 17—remains unknown. Can any concrete positive integer be proved to diverge in base 8?
4. *Universal capture in odd bases (base 25):* Theorem 4 certifies $\tau_{25} = 1$, so almost every start reaches 1 in base 25, yet unlike base 5 no proof rules out a density-zero exceptional cycle or escape. Does every positive integer reach 1 under f_{25}?

Standalone verification and Lean 4 formalization. Every computer-assisted result separates *search* from *certification*: the offline archive `checker.zip` runs under Python 3.10+ with no external libraries (`python3 checker.zip`) and verifies solely by exact integer and rational arithmetic:

- both verbatim Python boxes printed in Sections 1 and 6, plus the base-4 conjugacy through $y < 10{,}000$;
- exact even-base drifts through $b = 208$, all 14 tilted stopping-time vectors, and 78 complete integer-block comparisons of $p_k(b)$;
- all 999 odd-base cylinder surveys through $b = 1{,}999$ and exact rational absorption solves;
- all 128,625-row decimal basin-1 vectors (104,125 positive lower weights, 2,038,400 lower moves, 3,057,600 moves plus tail per upper vector) and all 70,875 rows (81,072 moves, $L = 2{,}269{,}527$) of basin-5;
- both short-period enumerations through $m = 8$ (Python and C++, 7,811,375 words), the 4D lattice screen (19 survivors), and all 10,000 sweep windows (`-r` replays both records per window);
- Figures 5, 8, 9, 11, and 16, the four-state automaton, 45,756-class quotient checks, and the Lean 4 axiom SHA-256.

The exhaustive $[7, 10^{15})$ sweep and 2^{72}-capped inverse-tree shards are archived separately. The `Ouroboros` Lean 4 library (revision `d0d72b27`, Lean/Mathlib 4.35.0) has 424 audited public declarations across 30 source files (0 `sorry`, 0 `native_decide`, axioms `propext`, `Classical.choice`, `Quot.sound`); Table 3 divides formal and external checks.

Prior work and authorship. Terras introduced parity-word coding for density-one stopping times in $3x+1$ [1]; Matthews and Watts extended congruence-cylinder coding to generalized Syracuse maps [2]; and Leigh developed Markov residue chains when multipliers share factors with the modulus [3]. The affine change of variables linking f_4 to Collatz parallels Yolcu, Aaronson, and Heule's analysis of Farkas's map [7], and the modular affine-translation mechanism behind (5) parallels Monks, Monks, Monks, and Monks's group-action analysis of arithmetic progressions in the $3x + 1$ graph [6]. For base 10, Krasikov and Lagarias's difference-inequality framework motivates our backward tree bounds [4], Simons's logarithmic cycle obstructions parallel our four-dimensional lattice screen [5], and Santi previously recorded the six-cycle (2) in his alternating-3/5 map [8]. We have not found an earlier study of the last-digit family f_b itself. The first author designed and directed the AI-assisted search that discovered f_{10} among 16 digit-driven candidates; AI agents developed the proofs, Lean formalization, computations, figures, and draft exposition under his direction. The manuscript awaits human mathematical review.

Topic	Lean 4 kernel (`d0d72b27`)	Exact verifier or archival record
Base 4 & even bases	Base-4 conjugacy, finite coding, drift growth, sign classification, tilting inequalities	Finite-state ergodic transfer and last-minimum step; 14 explicit tilted vectors
Odd bases	Trapdoor congruence, cylinder bit rules, base-17 escape, density squeeze	Survey of all 999 odd bases through 1,999 and exact absorption solves
Decimal basins	Mod-30 law, binary runs, quartic bounds ($N_1(X) \le 2.44\,X^{\beta}$), contracting double move, basin-5 bound (0.000605)	Analytic 2-adic Cantor closure, congruence-class & written-base dimension checks (45,756 classes), $N_1(X) \le X^{\beta}$; weighted basin-5 (0.02134); three 128,625-row basin-1 vectors
Decimal cycles	Cancellation law, halving budget, $m \le 3$ classification, lattice-box inclusion, $K > 64{,}000$ tail	Exhaustive $m \le 8$ search; 4D lattice enumeration (19 survivors); archived $[7, 10^{15})$ sweep

Table 3: Division of evidence between the pure-kernel Lean 4 formalization and exact external arithmetic.

References

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