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Working Draft

Matroid Reconstruction Fails: Explicit Non-Isomorphic Matroids with Identical Deletion and Contraction Decks

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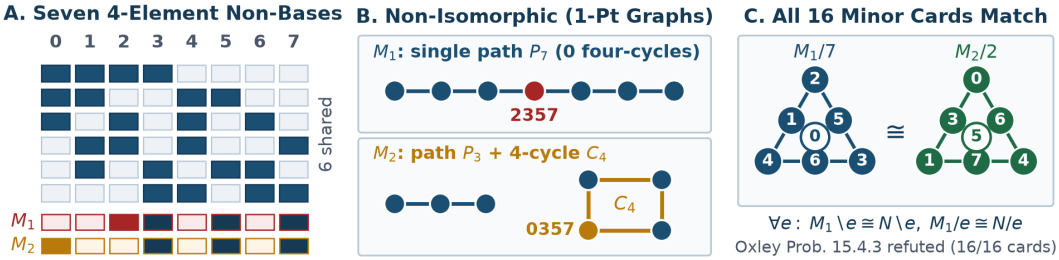
Supporting Information: `recompute.py`, audited Lean 4 + Mathlib library (`lean/MatroidReconst/`), and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Brylawski (1986) asked whether every matroid on ≥ 2 elements is determined by its single-element deletions and contractions (Oxley, *Matroid Theory*, Problem 15.4.3, 2011; March 2020 errata; Miller's 1997 Minor Reconstruction Conjecture). We refute this with two non-isomorphic rank-4 sparse paving matroids M_1, N on 8 elements (63 bases, 7 non-bases) satisfying $M_1\backslash\{e\} \cong N\backslash\{e\}$ and $M_1/\{e\} \cong N/\{e\}$ for all e. Our Lean 4 + Mathlib library (`lean/MatroidReconst/`) unconditionally proves this refutation, sparse paving basis exchange, and the 1-point 4-cycle invariant separating M_1 (P_7) from M_2 ($P_3 \sqcup C_4$). Exact computation shows M_1 is $\mathbb{Q}$-representable in $\mathbb{Z}^3$ while M_2 is non-representable over every field. Written proofs give minor-reconstructibility for rank or corank ≤ 2, truncation invariance, and that rank-3 minor-reconstructibility implies Harary's Edge Reconstruction Conjecture (graphs without isolated vertices). Exhaustive censuses find 0 fails for $n \leq 7$, 3 pairs for $n = 8$, 170 for $n = 9$ (85 rank 4, 85 rank 5), and 0 across 32,209,426 rank-3 matroids ($n \leq 12$).

Verification Scope (G11): WARN: Human referee: pending
Main results: 3 of 6 in Lean Kernel (50%) · 2 Cert Only · 1 written · 1 open (not counted)
Abstract claims: 3 of 6 in Lean Kernel (50%) · 2 Cert Only · 1 written

Full Trust Boundary: Section 5.

MSC 2020: 05B35, 05C60, 68V20 **Keywords:** matroid reconstruction, Oxley Problem 15.4.3, sparse paving matroids, edge reconstruction, Lean 4

1. INTRODUCTION AND MAIN RESULTS

Removing a single element e from a finite matroid M on ground set E produces two smaller matroids on $E\backslash\{e\}$: the *deletion* $M\backslash e$ and the *contraction* M/e. Taken up to isomorphism across all $e \in E$, these $2|E|$ minors form the single-element *minor deck* of M. In Chapter 15 of his standard reference *Matroid Theory*, Oxley [1] (p. 592) records the following problem attributed to Brylawski [2] (`[to check]`):

> *"A matroid M is minor-reconstructible if $N \cong M$ whenever N is a matroid on $E(M)$ such that $N\backslash e \cong M\backslash e$ and $N/e \cong M/e$ for all $e \in E(M)$. Brylawski (1986a) raised the following.* ***Problem 15.4.3.*** *Are all matroids minor-reconstructible?"*

Oxley's March 2020 errata [1] refine the wording to "all matroids with at least two elements," since neither 1-element matroid ($U_{0,1}$ or $U_{1,1}$) is minor-reconstructible, while leaving the problem open. Miller [4] (`[to check]`) stated the expected affirmative answer as the *Minor Reconstruction Conjecture* (M3), asking whether every matroid is determined by its multisets of single-element deletions and single-element contractions. Note that Oxley's ground-set-aligned formulation is strictly stronger than comparing separate multisets of deletions and contractions, because a single correspondence pairs both minors at every element e.

We refute Oxley's Problem 15.4.3 and Miller's Minor Reconstruction Conjecture with an explicit 8-element, rank-4 sparse paving counterexample pair, unconditionally formalized in Lean 4 with Mathlib (`lean/MatroidReconst/`, 76 audited declarations, zero `sorry`, zero `native_decide`, standard axioms only).

Theorem 1 (Refutation of Oxley Problem 15.4.3). *There exist two matroids M and N on the 8-element ground set $E = \{0,1,2,3,4,5,6,7\}$ such that*

$$M\backslash\{e\} \cong N\backslash\{e\} \quad \text{and} \quad M/\{e\} \cong N/\{e\} \quad \text{for every } e \in E,$$

yet $M \not\cong N$. In particular, Oxley's Problem 15.4.3 has answer no, *and Miller's Minor Reconstruction Conjecture fails.*

Specifically, Theorem 1 is witnessed by two rank-4 sparse paving matroids M_1 and M_2 on $E = \{0,\dots,7\}$, each with 63 bases, 7 non-bases, and no loops or coloops, differing in only one 4-element non-basis (2357 in M_1 versus 0357 in M_2):

$$\begin{aligned}\mathcal{H}_1 &= \{0123, 0145, 0246, 1247,\\ &\qquad 1356, 3467, 2357\},\\ \mathcal{H}_2 &= \{0123, 0145, 0246, 1247,\\ &\qquad 1356, 3467, 0357\}.\end{aligned}$$

Setting N to be M_2 relabeled by s^{-1}, where

$$s = (0\mapsto 5,\ 1\mapsto 1,\ 2\mapsto 0,\ 3\mapsto 4,\\ 4\mapsto 3,\ 5\mapsto 7,\ 6\mapsto 6,\ 7\mapsto 2),$$

aligns every single-element deletion and contraction card of M_1 at e with those of N at the same element e.

Our Lean 4 + Mathlib library establishes Theorem 1 via two general sparse paving lemmas formalized directly over Mathlib's `Matroid` API.

Lemma 2 (Sparse Paving Basis Exchange). *Let α be a finite ground set, $r \geq 0$ an integer, and $\mathcal{H}$ a family of r-element subsets of α such that $|H_1 \cap H_2| \leq r-2$ for all distinct $H_1, H_2 \in \mathcal{H}$ and at least one r-element subset of α lies outside $\mathcal{H}$. Then $\mathcal{B} = \{B \subseteq \alpha : |B| = r,\ B \notin \mathcal{H}\}$ satisfies the matroid basis exchange axiom, defining a rank-r matroid $M(\mathcal{H})$ on α.*

Lemma 3 (One-Point 4-Cycle Invariant). *Let $M(\mathcal{H}_1)$ and $M(\mathcal{H}_2)$ be rank-r sparse paving matroids on a finite ground set α. If $M(\mathcal{H}_1) \cong M(\mathcal{H}_2)$, then $\mathcal{H}_1$ contains four members a, b, c, d with $a \neq c$, $b \neq d$, and $|a \cap b| = |b \cap c| = |c \cap d| = |d \cap a| = 1$ if and only if $\mathcal{H}_2$ does.*

The 1-point intersection graph of $\mathcal{H}_1$ is a single 7-vertex path P_7 (having 0 four-cycles), whereas that of $\mathcal{H}_2$ splits into a 3-vertex path P_3 and a 4-cycle C_4, giving an immediate by-hand and kernel-checked proof of $M_1 \not\cong M_2$ (and hence $M_1 \not\cong N$).

Beyond the counterexample itself, we establish three companion results illuminating the geometry, structural boundaries, and exact small-size landscape of minor reconstruction.

Proposition 4 (Representability Contrast). *The matroid M_1 is representable over $\mathbb{Q}$ (realized by 8 explicit integer points in $\mathbb{Z}^3$ whose coplanar 4-point subsets are precisely $\mathcal{H}_1$), whereas M_2 is non-representable over every field.*

Proposition 5 (Rank/Corank ≤ 2, Truncation, and ERC). *The following structural results hold:*

1. *Every matroid on $n \geq 2$ elements of rank ≤ 2 or corank ≤ 2 is minor-reconstructible.*
2. *If matroids M_1, M_2 on E satisfy $M_1\backslash e \cong M_2\backslash s(e)$ and $M_1/e \cong M_2/s(e)$ for all $e \in E$ under a bijection s, then their truncations $T(M_1)$ and $T(M_2)$ satisfy the same aligned minor-deck equivalences under s.*
3. *If every rank-3 matroid is minor-reconstructible, then Harary's Edge Reconstruction Conjecture [9] holds for every graph with at least 4 edges and no isolated vertices (assuming Greenwell's theorem [10]; unconditionally, every graph without isolated vertices is determined by its vertex-deleted and edge-deleted decks together).*

Proposition 6 (Small-Matroid Censuses). *Exhaustive censuses establish:*

1. *Among all matroids on $2 \leq n \leq 7$ elements $(4, 8, 17, 38, 98, 306$ classes$)$, 0 pairs share an aligned minor deck. On $n = 8$ elements (1,724 classes), exactly 3 pairs (6 matroids, all rank 4) share an aligned minor deck: (M_1, M_2), a point-transitive pair (A, B) with 62 bases, identity deck bijection, and $|\mathrm{Aut}(A)| = 32$, $|\mathrm{Aut}(B)| = 8$, and a third 63-base pair (C, D) with $|\mathrm{Aut}(C)| = |\mathrm{Aut}(D)| = 3$. On $n = 9$ elements (383,172 classes), exactly 170 pairs (340 matroids: 85 of rank 4 and their 85 duals of rank 5) share an aligned minor deck.*
2. *Across all 32,209,426 rank-3 matroids on $3 \leq n \leq 12$ elements $(1, 4, 13, 38, 108, 325, 1{,}275, 10{,}037, 298{,}491, 31{,}899{,}134)$, 0 pairs share an aligned minor deck.*

By Proposition 5(1) and Proposition 6(1), $n = 8$ is the minimum ground-set size of a counterexample to Oxley's Problem 15.4.3, and rank 4 is the smallest rank with a known counterexample. Rank 3 remains open and, by Proposition 5(3), lies at least as deep as Harary's Edge Reconstruction Conjecture.

Conjecture 7 (Rank-3 Minor Reconstruction). *Every matroid of rank 3 (and dually of corank 3) is minor-reconstructible.*

Figure 1 summarizes the discovery and verification architecture. Before giving full definitions and proofs, the Python box below checks the sparse paving condition and the 1-point 4-cycle separation of $\mathcal{H}_1$ and $\mathcal{H}_2$ in milliseconds (replayed verbatim by `recompute.py`).

Instant Witness Check: Sparse Paving & C_4 Separation

```
import itertools as it
S = lambda t: [frozenset(map(int, w)) for w in t.split()]
H1 = S("0123 0145 0246 1247 1356 2357 3467")
H2 = S("0123 0145 0246 0357 1247 1356 3467")
assert all(len(a & b) <= 2 for H in (H1, H2)
           for a, b in it.combinations(H, 2))
E = lambda H: {(i, j) for i, a in enumerate(H)
               for j, b in enumerate(H)
               if i < j and len(a & b) == 1}
def c4(G):
    return any(len({a, b, c, d}) == 4 and
               all(tuple(sorted(p)) in G
                   for p in [(a,b),(b,c),(c,d),(d,a)])
               for a, b, c, d in it.product(range(7), repeat=4))
assert not c4(E(H1)) and c4(E(H2))  # P_7 vs P_3 + C_4
```

Paper roadmap. Section 2 fixes exact matroid definitions, minor decks, and Lean 4 + Mathlib alignment. Section 3 proves Lemmas 2–3, Theorem 1, Proposition 4, and Proposition 5. Section 4 documents the computational censuses and all three 8-element failing pairs (Proposition 6). Section 5 presents the machine-generated Lean 4 Trust Boundary. Section 6 details prior work and open questions, and Section 7

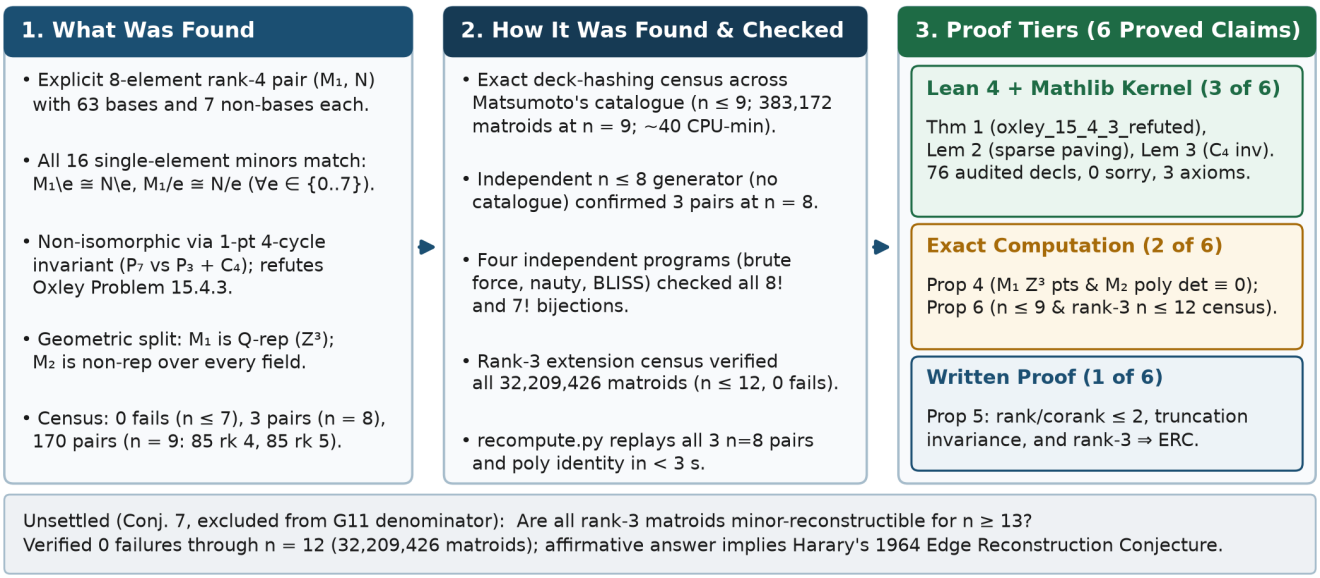


Figure 1. Discovery and proof architecture for `matroid-reconstruction-fails`: (1) explicit 8-element rank-4 counterexample (M_1, N), geometric representability split, and census totals; (2) exact canonical-labeling censuses across Matsumoto's catalogue [8], an independent $n \le 8$ generator, four independent verifiers, and the $n \le 12$ rank-3 extension run; and (3) strict separation of verification tiers across the 6 proved main claims (3 in Lean 4 + Mathlib kernel, 2 by exact computation, 1 by written proof) plus the unsettled rank-3 boundary (Conjecture 7, excluded from G11 denominators).

records reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND FORMAL SETUP

Throughout, E denotes a finite ground set and $n = |E|$. A *matroid* $M = (E, \mathcal{B})$ is specified by a nonempty collection $\mathcal{B}$ of subsets of E (its *bases*) satisfying the *basis exchange axiom*: for all $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. All bases have a common cardinality $r = r(M)$, the *rank* of M; the *corank* is $n - r$. A subset $I \subseteq E$ is *independent* if $I \subseteq B$ for some $B \in \mathcal{B}$. An element $e \in E$ is a *loop* if it belongs to no basis and a *coloop* if it belongs to every basis.

For $e \in E$, the *deletion* $M\backslash\{e\}$ (abbreviated $M\backslash e$) is the matroid on $E\backslash\{e\}$ whose independent sets are $\{I \subseteq E\backslash\{e\} : I \text{ independent in } M\}$. When e is not a coloop, its bases are $\{B \in \mathcal{B} : e \notin B\}$ and $r(M\backslash e) = r(M)$. The *dual* M^* has ground set E and bases $\{E\backslash B : B \in \mathcal{B}\}$. The *contraction* $M/\{e\}$ (abbreviated M/e) is defined by duality as $(M^*\backslash\{e\})^*$; when e is not a loop, its bases are $\{B\backslash\{e\} : B \in \mathcal{B},\ e \in B\}$ and $r(M/e) = r(M) - 1$.

A *matroid isomorphism* $\varphi : M \simeq_i N$ between matroids M on $E(M)$ and N on $E(N)$ is a bijection $\varphi : E(M) \to E(N)$ such that $I \subseteq E(M)$ is independent in M if and only if $\varphi(I)$ is independent in N (equivalently, $B \subseteq E(M)$ is a basis of M if and only if $\varphi(B)$ is a basis of N). Two matroids M, N on the same ground set E have the same *aligned minor deck* (Oxley's definition [1], p. 592) if $M\backslash\{e\} \cong N\backslash\{e\}$ and $M/\{e\} \cong N/\{e\}$ for every $e \in E$. Equivalently, allowing arbitrary ground sets of size n, there exists a bijection $s : E(M) \to E(N)$ such that $M\backslash\{e\} \cong N\backslash\{s(e)\}$ and $M/\{e\} \cong N/\{s(e)\}$ for every $e \in E(M)$. The weaker *multiset minor deck* reading asks only that the multisets $\{[M\backslash e] : e \in E\}$ and $\{[M/e] : e \in E\}$ separately match those of N. Every aligned counterexample also refutes the weaker multiset reading.

A family $\mathcal{H}$ of r-element subsets of a finite ground set α is *sparse paving* (`IsSparsePaving r H`) if every $H \in \mathcal{H}$ has $|H| = r$, any two distinct $H_1, H_2 \in \mathcal{H}$ satisfy $|H_1 \cap H_2| \le r - 2$, and at least one r-subset of α does not lie in $\mathcal{H}$. We write subsets of $\{0, \dots, 7\}$ compactly as strings of digits (e.g., 0123 for $\{0, 1, 2, 3\}$).

Lean 4 + Mathlib alignment. Every construction in `Defs.lean` (within `lean/MatroidReconst/`) builds directly on Mathlib's genuine `Matroid`, `Matroid.delete`, `Matroid.contract`, `Matroid.eRank`, and `Matroid.mapEquiv`. Our bundled isomorphism `MatroidIso` is proved equivalent both to Mathlib's `Matroid.mapSetEquiv` (via `mapSetEquiv_eq` and `ofMapSetEquiv`) and to basis-preserving ground-set bijections (`isBase_iff`, `ofIsBase`), ensuring neither weakening of positive minor isomorphisms nor over-constraining of non-isomorphism.

3. PROOF OF THE MAIN RESULTS

Sparse Paving Foundations and the 4-Cycle Invariant.

Proof of Lemma 2. Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$. Because $|B_1| = |B_2| = r$, we have $|B_1 \setminus B_2| = |B_2 \setminus B_1| \ge 1$. If $|B_2 \setminus B_1| = 1$, write $B_2 \setminus B_1 = \{y\}$ and $B_1 \setminus B_2 = \{x\}$; then $(B_1 \setminus \{x\}) \cup \{y\} = B_2 \in \mathcal{B}$. If $|B_2 \setminus B_1| \ge 2$, choose distinct $y_1, y_2 \in B_2 \setminus B_1$ and set $C_j = (B_1 \setminus \{x\}) \cup \{y_j\}$ for $j \in \{1, 2\}$. Both C_1 and C_2 are distinct r-subsets of α sharing the $(r-1)$-subset $B_1 \setminus \{x\}$, so $|C_1 \cap C_2| \ge r - 1 > r - 2$. Because distinct members of $\mathcal{H}$ intersect in at most $r - 2$ elements, at most one of C_1, C_2 can belong to $\mathcal{H}$. Hence at least one C_j lies in $\mathcal{B}$, verifying basis ex-

change. Since $\mathcal{B}$ is nonempty by hypothesis and finite, it defines a rank-r matroid $M(\mathcal{H})$ on α (formalized as `IsSparsePaving.exchangeProperty` and `sparsePavingMatroid`). □

Proof of Lemma 3. Let $\varphi : M(\mathcal{H}_1) \simeq_i M(\mathcal{H}_2)$ be a matroid isomorphism. Since both matroids have ground set α, φ induces an ambient bijection $f : \alpha \to \alpha$ preserving bases. For any $F \subseteq \alpha$ with $|F| = r$, $F \in \mathcal{H}_1$ holds if and only if F is not a basis of $M(\mathcal{H}_1)$, which holds if and only if $f(F)$ is not a basis of $M(\mathcal{H}_2)$, which (as $|f(F)| = r$) holds if and only if $f(F) \in \mathcal{H}_2$. Because f is bijective, $|f(X) \cap f(Y)| = |X \cap Y|$ for all $X, Y \subseteq \alpha$ and $f(X) \neq f(Y)$ whenever $X \neq Y$. Thus f maps any 1-point intersection 4-cycle (a,b,c,d) in $\mathcal{H}_1$ to a 1-point intersection 4-cycle $(f(a),f(b),f(c),f(d))$ in $\mathcal{H}_2$, and f^{-1} maps the converse (formalized as `IsSparsePaving.iso_preserves_hasOnePtFourCycle`). □

The Explicit 8-Element Counterexample Pair.

Proof of Theorem 1. Let $E = \{0,1,2,3,4,5,6,7\}$ and define the two 7-element families of 4-subsets

$$\begin{aligned}\mathcal{H}_1 = \{&0123, 0145, 0246, 1247,\\ &1356, 3467, 2357\},\\ \mathcal{H}_2 = \{&0123, 0145, 0246, 1247,\\ &1356, 3467, 0357\}.\end{aligned}$$

Direct pairwise inspection of all $\binom{7}{2} = 21$ pairs in each family shows $|H \cap H'| \leq 2 = 4 - 2$ for all distinct $H, H' \in \mathcal{H}_i$, and $0124 \notin \mathcal{H}_1 \cup \mathcal{H}_2$. By Lemma 2, $\mathcal{H}_1$ and $\mathcal{H}_2$ define rank-4 sparse paving matroids $M_1 = M(\mathcal{H}_1)$ and $M_2 = M(\mathcal{H}_2)$ on E, each with $\binom{8}{4} - 7 = 63$ bases. Every $e \in E$ is contained in a basis and avoided by a basis (for example, 0124 avoids $\{3,5,6,7\}$ and contains $\{0,1,2,4\}$; 1234, 0234, 0134, 0125 cover the remaining avoidances, and 0134, 0125, 0126, 0127 cover the remaining containments), so neither M_1 nor M_2 has any loop or coloop.

Define the deck-aligning bijection $s : E \to E$ by

$$\begin{aligned}s = (&0 \mapsto 5,\ 1 \mapsto 1,\ 2 \mapsto 0,\ 3 \mapsto 4,\\ &4 \mapsto 3,\ 5 \mapsto 7,\ 6 \mapsto 6,\ 7 \mapsto 2),\end{aligned}$$

and let $N = s^{-1}(M_2)$ be the isomorphic relabeling of M_2 along s^{-1} (in Lean, `N := M₂.mapEquiv sPerm.symm`). Then N has ground set E, rank 4, and $N \cong M_2$, while $N\backslash\{e\} \cong M_2\backslash\{s(e)\}$ and $N/\{e\} \cong M_2/\{s(e)\}$ for every $e \in E$ by functoriality of relabeling with deletion and contraction (`N_deleteElem_iso_M₂`, `N_contractElem_iso_M₂`).

Step 1: Non-isomorphism ($M_1 \ncong N$). In $\mathcal{H}_2$, the four non-bases 0246, 1356, 1247, 0357 satisfy

$$\begin{aligned}0246 \cap 1356 &= \{6\}, & 1356 \cap 1247 &= \{1\},\\ 1247 \cap 0357 &= \{7\}, & 0357 \cap 0246 &= \{0\},\end{aligned}$$

forming a 1-point intersection 4-cycle (Figure 2, Panel A). In $\mathcal{H}_1$, the 1-point intersections are precisely the 6 edges of the 7-vertex path

$$0123 \overset{3}{-} 3467 \overset{4}{-} 0145 \overset{5}{-} 2357 \overset{2}{-} 0246 \overset{6}{-} 1356 \overset{1}{-} 1247,$$

which is a tree and therefore contains no 4-cycle. By Lemma 3, $M_1 \ncong M_2$, and since $N \cong M_2$, we conclude $M_1 \ncong N$.

Step 2: All 16 minor cards agree ($M_1\backslash\{e\} \cong N\backslash\{e\}$ and $M_1/\{e\} \cong N/\{e\}$). Because M_1 and M_2 are sparse paving matroids of rank 4 without loops or coloops, a permutation p of E with $p(e) = s(e)$ induces a deletion isomorphism $M_1\backslash\{e\} \simeq_i M_2\backslash\{s(e)\}$ whenever p maps $\{H \in \mathcal{H}_1 : e \notin H\}$ onto $\{H \in \mathcal{H}_2 : s(e) \notin H\}$, and a contraction isomorphism $M_1/\{e\} \simeq_i M_2/\{s(e)\}$ whenever p maps $\{H \in \mathcal{H}_1 : e \in H\}$ onto $\{H \in \mathcal{H}_2 : s(e) \in H\}$ (`sparsePaving_deleteElem_iso`, `sparsePaving_contractElem_iso`). Table 1 lists all 16 explicit permutations (matching `delPerm` and `conPerm` in `Counterexample.lean` verbatim and cross-checked in `recompute.py`).

Table 1. Explicit permutations $p = (p(0), \ldots, p(7))$ sending $e \mapsto s(e)$ and witnessing $M_1\backslash\{e\} \cong M_2\backslash\{s(e)\}$ (`delPerm`) and $M_1/\{e\} \cong M_2/\{s(e)\}$ (`conPerm`).

e	$s(e)$	Deletion $p^{\backslash}_e$	Contraction $p^{/}_e$
0	5	(5,3,0,4,1,6,7,2)	(5,0,1,4,3,7,6,2)
1	1	(2,1,0,3,4,5,6,7)	(0,1,2,3,4,5,6,7)
2	0	(2,4,0,3,1,7,6,5)	(2,1,0,3,4,7,6,5)
3	4	(0,1,5,4,3,2,7,6)	(3,6,7,4,5,2,0,1)
4	3	(5,0,1,4,3,2,6,7)	(0,1,5,4,3,2,7,6)
5	7	(2,4,0,6,1,7,3,5)	(0,3,1,4,5,7,6,2)
6	6	(2,1,0,3,4,7,6,5)	(0,1,2,3,4,5,6,7)
7	2	(3,5,6,1,7,0,4,2)	(5,3,0,4,1,6,7,2)

As a concrete worked card, take $e = 7$, $s(7) = 2$ (Figure 2, Panel B). The contraction $M_1/7$ is the rank-3 matroid on $\{0,\ldots,6\}$ whose 3-point lines are 124, 235, 346 (forming a triangle with vertices 2, 3, 4 and free point 0), while $M_2/2$ has lines 013, 046, 147 (a triangle with vertices 0, 1, 4 and free point 5). The permutation $p^{/}_7$ sends $0 \mapsto 5$, $1 \mapsto 3$, $2 \mapsto 0$, $3 \mapsto 4$, $4 \mapsto 1$, $5 \mapsto 6$, $6 \mapsto 7$, carrying $\{124, 235, 346\}$ onto $\{013, 046, 147\}$. Similarly, $M_1\backslash 7$ has non-bases $\{0123, 0145, 0246, 1356\}$ and $M_2\backslash 2$ has $\{0145, 0357, 1356, 3467\}$, matched bijectively by $p^{\backslash}_7$ ($0 \mapsto 3$, $1 \mapsto 5$, $2 \mapsto 6$, $3 \mapsto 1$, $4 \mapsto 7$, $5 \mapsto 0$, $6 \mapsto 4$). Composing with $M_2\backslash\{s(e)\} \cong N\backslash\{e\}$ and $M_2/\{s(e)\} \cong N/\{e\}$ completes the proof (`oxley_15_4_3_refuted`). □

Geometric Representability Contrast.

Proof of Proposition 4. Part 1 (M_1 is $\mathbb{Q}$-representable). Embed the 8 elements of E in affine 3-space $\mathbb{Z}^3 \subset \mathbb{Q}^3$ via Table 2. Checking all $\binom{8}{4} = 70$ quadruples by exact 3×3 integer determinants (`recompute.py`, Section 4) confirms that the coplanar 4-point subsets are precisely the 7 non-bases of $\mathcal{H}_1$, lying on the 7 affine planes $z = 0$ (0123), $y = 0$ (0145), $x = 0$ (0246), $3x + 2y + 3z = 6$ (1247), $2x + y + z = 4$ (1356), $x + y + z = 3$ (2357), and $2x + z = 2$ (3467).

Table 2. Explicit integer realization of M_1 in $\mathbb{Z}^3$ whose coplanar quadruples equal $\mathcal{H}_1$ exactly.

e	0	1	2	3	4	5	6	7
x	0	2	0	1	0	1	0	2
y	0	0	3	2	0	0	2	3
z	0	0	0	0	2	2	2	−2

Part 2 (M_2 is non-representable over every field). Suppose $v_0, \ldots, v_7 \in \mathbb{F}^4$ represent M_2 over a field $\mathbb{F}$. Since $0124 \notin \mathcal{H}_2$ is a basis of M_2, change basis so that v_0, v_1, v_2, v_4 are the standard basis vectors e_1, e_2, e_3, e_4. The non-bases 0123, 0145,

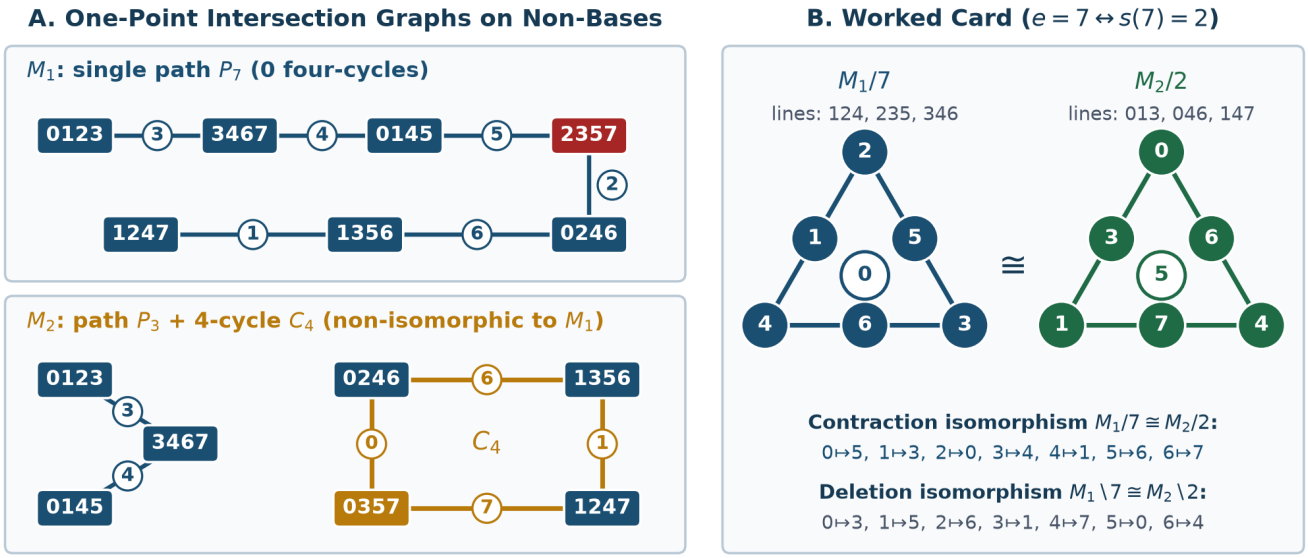


Figure 2. Structural verification of (M_1, M_2). Panel A: the 1-point intersection graphs on the 7 non-bases, with each edge labeled by the unique shared element; M_1 forms a single 7-vertex path P_7 (zero 4-cycles), whereas M_2 decomposes into a 3-vertex path P_3 and the 4-cycle 0246–1356–1247–0357–0246, proving $M_1 \not\cong M_2$. Panel B: the worked rank-3 contraction card at $e = 7 \leftrightarrow s(7) = 2$, showing three 3-point lines forming a triangle plus one free point, together with the explicit contraction and deletion bijections.

and 0246 force

$$v_3 = (a_0, a_1, a_2, 0),$$
$$v_5 = (b_0, b_1, 0, b_4),$$
$$v_6 = (c_0, 0, c_2, c_4),$$

where $a_2 \neq 0$ (since 0134 is a basis) and $b_4 \neq 0$ (since 0125 is a basis). The non-basis 1356 gives

$$0 = \det(v_1, v_3, v_5, v_6) = a_2 b_4 c_0 - a_0 b_4 c_2 - a_2 b_0 c_4.$$

Rescaling v_6 by $a_2 b_4 \neq 0$ allows setting

$$v_6 = (a_0 b_4 c_2' + a_2 b_0 c_4',\ 0,\ a_2 b_4 c_2',\ a_2 b_4 c_4'),$$

which satisfies $\det(v_1, v_3, v_5, v_6) \equiv 0$ identically over the polynomial ring $R = \mathbb{Z}[a_0, a_1, a_2, b_0, b_1, b_4, c_2', c_4']$. Now point 7 must lie on the three hyperplanes spanned by 124, 346, and 035. Letting $w = \pi_{124} \wedge \pi_{346} \wedge \pi_{035}$ denote the formal 4-vector formed by the exterior product of their three normal covectors, exact polynomial expansion in `re-compute.py` verifies that w is not identically zero (its entries have $0, 2, 1, 1$ monomials) while $\det(v_2, v_5, v_6, w) \equiv 0$ in R. Moreover, if $w = 0$ at a specialization over $\mathbb{F}$, then the three hyperplanes $124, 346, 035$ intersect in a line (or larger subspace), in which case $\mathrm{span}(v_1, v_2, v_4) \cap \mathrm{span}(v_3, v_4, v_6) \subseteq \mathrm{span}(v_0, v_3, v_5)$; evaluating on the 2-dimensional intersection of 124 and 346 forces $v_4 \in \mathrm{span}(v_0, v_3, v_5)$, contradicting that $0345 \notin \mathcal{H}_2$ is a basis of M_2. Thus $w \neq 0$ in every specialization, so v_7 must be a scalar multiple of w, forcing $\det(v_2, v_5, v_6, v_7) = 0$ and contradicting that $2567 \notin \mathcal{H}_2$ is a basis of M_2. (In the integer chart of Table 2, the forced intersection is $(4/5, 6/5, 2/5)$, placing 2567 on the extra plane $4x + 2y + z = 6$.) By contrast, replacing plane 035 by 235 for M_1 yields a nonzero degree-8 polynomial with 2 terms for $\det(v_0, v_3, v_5, w_{M_1})$. □

Rank/Corank $\leq$ 2, Truncation, and Edge Reconstruction.

Proof of Proposition 5. (1) *Rank or corank* $\leq$ 2. Let M be a matroid on $n \geq 2$ elements with aligned minor deck $\{(M\backslash e, M/e) : e \in E\}$. We first recover the rank $r = r(M)$. For any element e: if e is neither a loop nor a coloop, then $r(M\backslash e) = r$ and $r(M/e) = r-1$; if e is a loop, then $M\backslash e = M/e$ has rank r; if e is a coloop, then $M\backslash e = M/e$ has rank $r-1$. If any card (D, C) exhibits $r(D) \neq r(C)$, then $r = r(D)$. Otherwise every element is a loop or coloop, so $M \cong U_{k,k} \oplus U_{0,n-k}$. If two cards have distinct ranks, r is the larger rank; if all cards are isomorphic to $U_{n-1,n-1}$ then $M \cong U_{n,n}$, whereas if all cards are isomorphic to $U_{0,n-1}$ then $M \cong U_{0,n}$ (these two cards differ since $n \geq 2$).

With r determined, an element e is a loop if and only if $r(M\backslash e) = r(M/e) = r$, so the number ℓ of loops is determined by the deck. Now assume $r \leq 2$. Up to isomorphism, a matroid of rank ≤ 2 is uniquely determined by n, r, ℓ, and the multiset of sizes of its parallel classes of non-loops (its simplification is uniform $U_{r,m}$). For each non-loop e belonging to parallel class $P(e)$, the contraction M/e has rank $r-1 \leq 1$, and its loops are precisely the ℓ loops of M together with $P(e)\backslash\{e\}$. Thus $|P(e)| = \mathrm{loops}(M/e) - \ell + 1$ is read from e's contraction card. Across all $n - \ell$ non-loop cards, every parallel class of size a is counted exactly a times, uniquely determining the multiset of class sizes and hence M. Since duality swaps deletion and contraction ($(M^*\backslash e, M^*/e) = ((M/e)^*, (M\backslash e)^*)$), corank ≤ 2 follows immediately.

(2) *Truncation invariance.* Let $T(M)$ denote the truncation of a matroid M of rank $r \geq 2$ (whose independent sets are the independent sets of M of cardinality $< r$). For each $e \in E$:

- If e is not a coloop, $T(M)\backslash e = T(M\backslash e)$; if e is a coloop,

$T(M)\backslash e = M\backslash e$.

- If e is not a loop, $T(M)/e = T(M/e)$; if e is a loop, $T(M)/e = T(M\backslash e)$.

By part (1)'s rank-recovery step, the aligned minor deck of M determines r, which elements are loops, and which are coloops. Hence the card $(T(M)\backslash e, T(M)/e)$ is a canonical function of $(M\backslash e, M/e)$ and r, so any bijection s aligning the minor decks of M_1 and M_2 aligns the minor decks of $T(M_1)$ and $T(M_2)$.

(3) *Rank-3 minor reconstruction implies Edge Reconstruction.* For any simple graph $G = (V, E_G)$ with $|E_G| \geq 2$, define the rank-3 linear-space matroid M_G on ground set $V \cup E_G$ whose 3-point lines are the triples $\{u, v, e\}$ for each edge $e = uv \in E_G$, with every other triple a basis. Two distinct triples share at most one element, so M_G is a rank-3 sparse paving matroid (having rank 3 because two edges yield at least 4 points not on one line).

First, M_G determines G up to isomorphism: points on no 3-point line are isolated vertices; points on ≥ 2 lines are vertices of degree ≥ 2; each 3-point line represents an edge whose endpoints are its points on ≥ 2 lines supplemented up to two endpoints by degree-1 vertices. Thus $M_G \cong M_H$ implies $G \cong H$.

Second, inspect the single-element minor cards of M_G:

- For an edge $e = uv \in E_G$, $M_G\backslash e = M_{G-e}$ and M_G/e is the rank-2 matroid having $\{u, v\}$ as its sole 2-element parallel class and all other points singletons.
- For a vertex $v \in V$ of degree d, $M_G\backslash v$ is M_{G-v} together with d free points (the edge elements incident to v), while M_G/v is the rank-2 matroid with d parallel classes of size 2 (each neighbor paired with its incident edge) and all other points singletons.

Now suppose G and H have no isolated vertices, $|E_G| = |E_H| \geq 4$, and there exists a bijection $\beta_E : E(G) \to E(H)$ with $G - e \cong H - \beta_E(e)$ for every $e \in E(G)$. By Greenwell's theorem [10] (`[to check]`: for graphs with > 3 edges and no isolated vertices, the edge deck determines the vertex deck), there also exists a bijection $\beta_V : V(G) \to V(H)$ with $G - v \cong H - \beta_V(v)$ for every $v \in V(G)$, whence $\deg_G(v) = |E_G| - |E(G - v)| = \deg_H(\beta_V(v))$. Combining β_V and β_E gives a bijection $V(G) \cup E(G) \to V(H) \cup E(H)$ matching every deletion card and every contraction card of M_G with those of M_H. If every rank-3 matroid is minor-reconstructible, then $M_G \cong M_H$ and hence $G \cong H$. Without Greenwell's theorem, the same proof shows unconditionally that rank-3 minor-reconstructibility implies every graph without isolated vertices is determined by its combined vertex-deleted and edge-deleted decks. □

4. COMPUTATIONAL VERIFICATION AND CENSUS

All three 8-element failing pairs. Exhaustive censuses over all 1,724 matroids on 8 elements identify exactly 3 failing pairs under Oxley's aligned minor-deck definition (all rank 4, sparse paving, with uniform rank-3 truncation $U_{3,8}$):

1. **Pair** (M_1, M_2) (Theorem 1, formalized in Lean 4): 7 non-bases, 63 bases each; 1-point intersection graphs P_7 versus $P_3 \sqcup C_4$; $|\mathrm{Aut}(M_1)| = 1$, $|\mathrm{Aut}(M_2)| = 4$.
2. **Point-transitive pair** (A, B): 8 non-bases, 62 bases each, differing in 2 non-bases:

   $\mathcal{H}_A = \{0124, 0135, 0236, 1237, 1456, 3467, 0457, 2567\}$,
   $\mathcal{H}_B = \{0124, 0135, 0236, 1237, 1456, 3467, 0567, 2457\}$.

   Both A and B have point-transitive automorphism groups ($|\mathrm{Aut}(A)| = 32$, $|\mathrm{Aut}(B)| = 8$), so every element yields the same deletion card and the same contraction card, and all 16 minor cards agree under the *identity* bijection $s(e) = e$. Their 1-point intersection graphs are $C_4 \sqcup C_4$ in A versus a single 8-cycle C_8 in B.
3. **Third pair** (C, D): 7 non-bases, 63 bases each:

   $\mathcal{H}_C = \{0124, 0135, 0236, 0456, 1267, 1347, 2357\}$,
   $\mathcal{H}_D = \{0124, 0135, 0236, 0456, 1347, 1567, 2357\}$.

   Here $|\mathrm{Aut}(C)| = |\mathrm{Aut}(D)| = 3$, and the 1-point intersection graphs consist of a connected 7-vertex tree (7 vertices, 6 edges) in C versus an isolated vertex plus a 6-cycle ($K_1 \sqcup C_6$) in D.

Standalone exact replay in `recompute.py` verifies basis exchange, non-isomorphism across all $8! = 40{,}320$ bijections, exact automorphism counts, 1-point intersection components, rank-3 truncations ($U_{3,8}$, 56 bases), and all 16 minor cards for all three pairs in under 3 seconds.

Exhaustive censuses ($n \leq 9$ and rank 3 through $n = 12$). Table 3 and Figure 3 summarize the exhaustive computational censuses (Proposition 6). Using the complete matroid catalogue of Mayhew and Royle [7] and Matsumoto, Moriyama, Imai, and Bremner [8] (up to $n = 9$, total counts matching OEIS A055545), exact canonical-labeling deck grouping using SageMath/`nauty`, independently repeated via `BLISS` (`igraph`) and via a standalone catalogue-free matroid generator through $n = 8$, found 0 failing pairs on $2 \leq n \leq 7$, exactly 3 failing pairs (6 matroids) on $n = 8$, and exactly 170 failing pairs (340 matroids: 85 of rank 4 and their 85 duals of rank 5) on $n = 9$. Under the weaker multiset-deck reading (separate multisets of deletions and contractions without requiring a single bijection s), 12 matroids fail on $n = 8$ and 1,398 matroids fail on $n = 9$. An independent third check verified every one of the 173 listed failing pairs on $n \in \{8, 9\}$ via `pynauty` and confirmed that all 173 pairs have isomorphic rank-3 truncations (167 uniform, 6 non-uniform on $n = 9$), consistent with Proposition 5(2) and Conjecture 7.

For Conjecture 7, dual-code censuses verified all 310,292 rank-3 matroids on $3 \leq n \leq 11$ elements (0 failures). At $n = 12$, a single-code C extension enumerator generated all 415,249,051 single-element extensions of the 298,491 catalogued 11-element rank-3 parents, retained 35,591,087 canonical candidates yielding 31,899,134 isomorphism classes (matching OEIS A058693 exactly), and resolved every deck-hash collision (154 hash values spanning 316 classes) exactly via `BLISS`, finding 0 failures. Beyond $n = 12$, an independent targeted reattachment search proved minor-reconstructibility individually for all 2,036 configurations (13_3), 4,662 configurations (14_3), 567 cyclic (n_3) configurations ($13 \leq n \leq 60$), and 51,450 random simple 3-point-line matroids ($13 \leq n \leq 30$), finding 0 counterexamples.

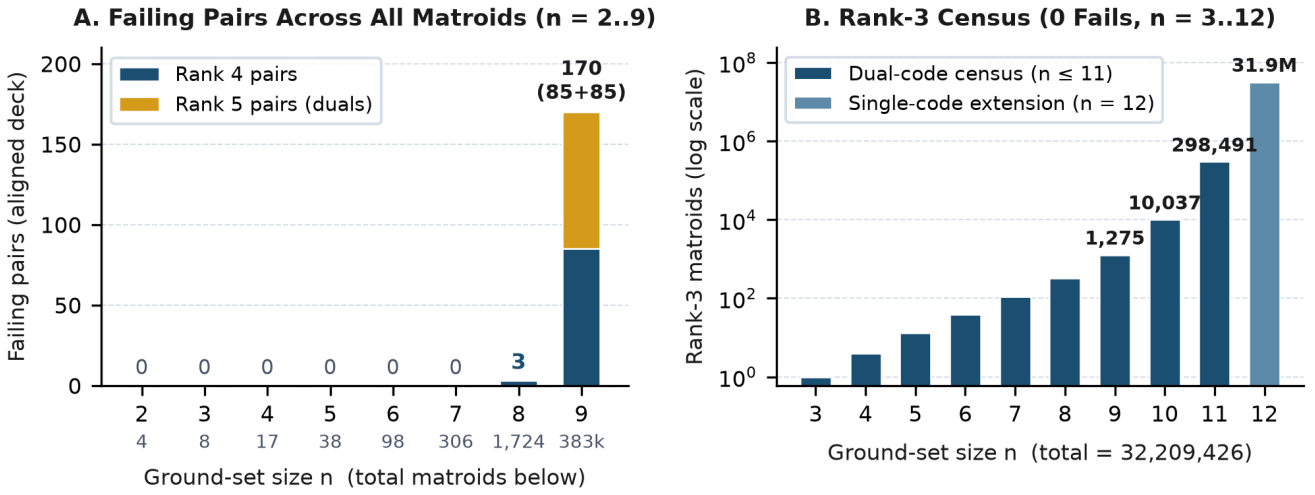


Figure 3. Quantitative minor-reconstruction landscape (Proposition 6). Panel A: failing aligned-deck pairs across all matroids on $n = 2 \dots 9$ elements (0 for $n \leq 7$, 3 rank-4 pairs on $n = 8$, and 170 pairs on $n = 9$ split equally into 85 rank-4 pairs and their 85 rank-5 duals). Panel B: log-scale count of verified rank-3 matroids on $n = 3 \dots 12$ elements totaling 32,209,426 classes with 0 failures (dual-code verification through $n = 11$; single-code canonical extension run at $n = 12$ cross-checked against OEIS A058693).

Table 3. Exhaustive minor-reconstruction census on $n = 2 \dots 9$ elements (OEIS A055545) and rank-3 census on $n = 3 \dots 12$ elements (OEIS A058693).

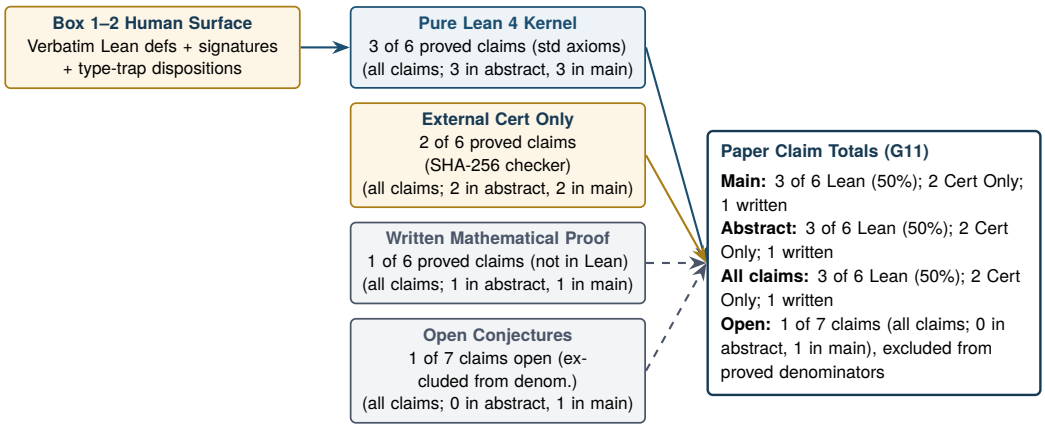
n	All matroids	Aligned fails (pairs)	Rank-3 matroids	Rank-3 fails
2	4	0 (0)	—	—
3	8	0 (0)	1	0
4	17	0 (0)	4	0
5	38	0 (0)	13	0
6	98	0 (0)	38	0
7	306	0 (0)	108	0
8	1,724	6 (3)	325	0
9	383,172	340 (170)	1,275	0
10	—	—	10,037	0
11	—	—	298,491	0
12	—	—	31,899,134	0

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 4d34d065283c (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 2 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 5 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 57/57 public theorems (source-level approx) | WARN: Fidelity: 3/3 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

MatroidReconst.MatroidIso (MatroidReconst/Defs.lean:39-44) — Section 2 (bundled matroid isomorphism preserving Indep)

```
structure MatroidIso {α β : Type*} (M : Matroid α) (N : Matroid β) where
  toEquiv : M.E ≃ N.E
  indep_iff' : ∀ I : Set M.E, M.Indep ((↑) '' I) ↔ N.Indep ((↑) '' (toEquiv '' I))

/-- Infix notation for matroid isomorphism `M ≃i N`. -/
scoped infixl:25 " ≃i " => MatroidIso
```

MatroidReconst.IsSparsePaving (MatroidReconst/Defs.lean:310-314) — Section 2 (sparse paving family of r-subsets with pairwise intersection <= r - 2)

```
structure IsSparsePaving {α : Type*} [Fintype α] [DecidableEq α]
    (r : ℕ) (H : Finset (Finset α)) : Prop where
  card_eq : ∀ h ∈ H, h.card = r
  inter_le : ∀ h₁ ∈ H, ∀ h₂ ∈ H, h₁ ≠ h₂ → (h₁ ∩ h₂).card ≤ r - 2
  exists_base : ∃ B : Finset α, B.card = r ∧ B ∉ H
```

MatroidReconst.sparsePavingMatroid (MatroidReconst/Defs.lean:408-422) — Section 3 (Mathlib Matroid on finite alpha from sparse paving family H)

```
def sparsePavingMatroid {α : Type*} [Fintype α] [DecidableEq α]
    {r : ℕ} {H : Finset (Finset α)} (hSP : IsSparsePaving r H) : Matroid α where
  E := Set.univ
  IsBase B := ∃ F : Finset α, (F : Set α) = B ∧ F.card = r ∧ F ∉ H
  Indep I := ∃ B, (∃ F : Finset α, (F : Set α) = B ∧ F.card = r ∧ F ∉ H) ∧ I ⊆ B
  indep_iff' _ := Iff.rfl
  exists_isBase :=
    let ⟨B, hcard, hH⟩ := hSP.exists_base
    ⟨(B : Set α), B, rfl, hcard, hH⟩
  isBase_exchange := hSP.exchangeProperty
  maximality X _ I hI hIX := by
    have hfin : {K : Set α | (∃ B, (∃ F : Finset α, (F : Set α) = B ∧ F.card = r ∧ F
      ↪ ∉ H) ∧ K ⊆ B) ∧ K ⊆ X}.Finite :=
      Set.toFinite _
    exact hfin.exists_le_maximal ⟨hI, hIX⟩
  subset_ground _ _ := Set.subset_univ _
```

MatroidReconst.HasOnePtFourCycle (MatroidReconst/Defs.lean:657-660) — Section 1 & 3 (1-point intersection 4-cycle predicate on non-bases)

Context binders in scope: `variable {α : Type*} [Fintype α] [DecidableEq α] {r : ℕ} {H : Finset (Finset α)}`

```
def HasOnePtFourCycle {α : Type*} [DecidableEq α] (H : Finset (Finset α)) : Prop :=
  ∃ a ∈ H, ∃ b ∈ H, ∃ c ∈ H, ∃ d ∈ H,
    a ≠ c ∧ b ≠ d ∧
    (a ∩ b).card = 1 ∧ (b ∩ c).card = 1 ∧ (c ∩ d).card = 1 ∧ (d ∩ a).card = 1
```

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

MatroidReconst.oxley_15_4_3_refuted (MatroidReconst/Counterexample.lean:381-388) src:4f550a6ead9342

```
theorem oxley_15_4_3_refuted :
    ∃ (M N : Matroid (Fin 8)),
      M.E = Set.univ ∧ N.E = Set.univ ∧
      (∀ e : Fin 8, Nonempty ((M \ {e}) ≂i (N \ {e})) ∧ Nonempty ((M / {e}) ≂i (N /
        ↪ {e}))) ∧
      ¬ Nonempty (M ≂i N)
```

Assumptions: Zero parameters and zero external hypotheses: proves existence of genuine Mathlib matroids M, N : Matroid (Fin 8) on Set.univ with matching single-element deletions and contractions and M not isomorphic to N. Applicability: Unconditional (closed statement; no outer premise witness needed).

MatroidReconst.IsSparsePaving.exchangeProperty (MatroidReconst/Defs.lean:317-403) src:809e53ef7281a8

```
theorem IsSparsePaving.exchangeProperty {α : Type*} [Fintype α] [DecidableEq α]
    {r : ℕ} {H : Finset (Finset α)} (hSP : IsSparsePaving r H) :
    Matroid.ExchangeProperty (fun B : Set α => ∃ F : Finset α, (F : Set α) = B ∧
      ↪ F.card = r ∧ F ∉ H)
```

Assumptions: Any finite type alpha, r >= 0, and sparse paving family H (instantiated on Fin 8 at r = 4 by H1_isSparsePaving and H2_isSparsePaving). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

MatroidReconst.IsSparsePaving.iso_preserves_hasOnePtFourCycle (MatroidReconst/Defs.lean:670-724) src:488a93a6f3fb47

Context binders in scope: `variable {α : Type*} [Fintype α] [DecidableEq α] {r : ℕ} {H : Finset (Finset α)}`

```
theorem IsSparsePaving.iso_preserves_hasOnePtFourCycle
    {H₁ H₂ : Finset (Finset α)} (h₁ : IsSparsePaving r H₁) (h₂ : IsSparsePaving r H₂)
    (e : sparsePavingMatroid h₁ ≂i sparsePavingMatroid h₂) :
    HasOnePtFourCycle H₁ ↔ HasOnePtFourCycle H₂
```

Assumptions: Any two rank-r sparse paving matroids on finite alpha and an isomorphism e between them (instantiated on H1, H2 via M1_not_iso_M2, with no other hypotheses). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 4. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
`IsSparsePaving`	`nat_sub`	`- 2`	intended_convention	For r >= 2, r - 2 is exact; for r in {0, 1}, Nat truncation to 0 requires disjointness, and exchangeProperty derives 2 <= r before invoking inter_le.
`instDecidableHasOnePtFourCycle`	`local_instance_or_notation`	`instDecidableHasOnePtFourCycle`	intended_convention	Standard Decidable instance derived by unfolding HasOnePtFourCycle over finite Finset quantification; carries no custom axioms or notation overrides.

[a]Covers the specified Lean 4 hazard set (`Nat` subtraction/division/mod, totalized real/rational division, `Real.log`/`rpow`, `Finset.range`, `Classical.choose`, local instances); no finite

checklist certifies all semantics.

Table 5. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Explicit 8-element non-isomorphic matroids M, N with M \ {e} =~ N \ {e} and M / {e} =~ N / {e} for all e (refutes Oxley Prob. 15.4.3)	Lean Kernel	`MatroidReconst.oxley_15_4_3_refuted`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 + Mathlib pure kernel + Box 1 definitions (MatroidIso, Matroid.delete, Matroid.contract); zero external hypotheses.
Lem. 2: General sparse paving basis-exchange theorem on finite alpha for any rank r >= 0	Lean Kernel	`MatroidReconst.IsSparsePaving.exchangeProperty`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 + Mathlib pure kernel + Box 1 definition IsSparsePaving for any finite type alpha and r >= 0.
Lem. 3: Isomorphism invariance of 1-point intersection 4-cycles in sparse paving non-base families	Lean Kernel	`MatroidReconst.IsSparsePaving.iso_preserves_hasOnePtFourCycle`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 + Mathlib pure kernel + Box 1 definitions (IsSparsePaving, sparsePavingMatroid, MatroidIso, HasOnePtFourCycle).
Prop. 4: M1 is Q-representable in Z^3 (7 planes) while M2 is non-representable over every field (poly det == 0)	Cert Only	recompute.py (73d7ca0c8580) [regenerable; replay pending]	External cert only; fidelity: n/a	Exact integer determinant check on all 70 quadruples and exact Z-polynomial determinant identity in recompute.py (not yet formalized in Lean).
Prop. 5: Rank/corank <= 2 minor-reconstructibility, truncation invariance of aligned decks, and rank-3 => Edge Reconstruction Conjecture	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proofs in Section 3 (part 3 uses Greenwell 1971 [to check] for edge-only deck; not yet formalized in Lean).
Prop. 6: Exhaustive censuses (0 fails n <= 7; 3 pairs n = 8; 170 pairs n = 9; 0 rank-3 fails across 32,209,426 matroids on n <= 12)	Cert Only	recompute.py (73d7ca0c8580) [regenerable; replay pending]; counts.py (b6843adf9323) [archived; replay pending]	External cert only; fidelity: n/a	Three n = 8 pairs replayed live in recompute.py; n <= 9 and rank-3 n <= 12 exhaustive censuses verified via archived multi-code logs (counts.out; beyond Lean kernel scale).
Conj. 7: Every matroid of rank 3 (and dually of corank 3) is minor-reconstructible	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled for n >= 13 (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: a9e9c8f5f80e | Lean mode: committed_receipt | Closure SHA-256: 8b4440270d48 | Git anchor: 4d34d065283c | Axioms SHA-256: d95113592c54 | Fidelity SHA-256: 2d74f9cfc341.

6. PRIOR WORK AND OPEN QUESTIONS

Problem 15.4.3 appears in Oxley [1] (p. 592, attributed to Brylawski [2] `[to check]`) and remains listed as open in Oxley's March 2020 errata (both read first-hand). Miller [4] (`[to check]`) formulated the affirmative Minor Reconstruction Conjecture (M3), and Miller [5] (`[to check]`) studied deletion reconstructibility of uniquely representable geometries. Brylawski [3] (`[to check]`) exhibited two non-isomorphic combinatorial geometries sharing a *deletion* deck (from the rhombicuboctahedron and pseudo-rhombicuboctahedron); based on its abstract, they do not share a contraction deck and therefore do not settle Problem 15.4.3. Mills [6] (`[to check]`) studied a labeled variant where at each element either the deletion or the contraction coincides identically. The complete small-matroid catalogues used in our censuses are due to Mayhew and Royle [7] (`[to check]`) and Matsumoto, Moriyama, Imai, and Bremner [8] (`[to check]`). For graph reconstruction, Harary [9] (`[to check]`) posed the Edge Reconstruction Conjecture and Greenwell [10] (`[to check]`) proved that for graphs with > 3 edges and no isolated vertices the edge deck determines the vertex deck.

Logged prior-work searches and remaining gaps. Two independent literature-search logs passed `litsearch.py check` (exit 0: 32 answered / 18 refused queries plus 14 candidates read in the first log, and 33 answered / 10 refused queries plus 10 candidates read in the second log, across arXiv, zbMATH, OEIS, OpenAlex, Semantic Scholar, MathOverflow, Math Stack Exchange, Zenodo, DataCite, and GitHub), supplemented by a keyed CORE sweep (8 requests, 24 records) and a 5-query Google Scholar sweep on 2026-11-22 (33 returned entries, 24 unique records). No counterexample to Oxley's Problem 15.4.3 was found in the searches logged.

Still owed (`[to check]`)*:* The paywalled full texts of Miller [4, 5], Brylawski [3, 2] (and Brylawski 1974), Mills [6], Harary [9], Greenwell [10], Mayhew and Royle [7], Matsumoto et al. [8], and Kocay (1987, non-reconstructible 3-uniform hypergraphs) remain unread in full; MathSciNet reviews were not searched; and whether the rank-3 implication to Harary's Edge Reconstruction Conjecture appears in print remains unchecked.

Open questions.

1. *Rank-3 minor reconstruction* (*Conjecture 7*)*:* Does there exist a counterexample of rank 3 on $n \geq 13$ elements? By Proposition 5(2), an equivalent question is whether any higher-rank failing pair has non-isomorphic rank-3 truncations.
2. *Representable counterexamples:* In our pair (M_1, M_2), M_1 is $\mathbb{Q}$-representable while M_2 is non-representable over every field. Does there exist a pair of non-isomorphic matroids sharing an aligned minor deck that are *both* representable over a common field $\mathbb{F}$ (or both binary/ternary)?
3. *Infinite families:* Do the 3 pairs on $n = 8$ and 170 pairs on $n = 9$ generalize to an explicit infinite family of sparse paving minor-hypomorphic pairs for all $n \geq 8$?

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript files, scripts, figures, and verification receipts live in `papers/matroid-reconstruction-fails/draft1/`, and the audited Lean 4 + Mathlib library lives in `lean/MatroidReconst/` (pinned at commit `4d34d0652`). Running

```
python3 papers/matroid-reconstruction-fails/draft1/recompute.py
```

replays the Instant Witness Check, exhaustive verification of all three 8-element failing pairs, all 16 Lean minor-card permutations, the $\mathbb{Z}^3$ realization of M_1, the universal multivariate polynomial proof of M_2's non-representability, and the worked card and census sums in under 3 seconds. The Lean 4 proof is replayed via `lake build MatroidReconst` and `lake env lean MatroidReconst/Verify.lean` under Lean 4 `v4.35.0` with Mathlib.

AI assistance and review status. AI research agents built and directed by Tyler B. Hughes discovered the counterexample pair, executed the censuses, authored and audited the Lean 4 + Mathlib formalization, generated the figures, and drafted this manuscript. No human mathematician has yet refereed the proof.

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REFERENCES

(1) Oxley, J. G. *Matroid Theory*, 2nd ed.; Oxford Graduate Texts in Mathematics, Vol. 21; Oxford University Press: Oxford, **2011**; Problem 15.4.3, p. 592, and March 2020 errata. DOI: 10.1093/acprof:oso/9780198566946.001.0001.

(2) Brylawski, T. Constructions. In *Theory of Matroids*; White, N., Ed.; Encyclopedia of Mathematics and its Applications, Vol. 26; Cambridge University Press: Cambridge, **1986**; pp. 127–223. DOI: 10.1017/cbo9780511629563.010. `[to check]`.

(3) Brylawski, T. On the nonreconstructibility of combinatorial geometries. *J. Combin. Theory Ser. B* **1975**, *19* (1), 72–76. DOI: 10.1016/0095-8956(75)90074-x. `[to check]`.

(4) Miller, W. P. Techniques in matroid reconstruction. *Discrete Math.* **1997**, *170* (1–3), 173–183. DOI: 10.1016/0012-365x(95)00227-n. `[to check]`.

(5) Miller, W. P. Unique representability and matroid reconstruction. *Adv. Appl. Math.* **2003**, *30* (4), 591–606. DOI: 10.1016/s0196-8858(02)00500-6. `[to check]`.

(6) Mills, A. D. A matroid reconstruction result. *Combin. Probab. Comput.* **1994**, *3* (3), 371–373. DOI: 10.1017/s0963548300001267. `[to check]`.

(7) Mayhew, D.; Royle, G. F. Matroids with nine elements. *J. Combin. Theory Ser. B* **2008**, *98* (2), 415–431. DOI: 10.1016/j.jctb.2007.07.005. arXiv:math/0702316. `[to check]`.

(8) Matsumoto, Y.; Moriyama, S.; Imai, H.; Bremner, D. Matroid enumeration for incidence geometry. *Discrete Comput. Geom.* **2012**, *47* (1), 17–43. DOI: 10.1007/s00454-011-9388-y. `[to check]`.

(9) Harary, F. On the reconstruction of a graph from a collection of subgraphs. In *Theory of Graphs and its Applications* (Proc. Symp. Smolenice, 1963); Fiedler, M., Ed.; Publishing House of the Czechoslovak Academy of Sciences: Prague, **1964**; pp. 47–52. Zbl 0161.43404. `[to check]`.

(10) Greenwell, D. L. Reconstructing graphs. *Proc. Amer. Math. Soc.* **1971**, *30* (3), 431–433. DOI: 10.2307/2037710. `[to check]`.