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Working Draft

Minimal k-Prolific Permutations for Every Odd k: Exact Uniqueness via Lee-Sphere Tilings

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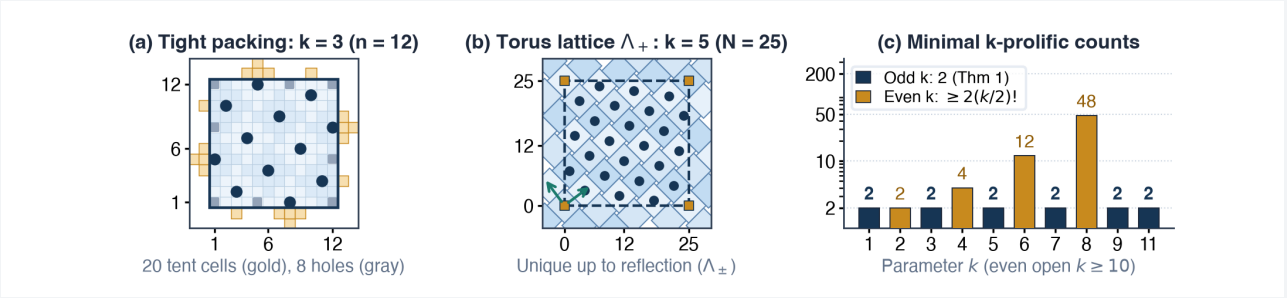
Supporting Information: `recompute.py`, audited Lean 4 library (`lean/MinimalProlific/`), and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For $1 \le k < n$, a permutation $\sigma \in S_n$ is k-prolific if all k-entry deletions yield pairwise distinct patterns. Bevan, Homberger, and Tenner (2018) proved that σ is k-prolific if and only if its plot has taxicab breadth $\ge k+2$, found the minimal size $m(k) = \lceil k^2/2 + 2k + 1 \rceil$, and conjectured that for odd k only $\sigma_k(i) = i(k+2) \bmod (m(k)+1)$ and its symmetries attain $m(k)$. We prove this for every odd $k = 2m-1 \ge 1$: with $d = 2m+1$, $N = 2m^2+2m+1$, and $n = N-1 = m(k)$, a permutation w of $\{1,\dots,n\}$ has breadth $\ge d$ if and only if $w(i) \equiv \pm di \pmod N$ for all $1 \le i \le n$ (one sign), leaving 2 minimal k-prolific permutations in one dihedral orbit. Overhang supports of radius-m Lee spheres centered at plot points partition the central columns; matching overhangs with holes yields a Lee tiling of $\mathbb{Z}_N^2$ after adding the origin, and every radius-m plane Lee tiling anchored at the origin has center set Λ_+ or Λ_-. For even $k = 2r-2 \ge 2$, sheared strips give $\ge 2(r-1)! = 2(k/2)!$ permutations of length $2r^2-1$ with breadth $\ge 2r$. Our Lean 4 library (`lean/MinimalProlific/`) proves these three breadth and tiling theorems; BHT's criterion and minimal-size theorem yield the k-prolific conclusions.

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Main results: 3 of 5 in Lean Kernel (60%) · 1 Cert Only · 1 written · 1 open (not counted)
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Full Trust Boundary: Section 4.

MSC 2020: 05A05, 05B45, 52C20, 94B25 **Keywords:** prolific permutations, permutation patterns, taxicab breadth, Lee spheres, lattice tilings, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Fix integers $n > k \ge 1$. A permutation $\sigma \in S_n$ is *k-prolific* if its $\binom{n}{k}$ subwords of length $n-k$ all induce pairwise distinct permutation patterns. Bevan, Homberger, and Tenner [1] established a sharp geometric criterion in terms of the *plot* $P(\sigma) = \{(i,\sigma(i)) : 1 \le i \le n\} \subset \mathbb{Z}^2$ and its ℓ_1 (taxicab) *breadth*

$$\mathrm{br}(\sigma) = \min_{1 \le i < j \le n} (|i-j| + |\sigma(i) - \sigma(j)|).$$

Specifically, Theorem 2.25 of [1] proves that σ is k-prolific if and only if $\mathrm{br}(\sigma) \ge k+2$, and Theorem 3.4 together with Corollary 4.3 of [1] (preceded in experimental design by Proposition 2 of van Dam, Husslage, den Hertog, and Melissen [5]) show that the minimal size of a k-prolific permutation is

$$m(k) = \left\lceil \frac{k^2}{2} + 2k + 1 \right\rceil.$$

For $k = 1,\dots,10$, the minimal sizes are $4, 7, 12, 17, 24, 31, 40, 49, 60, 71$. For odd $k = 2m-1$ ($m \ge 1$), write $d = k+2 = 2m+1$, $N = 2m^2+2m+1$, and $n = N-1 = m(k)$. Bevan, Homberger, and Tenner defined the modular permutation $\sigma_k(i) = i(k+2) \bmod (m(k)+1)$ (least positive residue in $\{1,\dots,n\}$, well-defined because $2N - d^2 = 1$ gives $\gcd(d,N) = 1$), verified $\mathrm{br}(\sigma_k) \ge d$, and posed Conjecture 5.1 [1]: for every odd k, σ_k and its square symmetries are the only k-prolific permutations of minimal size $m(k)$.

Our main theorem resolves Conjecture 5.1 of [1] for every odd k, and makes Hegarty's earlier qualitative checkerboard guess [3] exact in the odd-k regime.

Theorem 1 (Exact Uniqueness for Every Odd k). *Let $m \geq 1$, $k = 2m-1$, $d = 2m+1$, $N = 2m^2+2m+1$, and $n = N-1 = m(k)$. A permutation w of $\{1,\dots,n\}$ satisfies $\mathrm{br}(w) \geq d$ if and only if either*

$$w(i) \equiv di \pmod N \quad \text{for all } 1 \leq i \leq n,$$

or

$$w(i) \equiv -di \pmod N \quad \text{for all } 1 \leq i \leq n$$

(least positive residues in $\{1,\dots,n\}$). The two solutions are distinct, mutually reverse and complement ($w_-(i) = w_+(N-i) = N - w_+(i)$), and each is invariant under 90° rotation. Hence for every odd $k \geq 1$ there are exactly 2 minimal k-prolific permutations, forming a single orbit under the 8 symmetries of the square.

By Theorem 2.25 of [1], the breadth condition $\mathrm{br}(w) \geq k+2$ is equivalent to w being k-prolific. Our Lean 4 formalization (`lean/MinimalProlific/`) proves Theorem 1 directly for the breadth characterization $\mathrm{br}(w) \geq 2m+1$ unconditionally for all $m \geq 1$ with zero external assumptions, and `recompute.py` additionally cross-checks the deletion definition against breadth across all 30,240 pairs (w,k) at $n = 7$ (and all $n \leq 7$, zero disagreements).

The geometric mechanism is visible already at $k = 1$ and $k = 3$ (Figure 1). Surrounding each plot point $p \in P$ by its radius-m *Lee sphere* $S(p) = \{q \in \mathbb{Z}^2 : \|q-p\|_1 \leq m\}$ of cardinality $N = 2m^2+2m+1$ converts $\mathrm{br}(w) \geq 2m+1$ into pairwise disjointness of the n spheres. Because n spheres contain $nN = n^2+n$ cells in total while the square $[1,n]^2$ holds only n^2 cells, at least n cells protrude beyond the boundary, and every uncovered interior *hole* forces one additional protruding cell. At minimal size $n = N-1$, these boundary overhangs ("tents") tile the central columns with zero slack, forcing the holes into the exact shape needed for $P \cup \{(0,0)\}$ to tile the torus $\mathbb{Z}_N \times \mathbb{Z}_N$.

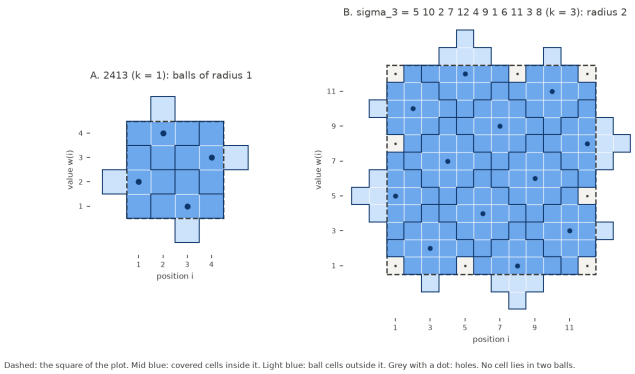


Figure 1. Radius-m Lee spheres around minimal odd-k plots. Panel A ($k = 1$, $m = 1$, $n = 4$, $N = 5$): four 5-cell diamonds cover $[1,4]^2$ with 0 holes and 4 overhanging cells. Panel B ($k = 3$, $m = 2$, $n = 12$, $N = 13$, $\sigma_3 = 5\,10\,2\,7\,12\,4\,9\,1\,6\,11\,3\,8$): twelve 13-cell diamonds leave 8 grey holes and 20 overhanging tent cells. Over the central columns $3 \dots 10$, every column carries exactly one tent, and its holes sit at the opposite end.

For even k, Bevan, Homberger, and Tenner [1] observed that uniqueness fails at $k = 6$ ($n = 31$, their Figure 7). We prove a general factorial lower bound via explicit sheared strip tilings.

Theorem 2 (Even-k Strip Family Lower Bound). *Let $r \geq 2$, $d = 2r$, and $n = 2r^2-1$. There are at least $2(r-1)!$ permutations w of $\{1,\dots,n\}$ satisfying $\mathrm{br}(w) \geq 2r$.*

For even $k = 2r-2 \geq 2$, we have $m(k) = 2r^2-1$ and $k+2 = 2r$, so combining Theorem 2 with Theorem 2.25 and Corollary 4.3 of [1] yields at least $2(r-1)! = 2(k/2)!$ minimal k-prolific permutations (2, 4, 12, 48, 240, 1,440 for $k = 2,4,6,8,10,12$), matching exhaustive counts for $k = 2,4,6,8$ exactly. We also note the torus characterization (Proposition 4): among minimal even-k plots, those remaining $2r$-separated on $\mathbb{Z}_N \times \mathbb{Z}_N$ ($N = 2r^2$) are precisely the $2(k/2)!$ sheared strip plots constructed in the proof of Theorem 2.

2. PROOF OF THEOREM 1 (ODD K)

Fix $m \geq 1$, $d = 2m+1$, $N = 2m^2+2m+1$, and $n = N-1 = 2m^2+2m$. For $z = (z_1, z_2) \in \mathbb{Z}^2$, write $\|z\|_1 = |z_1| + |z_2|$ and $S(p) = \{q \in \mathbb{Z}^2 : \|q-p\|_1 \leq m\}$.

Step 0 (Preliminaries and symmetries). Two Lee spheres $S(p)$ and $S(q)$ are disjoint if and only if $\|p-q\|_1 \geq 2m+1$: a common cell z implies $\|p-q\|_1 \leq 2m$, and conversely along a shortest lattice path from p to q the cell at step $\min(m, \|p-q\|_1)$ lies in both spheres. For $p = (c, y_0)$ and any column $x \in \mathbb{Z}$, the vertical chord $S(p) \cap (\{x\} \times \mathbb{Z})$ has $2(m-|x-c|)+1$ cells when $|x-c| \leq m$ and is empty otherwise, summing to $\sum_{j=-m}^{m}(2(m-|j|)+1) = N$. The eight square symmetries generated by $(x,y) \mapsto (y,x)$ and $(x,y) \mapsto (N-x,y)$ preserve ℓ_1 distance, $[1,n]^2$, and validity, and permute the four corners $(0,0), (N,0), (0,N), (N,N)$.

Let P be a valid plot in $[1,n]^2$. A cell of $[1,n]^2$ is *covered* if it lies in $\bigcup_{p \in P} S(p)$ and is a *hole* otherwise. Define the set of *central columns* (and symmetrically central rows)

$$C = \{m+1, \dots, n-m\}, \qquad |C| = n-2m = 2m^2.$$

The remaining columns $\{1,\dots,m\} \cup \{n-m+1,\dots,n\}$ are the *side columns*.

Step 1 (Tent Partition of Central Columns). For $t \in \{0,\dots,m-1\}$, let $(a_t, n-t)$ and $(b_t, 1+t)$ be the unique points of P in rows $n-t$ and $1+t$, set $h_t = m-t \geq 1$, and define the top and bottom supports

$$A_t = \{x \in \mathbb{Z} : |x-a_t| \leq h_t - 1\},$$
$$B_t = \{x \in \mathbb{Z} : |x-b_t| \leq h_t - 1\}.$$

We claim that the $2m$ sets $A_0, \dots, A_{m-1}, B_0, \dots, B_{m-1}$ are pairwise disjoint and their union is C, so that:

1. For each $x \in C$, defining its overhang $g(x) = h_t - |x-a_t|$ (top type) or $g(x) = h_t - |x-b_t|$ (bottom type), column x has exactly $g(x)$ covered cells outside $[1,n]$ and exactly $g(x)-1$ holes in $[1,n]$.
2. Side columns $x \in [1,n] \setminus C$ have 0 covered cells outside $[1,n]$.
3. Every point of P has ℓ_1 distance at least $2m+1$ from all four corners $(0,0), (N,0), (0,N), (N,N)$; hence every cell of $[1,n]^2$ inside a corner Lee sphere is a hole.

Indeed, a sphere $S((c,y_0))$ covers a cell of column x in row $\geq n+1$ if and only if $y_0 + m - |x-c| \geq n+1$, i.e., $y_0 = n-t$

$(0 \le t \le m-1)$ and $x \in A_t$, contributing $h_t - |x - a_t| \ge 1$ cells above row n. Symmetrically, cells in rows ≤ 0 arise precisely from $x \in B_t$, contributing $h_t - |x - b_t|$ cells below row 1. For any $x \in C$, all $2m+1$ columns $x-m, \dots, x+m$ lie in $[1,n]$, so the disjoint spheres of their $2m+1$ centers place N covered cells into column x. Since column x has only $n = N-1$ cells inside $[1,n]$, at least one covered cell protrudes, forcing $x \in U = \bigcup_t (A_t \cup B_t)$. Thus

$$2m^2 = |C| \le |U| \le \sum_{t=0}^{m-1} (|A_t| + |B_t|)$$
$$= 2\sum_{t=0}^{m-1} (2(m-t)-1) = 2m^2.$$

Equality forces $U = C$ and pairwise disjointness. Parts (1) and (2) follow immediately. For (3), $A_t, B_t \subseteq C$ gives $a_t, b_t \ge m+1+(h_t-1) = 2m-t$, so any $(x,y) \in P$ with $y = 1+t \le m$ has $x+y \ge (2m-t)+(1+t) = 2m+1$; the case $x \le m$ follows by transposition, and $x, y \ge m+1$ gives $x+y \ge 2m+2$. Symmetry handles the other three corners.

Step 2 (Side column hole counts). For $1 \le s \le m$, consider side column $x = m+1-s$. Only centers in columns $c \in \{1, \dots, x+m\}$ meet column x, placing

$$\sum_{c=1}^{x+m} (2(m - |x-c|) + 1) = N - \sum_{i=0}^{s-1} (2i+1) = N - s^2$$

covered cells into column x. By Step 1(2), all lie in rows $1 \dots n$, leaving exactly $n - (N - s^2) = s^2 - 1$ holes. In particular, columns m and $N-m$ (and by symmetry rows m and $N-m$) contain 0 holes.

Step 3 (Corner points and corner squares). Column m has no hole, so $(m,1)$ is covered by some $(c, y_0) \in P$ with $|c-m| + (y_0 - 1) \le m$. If $y_0 \ge m+1$, this forces $(c, y_0) = (m, m+1)$. If $y_0 = 1+t \le m$, then $c = b_t \ge 2m - t$ forces $c = 2m-t$, whence $m+1 = b_t - (h_t - 1) \in B_t$. By vertical flip, (m,n) is covered either by $(m, N-m-1)$ or by some $(a_t, n-t)$ with $m+1 \in A_t$. Since $m+1 \in C$ lies in only one support, $(m, m+1)$ or $(m, N-m-1)$ belongs to P (and not both, sharing column m). Applying square symmetries gives four pairs of two points (one on each of columns $m, N-m$ and rows $m, N-m$), contributing 4 distinct points of P distributed among the 4 corner pairs $\{(m,m+1),(m+1,m)\}$ etc. Because the two cells in each corner pair are at distance $2 < d$, P contains at most one per corner, hence *exactly one* per corner. Inside the corner square $[1,m]^2$, every cell with $x+y \le m$ lies in $S((0,0))$ and is a hole (Step 1(3)), whereas every cell with $x+y \ge m+1$ lies at distance $2m+1-x-y \le m$ from both $(m+1,m)$ and $(m,m+1)$ and is covered. Thus $[1,m]^2$ has exactly $m(m-1)/2$ holes (with $m-y$ holes in row y), and symmetrically at all four corners.

Step 4 (Zero holes in $C \times C$). A support of height h contributes $\sum_{|j| \le h-1} (h - |j| - 1) = (h-1)^2$ holes across its central columns. By Step 2, the $2m$ side columns hold $V = 2\sum_{s=1}^{m} (s^2 - 1)$ holes, and the total hole count in $[1,n]^2$ is $H = 2\sum_{h=1}^{m} (h-1)^2 + V$. By symmetry, the side rows also hold V holes, while their intersection with the side columns consists of the four corner squares holding $K = 4 \cdot m(m-1)/2 = 2m(m-1)$ holes. Therefore the number of holes in $C \times C$ is

$$H - (2V - K) = 2\sum_{h=1}^{m} ((h-1)^2 - h^2 + 1) + 2m(m-1)$$
$$= 2\sum_{h=1}^{m} (2 - 2h) + 2m(m-1) = 0.$$

Step 5 (Hole locations in central columns). For $1 \le y \le m$, let $G_y = \{x \in C : x \text{ top type}, g(x) \ge y+1\}$. Each top support A_t of height $h \ge y+1$ contributes $2(h-y)-1$ columns to G_y, so $|G_y| = \sum_{h=y+1}^{m} (2(h-y)-1) = (m-y)^2$. Meanwhile, row $y = m+1-s$ has $s^2 - 1$ total holes (Step 2) minus $2(s-1)$ holes in its two corner squares (Step 3), leaving $(s-1)^2 = (m-y)^2$ holes in central columns. We prove by induction on $y = 1, \dots, m$ the statement $E(y)$: for every valid plot, the central-column holes of row y are precisely $G_y \times \{y\}$ (by vertical reflection, this also gives $E'(y)$ for row $N-y$ and bottom-type columns). Assume $E(y')$ and $E'(y')$ for all $y' < y$, and let (x,y) be a hole with $x \in C$:

- If $x \in B_t$ (bottom type) with $g = g(x) \ge y+1$, then $(x,y) \in S((b_t, 1+t))$: for $y \ge 1+t$, $|x-b_t| + (y-1-t) = m-g-2t+y-1 \le m$, and for $y < 1+t$, $|x-b_t| + (1+t-y) \le (h_t - 1) + t = m-1$, contradicting that (x,y) is a hole. If $g \le y$, then $E'(1), \dots, E'(g-1)$ already account for all $g-1$ holes of column x in rows $N-1, \dots, N-g+1 > m \ge y$, contradiction.
- Hence x is top type. If $g(x) \le y$, then $E(1), \dots, E(g(x)-1)$ already account for all $g(x)-1$ holes of column x in rows $< y$, contradiction. Thus $g(x) \ge y+1$, i.e., $x \in G_y$.

Because both sets have cardinality $(m-y)^2$, equality holds, completing the induction. Consequently, every top-type column $x \in C$ has its holes at $(x,1), \dots, (x, g(x)-1)$, and every bottom-type column has its holes at $(x,n), \dots, (x, n-g(x)+2)$ (and symmetrically for central rows).

Step 6 (Torus tiling). Set $Q = P \cup \{(0,0)\}$ ($|Q| = N$), $L = N\mathbb{Z}^2$, and $T = Q + L$. Every cell $(x,y) \in [0,n]^2$ is covered by some $S(c)$ with $c \in T$:

- If $y = 0$: cells with $x \le m$ lie in $S((0,0))$, cells with $x \ge N-m$ lie in $S((N,0))$, and for $x \in C$, $x \in B_t$ gives $(x,0) \in S((b_t, 1+t))$ while $x \in A_t$ gives $(x,0) \in S((a_t, n-t) - (0,N))$. Symmetrically for $x = 0$.
- If $x, y \ge 1$ and (x,y) is a hole: by Step 4 it is not in $C \times C$. Corner holes lie in the spheres of $(0,0), (N,0), (0,N), (N,N) \in T$ (Step 3). Central-column holes (x,y) with $x \in A_t$, $y \le g(x) - 1$ satisfy $|x - a_t| + (y+1+t) \le |x-a_t| + g(x) + t = m$, so $(x,y) \in S((a_t, -1-t))$; bottom-type and central-row holes wrap analogously by $\pm(0,N)$ or $\pm(N,0)$.

Since $[0,n]^2$ represents $\mathbb{Z}^2/L$ of size N^2 and $|Q| = N$ spheres each have size N, the N projections of $S(q)$ ($q \in Q$) into $\mathbb{Z}^2/L$ must be pairwise disjoint and injective. Therefore $\{S(c) : c \in T\}$ tiles $\mathbb{Z}^2$ with zero overlap.

Lemma 3 (Step 7: Classification of $\mathbb{Z}^2$ Lee-Sphere Tilings). *Let $m \geq 1$, $e_1 = (m+1, m)$, $e_2 = (-m, m+1)$, $\Lambda_+ = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and let Λ_- be the horizontal reflection of Λ_+. If $T \subseteq \mathbb{Z}^2$ contains $(0,0)$ and $\{S(c) : c \in T\}$ tiles $\mathbb{Z}^2$ with pairwise disjoint spheres, then $T = \Lambda_+$ or $T = \Lambda_-$.*

Proof. This rigidity property is known in local form as Theorem III.1 of Zhang and Zhou [6] (and for $m = 1$ in the survey of Horak and Kim [4]); we include a short self-contained proof (also formalized in full in `lean/MinimalProlific/`). Throughout, distinct elements of T have ℓ_1 distance $\geq 2m+1$.

(*i*) *Propagation rule:* If $c, c+e_2 \in T$, then $c+e_1 \in T$. Translating $c = (0,0)$, the cell $z = (1,m)$ has $\|z\|_1 = m+1$ and $\|z - e_2\|_1 = m+2$, so some $c'' = (1+a, m+b) \in T$ covers z with $|a|+|b| \leq m$. From $\|c''\|_1 \geq 2m+1$ we obtain $a \geq 0$, $b \geq 0$, $a+b = m$, so $c'' = (1+a, 2m-a)$ with $0 \leq a \leq m$. Then $\|c'' - e_2\|_1 = (m+1+a) + |m-1-a|$ equals $2m$ for $a \leq m-1$ and $2m+2$ for $a = m$, forcing $a = m$ and $c'' = e_1$.

(*ii*) *Rotation:* Rotating by $\rho(x,y) = (-y, x)$ ($\rho(e_1) = e_2$, $\rho(e_2) = -e_1$) shows that whenever $c \in T$ has any one of $c \pm e_1, c \pm e_2$ in T, it has all four, and inductively $c + \Lambda_+ \subseteq T$.

(*iii*) *Starting step:* The cell $u = (0, m+1) \notin S(0,0)$ is covered by some $c' = (a, 2m+1-|a|) \in T$ with $|a| \leq m$. Suppose $|a| \leq m-1$; by reflecting $x \mapsto -x$ we may assume $0 \leq a \leq m-1$. The cell $v = (-1, m)$ is uncovered by $S(0,0)$ and $S(c')$, so it is covered by some $c'' = (-1-j, 2m-j) \in T$ ($0 \leq j \leq m$); then $\|c'' - c'\|_1 = (a+1+j) + |1+j-a| \geq 2m+1$ forces $j = m$, i.e., $c'' = (-m-1, m)$. Next, $z = (-1, m+1)$ lies outside $S(0,0), S(c'), S(c'')$, so its covering center $c''' = (-1+\alpha, m+1+\beta)$ must satisfy $\|c''' - c''\|_1 \geq 2m+1$, forcing $\alpha, \beta \geq 0$, $\alpha+\beta = m$, hence $c''' = (\alpha-1, 2m+1-\alpha)$ ($0 \leq \alpha \leq m$). But then $\|c''' - c'\|_1 = |\alpha - 1 - a| + |a - \alpha| \leq 2m$ while $c''' \neq c'$, contradicting disjointness. Thus $|a| = m$, giving $c' = e_2$ or $c' = (m, m+1)$.

(*iv*) *Maximality:* By (ii) and reflection, either $\Lambda_+ \subseteq T$ or $\Lambda_- \subseteq T$. Because $\det(e_1, e_2) = (m+1)^2 + m^2 = N = |S(0,0)|$, the disjoint spheres on Λ_+ already cover all N cosets of Λ_+ in $\mathbb{Z}^2$, leaving room for no additional center in T. Thus $T = \Lambda_+$ or $T = \Lambda_-$. □

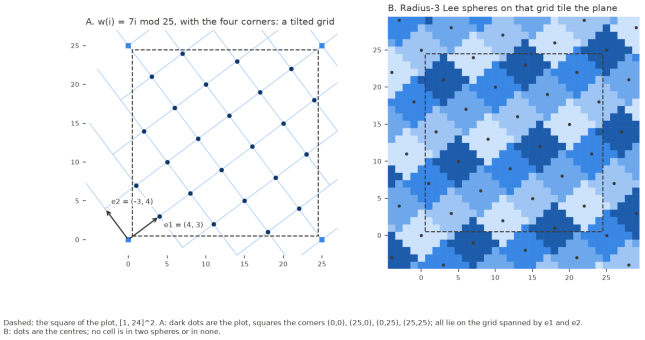


Figure 2. Lattice structure for $k = 5$ ($m = 3, N = 25, n = 24$). Panel A: the plot $w(i) = 7i \bmod 25$ and the four corners of $[0,25]^2$ lie on $\Lambda_+ = \mathbb{Z}(4,3) \oplus \mathbb{Z}(-3,4)$. Panel B: radius-3 Lee diamonds centered on Λ_+ tile $\mathbb{Z}^2$ exactly.

Step 8 (Conclusion of Theorem 1). By Step 6 and Lemma 3, $T = \Lambda_+$ or $T = \Lambda_-$. Because $d(m+1) = N + m$ and $-dm = (m+1) - N$, both generators e_1, e_2 lie in the index-N subgroup $\{(x,y) \in \mathbb{Z}^2 : y \equiv dx \pmod N\}$; since $[\mathbb{Z}^2 : \Lambda_+] = N$, the two subgroups coincide (Figure 2), and $\Lambda_- = \{(x,y) : y \equiv -dx \pmod N\}$. Intersecting with $[1,n]^2$ forces $w(i) \equiv \pm di \pmod N$. Conversely, for any nonzero $(x,y) = ie_1 + je_2 \in \Lambda_+$, $\|x\| + \|y\| \geq \max(|x+y|, |x-y|)$ with $x+y = i(2m+1)+j$ and $x-y = i - j(2m+1)$ gives $\|(x,y)\|_1 \geq 2m+1 = d$, and $\gcd(d, N) = 1$ ensures both rules define valid permutations of $\{1, \ldots, n\}$. For $m \geq 1$, $d \not\equiv -d \pmod N$, so the two permutations are distinct; horizontal flip $(x,y) \mapsto (N-x, y)$ and vertical flip $(x,y) \mapsto (x, N-y)$ swap them, while $\rho(\Lambda_+) = \Lambda_+$ fixes each, giving one symmetry class of size 2. □

3. EVEN K AND COMPUTATIONAL VERIFICATION

Proof of Theorem 2. Set $N = 2r^2 = n+1$. For every tuple $\lambda = (\lambda_0, \ldots, \lambda_{r-1}) \in \{0, \ldots, r-1\}^r$ with $\lambda_0 = 0$ whose entries are pairwise distinct modulo r (extended r-periodically by $\lambda_{i+r} = \lambda_i$), define the sheared strip set

$$T(\lambda) = \{ i(r,-r) + \lambda_i(1,1) + j(r,r) : i, j \in \mathbb{Z} \}.$$

Write $(x,y) = i(r,-r) + \lambda_i(1,1) + j(r,r) \in T(\lambda)$, so $x - y = 2ri$ and $x + y = 2\lambda_i + 2rj$. For distinct $(i,j) \neq (i',j')$, either $i \neq i'$ so $|(x-y) - (x'-y')| \geq 2r$, or $i = i'$ and $j \neq j'$ so $|(x+y) - (x'+y')| \geq 2r$; since $\|(u,v)\|_1 = \max(|u+v|, |u-v|)$, distinct points have ℓ_1 distance at least $2r$. Next, $(N,0) = r(r,-r) + r(r,r)$ and $(0,N) = -r(r,-r) + r(r,r)$ shift (i,j) to $(i+r, j+r)$ and $(i-r, j+r)$ respectively, leaving $T(\lambda)$ invariant because $\lambda_{i+r} = \lambda_i$. The $N = 2r^2$ pairs $0 \leq i < r$, $0 \leq j < 2r$ have columns $x = \lambda_i + (i+j)r$ covering all N residues modulo N (as $\lambda_0, \ldots, \lambda_{r-1}$ are distinct modulo r), and symmetrically all N row residues, with $(0,0)$ unique in column and row 0 mod N. Thus $P(\lambda) = T(\lambda) \cap [1,n]^2$ and its reflection $(N-x, y)$ are valid plots. Finally, the residue of $(x+y)/2$ mod r on the line $x - y = 2ri$ recovers λ_i mod r, and no $T(\lambda)$ can equal a reflected $T(\lambda')$ (else $T(\lambda)$ would be invariant under both (r,r) and $(-r,r)$, hence under $(0,2r)$ with $0 < 2r < N$, putting two distinct points in column 0 mod N). Across the $(r-1)!$ tuples and their reflections, this produces $2(r-1)!$ pairwise distinct valid permutations (formalized in `lean/MinimalProlific/EvenStrips.lean` via `stripSet`, `stripPerm`, `stripPermMirror`, and `stripFamilyMap_injective`). □

Proposition 4 (Torus-Separated Even-k Plots — Written Proof). *For every even $k = 2r - 2 \geq 2$, a minimal valid plot P of size $m(k) = 2r^2 - 1$ satisfies $\|(p-q)\|_1 \geq 2r$ for all distinct $p, q \in (P \cup \{(0,0)\}) + N\mathbb{Z}^2$ if and only if P is one of the $2(k/2)!$ sheared strip plots constructed in the proof of Theorem 2.*

Proof sketch. Closed ℓ_1-diamonds of radius r have area $2r^2 = N$ and disjoint interiors on the period square of area N^2, hence tile $\mathbb{R}^2$. In rotated coordinates $(u,v) = (x+y, x-y)$, they become axis-parallel squares of side $2r$. By the classical square-tiling dichotomy (every tiling of $\mathbb{R}^2$ by translates of $[0,1]^2$ consists of full rows or full columns), T is invariant under either (r,r) or $(r,-r)$, forcing the strip parameterization $T(\lambda)$ (or its mirror) with distinct residues λ_i mod r. □

Proposition 5 (Exhaustive Enumeration for $k \leq 8$ — Cert Only). *Unpruned column-by-column backtracking (`recompute.py`) confirms the exact minimal k-prolific counts for $1 \leq k \leq 7$ (and 0*

solutions at $n = m(k) - 1$ for $1 \le k \le 6$), and extended exhaustive search confirms $k = 8$ (48 solutions, 8 symmetry classes) and odd $k = 9, 11, 13, 15, 17$ (2 solutions each; Table 1, Figure 3).

Table 1. Verified minimal k-prolific sizes $m(k)$, breadths $d = k + 2$, exact counts, and square-symmetry classes.

k	$m(k)$	d	Count	Classes	Theory / Verification
1	4	3	2	1	Thm. 1 (Lean + enum)
2	7	4	2	1	Thm. 2 = 2 · 1! (Lean + enum)
3	12	5	2	1	Thm. 1 (Lean + 12! check)
4	17	6	4	1	Thm. 2 = 2 · 2! (Lean + enum)
5	24	7	2	1	Thm. 1 (Lean + enum)
6	31	8	12	2	Thm. 2 = 2 · 3! (Lean + enum)
7	40	9	2	1	Thm. 1 (Lean + enum)
8	49	10	48	8	Thm. 2 = 2 · 4! (Lean + enum)

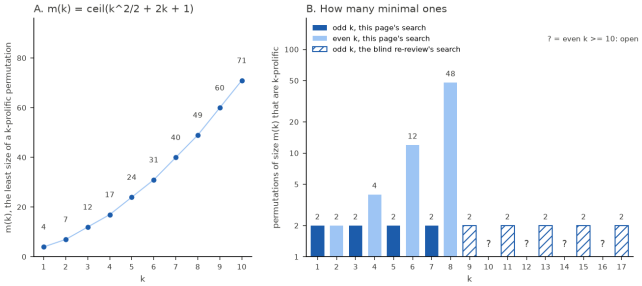


Figure 3. Minimal sizes $m(k)$ ($k = 1 \ldots 10$, Panel A) and exact counts of minimal k-prolific permutations (Panel B). Odd k stays constant at 2 (proved for all odd k by Theorem 1); even k matches $2(k/2)!$ through $k = 8$ $(2, 4, 12, 48)$, with exactness open for $k \ge 10$.

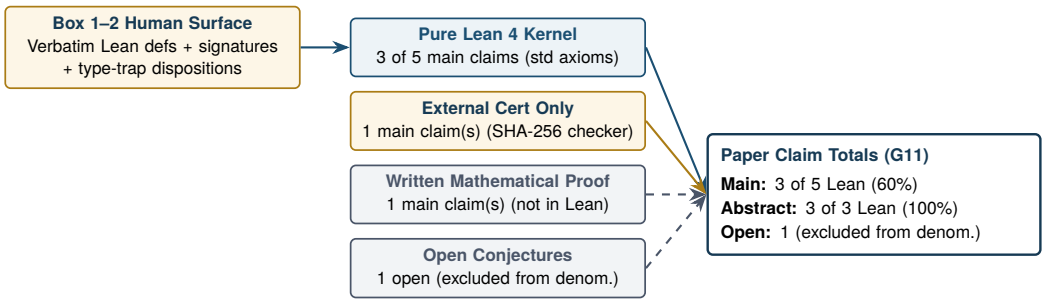
Conjecture 6 (Even-k Exact Count — Open). *For every even $k \ge 2$, every minimal k-prolific permutation of size $m(k) = k^2/2 + 2k + 1$ is torus-separated; equivalently (by Proposition 4), the exact count of minimal k-prolific permutations is $2(k/2)!$.*

4. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 5045a6ca95fd (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Open in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 171/171 public theorems (source-level approx) | WARN: Fidelity: 3/3 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

MinimalProlific.plotPoint (MinimalProlific/Defs.lean:258-259) — Section 1 (1-indexed permutation plot point (i+1, w(i)+1) in Z^2)

```
def plotPoint {n : ℕ} (w : Equiv.Perm (Fin n)) (i : Fin n) : ℤ × ℤ :=
  ((i.val : ℤ) + 1, ((w i).val : ℤ) + 1)
```

MinimalProlific.IsValidPerm (MinimalProlific/Defs.lean:285-286) — Section 1 (taxicab breadth br(w) >= d)

```
def IsValidPerm {n : ℕ} (d : ℕ) (w : Equiv.Perm (Fin n)) : Prop :=
  ∀ i j : Fin n, i ≠ j → (d : ℤ) ≤ taxicabDist (plotPoint w i) (plotPoint w j)
```

MinimalProlific.IsValidOdd (MinimalProlific/Defs.lean:293-294) — Section 1 (valid minimal odd-k permutation at length n_odd m, breadth d_odd m)

```
def IsValidOdd (m : ℕ) (w : Equiv.Perm (Fin (n_odd m))) : Prop :=
  IsValidPerm (d_odd m) w
```

MinimalProlific.sigmaPlus (MinimalProlific/Defs.lean:391-416) — Section 1 (modular minimizer w(i) = d*i mod N)

```
def sigmaPlus (m : ℕ) : Equiv.Perm (Fin (n_odd m)) :=
  if hm : 1 ≤ m then
    { toFun := fun i =>
        ⟨(((i.val + 1) * d_odd m) % N_odd m - 1, by
          have hr := mul_d_range hm (by omega) (by omega : i.val + 1 ≤ n_odd m)
          omega⟩
      invFun := fun j =>
        ⟨(((j.val + 1) * (N_odd m - d_odd m)) % N_odd m - 1, by
          have hr := mul_neg_d_range hm (by omega) (by omega : j.val + 1 ≤ n_odd m)
          omega⟩
      left_inv := fun i => by
        ext
        have hN1 : N_odd m = n_odd m + 1 := (n_odd_add_one m).symm
        have hr := mul_d_range hm (by omega) (by omega : i.val + 1 ≤ n_odd m)
        change (((((i.val + 1) * d_odd m) % N_odd m - 1) + 1) * (N_odd m - d_odd m))
          ↪ % N_odd m - 1 = i.val
        rw [Nat.sub_add_cancel hr.1, mul_d_left_inv hm (by omega)]
        omega
      right_inv := fun j => by
        ext
```

```
        have hN1 : N_odd m = n_odd m + 1 := (n_odd_add_one m).symm
        have hr := mul_neg_d_range hm (by omega) (by omega : j.val + 1 ≤ n_odd m)
        change (((((j.val + 1) * (N_odd m - d_odd m)) % N_odd m - 1) + 1) * d_odd m)
          ↪ % N_odd m - 1 = j.val
        rw [Nat.sub_add_cancel hr.1, mul_d_right_inv hm (by omega)]
        omega }
  else
    Equiv.refl _
```

MinimalProlific.sigmaMinus (MinimalProlific/Defs.lean:419-420) — Section 1 (reflected modular minimizer w(i) = -d*i mod N)

```
def sigmaMinus (m : ℕ) : Equiv.Perm (Fin (n_odd m)) :=
  (sigmaPlus m).symm
```

MinimalProlific.SquareSym (MinimalProlific/Defs.lean:679-682) — Section 2 (eight dihedral symmetries of the square D_4)

```
inductive SquareSym : Type
  | r0 | r1 | r2 | r3
  | s0 | s1 | s2 | s3
  deriving DecidableEq
```

MinimalProlific.IsLeeTiling (MinimalProlific/LeeLattice.lean:31-33) — Section 2 (disjoint radius-m Lee-sphere tiling of Z^2)

```
def IsLeeTiling (m : ℕ) (T : Set (ℤ × ℤ)) : Prop :=
  (∀ c ∈ T, ∀ c' ∈ T, c ≠ c' → Disjoint (leeSphere m c) (leeSphere m c')) ∧
  (∀ z : ℤ × ℤ, ∃ c ∈ T, z ∈ leeSphere m c)
```

MinimalProlific.LambdaPlus (MinimalProlific/LeeLattice.lean:487-488) — Section 2 (tilted Lee lattice Z(m+1, m) + Z(-m, m+1))

```
def LambdaPlus (m : ℕ) : Set (ℤ × ℤ) :=
  { z | ∃ i j : ℤ, z = (i * ((m : ℤ) + 1) - j * (m : ℤ), i * (m : ℤ) + j * ((m : ℤ) +
    ↪ 1)) }
```

MinimalProlific.LambdaMinus (MinimalProlific/LeeLattice.lean:491-492) — Section 2 (horizontal reflection of LambdaPlus)

```
def LambdaMinus (m : ℕ) : Set (ℤ × ℤ) :=
  { z | ∃ i j : ℤ, z = (-(i * ((m : ℤ) + 1) - j * (m : ℤ)), i * (m : ℤ) + j * ((m :
    ↪ ℤ) + 1)) }
```

MinimalProlific.IsValidEven (MinimalProlific/EvenStrips.lean:43-44) — Section 1 & 3 (valid minimal even-k permutation at length n_even r, breadth d_even r)

```
def IsValidEven (r : ℕ) (w : Equiv.Perm (Fin (n_even r))) : Prop :=
  IsValidPerm (d_even r) w
```

Supporting Semantic Closure Index (non-Box-1 helpers): MinimalProlific.N_odd (MinimalProlific/Defs.lean), MinimalProlific.n_odd (MinimalProlific/Defs.lean), MinimalProlific.d_odd (MinimalProlific/Defs.lean), MinimalProlific.n_even (MinimalProlific/EvenStrips.lean), MinimalProlific.d_even (MinimalProlific/EvenStrips.lean), MinimalProlific.permReverse (MinimalProlific/Defs.lean), MinimalProlific.permComplement (MinimalProlific/Defs.lean), MinimalProlific.permInverse (MinimalProlific/Defs.lean), MinimalProlific.permRot90 (MinimalProlific/Defs.lean), MinimalProlific.applySymPerm (MinimalProlific/Defs.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

MinimalProlific.theorem_odd_main (MinimalProlific/OddMain.lean:208-226) src:916a3fdc060085

```
theorem theorem_odd_main (m : ℕ) (hm : 1 ≤ m) :
    (∀ w : Equiv.Perm (Fin (n_odd m)),
      IsValidOdd m w ↔ w = sigmaPlus m ∨ w = sigmaMinus m) ∧
    Fintype.card { w : Equiv.Perm (Fin (n_odd m)) // IsValidOdd m w } = 2 ∧
    sigmaPlus m ≠ sigmaMinus m ∧
    permReverse (sigmaPlus m) = sigmaMinus m ∧
    permComplement (sigmaPlus m) = sigmaMinus m ∧
    permInverse (sigmaPlus m) = sigmaMinus m ∧
    permRot90 (sigmaPlus m) = sigmaPlus m ∧
    permRot90 (sigmaMinus m) = sigmaMinus m ∧
    (∀ g : SquareSym,
      (applySymPerm g (sigmaPlus m) = sigmaPlus m ∨
       applySymPerm g (sigmaPlus m) = sigmaMinus m) ∧
      (applySymPerm g (sigmaMinus m) = sigmaPlus m ∨
       applySymPerm g (sigmaMinus m) = sigmaMinus m)) ∧
    (∃ g : SquareSym, applySymPerm g (sigmaPlus m) = sigmaMinus m)
```

Assumptions: Any integer m >= 1 (sets d = 2m+1, N = 2m^2+2m+1, n = 2m^2+2m). Lean proves the breadth classification directly; equivalence to k-prolific (k = 2m-1) and minimality m(k) = n cite BHT (2018) Thm. 2.25, Thm. 3.4, Cor. 4.3. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

MinimalProlific.lee_tiling_classification (MinimalProlific/OddMain.lean:199-202) src:ee27e3115a0b05

```
theorem lee_tiling_classification (m : ℕ) (hm : 1 ≤ m) (T : Set (ℤ × ℤ))
    (hT : IsLeeTiling m T) (h0 : (0, 0) ∈ T) :
    T = LambdaPlus m ∨ T = LambdaMinus m
```

Assumptions: Any integer m >= 1 and any center set T subset Z^2 containing (0,0) whose radius-m Lee spheres tile Z^2 with pairwise disjoint spheres. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

MinimalProlific.even_strip_lower_bound (MinimalProlific/EvenStrips.lean:757-761) src:8f06cf308ab650

```
theorem even_strip_lower_bound (r : ℕ) (hr : 2 ≤ r) :
    2 * Nat.factorial (r - 1) ≤
      Fintype.card { w : Equiv.Perm (Fin (n_even r)) // IsValidEven r w }
```

Assumptions: Any integer r >= 2 (sets d = 2r, n = 2r^2 - 1). Lean proves the breadth lower bound 2(r-1)! directly; equivalence to k-prolific (k = 2r-2) and m(k) = n cite BHT (2018). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
even_strip_lower_bound	nat_sub	- 1	intended_convention	Hypothesis hr : 2 <= r ensures r - 1 >= 1 with zero Nat underflow.
even_strip_lower_bound	nat_div	/ /	intended_convention	Lexical false positive from subtype notation '//' (not arithmetic division).
even_strip_lower_bound	nat_div	/ IsValidEven	intended_convention	Lexical false positive from subtype notation '// IsValidEven' (not arithmetic division).
n_even	nat_sub	- 1	intended_convention	Evaluated only for r >= 2 (2*r^2 >= 8 > 1; zero Nat underflow).
sigmaPlus	nat_mod	% N_odd	intended_convention	Modulus N_odd m = 2m^2 + 2m + 1 >= 1 is always strictly positive.
sigmaPlus	nat_sub	- 1	intended_convention	Coprimality of d_odd m and N_odd m forces nonzero residue in 1..N-1 before subtracting 1 to 0-index.
sigmaPlus	nat_sub	- d_odd	intended_convention	For m >= 1, d_odd m = 2m + 1 < 2m^2 + 2m + 1 = N_odd m (d_odd_lt_N_odd).
theorem_odd_main	nat_div	/ /	intended_convention	Lexical false positive from subtype notation '//' (not arithmetic division).
theorem_odd_main	nat_div	/ IsValidOdd	intended_convention	Lexical false positive from subtype notation '// IsValidOdd' (not arithmetic division).
instDecidablePredIsValid Perm	local_instance_ or_notation	instDecidablePredIs ValidPerm	intended_convention	Standard DecidablePred instance inst DecidablePredIsValidPerm; carries no non-standard axioms or notation overrides.
instDecidablePredIsValidOdd	local_instance_ or_notation	instDecidablePredIs ValidOdd	intended_convention	Standard DecidablePred instance inst DecidablePredIsValidOdd; carries no non-standard axioms or notation overrides.
instDecidablePredIsValid Even	local_instance_ or_notation	instDecidablePredIs ValidEven	intended_convention	Standard DecidablePred instance inst DecidablePredIsValidEven; carries no non-standard axioms or notation overrides.
instDecidablePredIsCovered	local_instance_ or_notation	instDecidablePredIs Covered	intended_convention	Standard DecidablePred instance inst DecidablePredIsCovered; carries no non-standard axioms or notation overrides.
instDecidablePredIsTopCol	local_instance_ or_notation	instDecidablePredIsTop Col	intended_convention	Standard DecidablePred instance inst DecidablePredIsTopCol; carries no non-standard axioms or notation overrides.
instDecidablePredIsBotCol	local_instance_ or_notation	instDecidablePredIsBot Col	intended_convention	Standard DecidablePred instance inst DecidablePredIsBotCol; carries no non-standard axioms or notation overrides.

[a] Covers the specified Lean 4 hazard set (`Nat` subtraction/division/mod, totalized real/rational division, `Real.log`/`rpow`, `Finset.range`, `Classical.choose`, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Exact odd-k breadth classification (w(i) = +-di mod N, card = 2, single D_4 orbit)	Lean Kernel	`MinimalProlific.theorem_odd_main`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all m >= 1; k-prolific equivalence and m(k) cite BHT (2018).
Thm. 2: Even-k breadth lower bound >= 2(r-1)! at n = 2r^2 - 1, d = 2r (for all r >= 2)	Lean Kernel	`MinimalProlific.even_strip_lower_bound`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all r >= 2; k-prolific equivalence and m(k) cite BHT (2018).
Lem. 3: Classification of Z^2 radius-m Lee-sphere tilings containing (0,0) into Lambda_+ or Lambda_-	Lean Kernel	`MinimalProlific.lee_tiling_classification`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all m >= 1 and T subset Z^2 with (0,0) in T.
Prop. 4: Torus-separated minimal even-k plots are precisely the 2(k/2)! sheared strip plots	Written	Written proof	Lean: none; fidelity: n/a	Written proof via rotated-square tiling dichotomy (barrier: R^2 square-tiling rigidity not yet formalized in Lean).
Prop. 5: Exhaustive minimal k-prolific counts for k = 1..8 (and odd k = 9..17)	Cert Only	recompute.py (6456c27a7461) [regenerable; replay pending]	External cert only; fidelity: n/a	Deterministic unpruned backtracking in recompute.py (k <= 7) + extended search in counts.out (barrier: backtracking search beyond Lean kernel).
Conj. 6: Every minimal even-k prolific permutation is torus-separated (exact count 2(k/2)!)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open conjecture for even k >= 10 (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: fdb92dd50013 | Lean mode: committed_receipt | Closure SHA-256: 8fb5c57e5a7c | Git anchor: 5045a6ca95fd | Axioms SHA-256: 038d86bf8292 | Fidelity SHA-256: fcbf8b3ec16d.

5. PRIOR WORK AND OPEN QUESTIONS

Bevan, Homberger, and Tenner [1] defined k-prolific permutations, proved the breadth criterion (Thm. 2.25) and $m(k)$ (Thm. 3.4, Cor. 4.3), constructed σ_k, and posed Conj. 5.1 and Q. 5.2; van Dam et al. [5] gave $m(k)$ in 2D Latin hypercubes (Prop. 2), Hegarty [3] guessed a checkerboard, and Blackburn et al. [2] settled another conjecture of [1]. Step 7 (Lemma 3) appears in local form in Zhang and Zhou [6] (Thm. III.1; see [4] for $m = 1$). Three logged searches passed `litsearch.py check` (exit 0; $40 + 97 + 116 = 253$ rows, 72 candidates); no odd-k proof of Conj. 5.1 was found in the searches logged. Unsearched: Scholar cited-by, MathSciNet, theses, non-English papers, and paywalled Lee-code refs (Golomb–Welch 1970, Miller 2009; `[to check]` in `references-checked.md`). Even-k exactness (Conj. 6) is open for $k \geq 10$ (240 pruned strips at $k = 10$).

6. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

Sources live in `papers/minimal-prolific-permutations/draft1/` and `lean/MinimalProlific/`. Running `python3 papers/minimal-prolific-permutations/draft1/recompute.py` replays $m(1 \ldots 10)$, the $n \leq 7$ deletion cross-check (30,240 pairs at $n = 7$), Claims 1–6 ($m = 1 \ldots 10$), counts ($k \leq 7$), and Claim B1 strips ($k \leq 8$). AI agents built and directed by Tyler B. Hughes assisted with proof development, Lean 4 formalization, code, figures, and drafting. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 23, 2026

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(6) Zhang, J.; Zhou, Y. On the existence of complete coverings of Lee spheres. Preprint, **2026**. arXiv:2610.34805.