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Working Draft

# Flammenkamp's Twelve Losing Lengths in $.0^{k-1}7$ for Every $k \geq 4$

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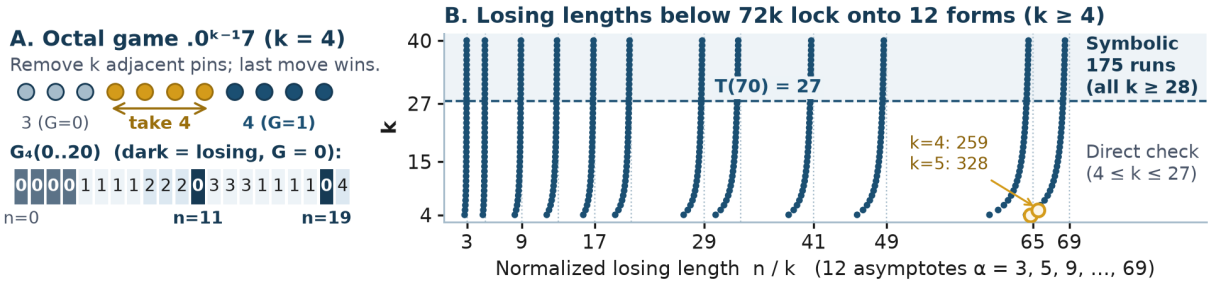
Ⓢ *Supporting Information:* `recompute.py` *(symbolic-interval solver, certificate verifier, and concrete checks)*

**Disclosure:** AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.




**ABSTRACT:** In the normal-play octal game $.0^{k-1}7$, a move removes $k$ adjacent pins from a row. Achim Flammenkamp conjectured (checked for $3 < k < 32$) that for every $k \geq 4$, the first twelve losing lengths $n \geq k$ are $3k-1, 5k-1, 9k-3, 13k-5, 17k-5, 21k-7, 29k-9, 33k-11, 41k-11, 49k-13, 65k-19, 69k-19$. We prove this for every $k \geq 4$ by combining three written results with exact symbolic-interval computation: a hand proof for $3k-1$ ($k \geq 2$) and a candidate parity lemma; soundness of symbolic Sprague–Grundy evaluation on $[0, Mk)$ for $k > T(M)$; and the invariant $a + 2b \geq 1$ proving $T(M) \leq \lfloor M/2 \rfloor$ (exact on $[0, k^2)$). At $M = 70$, symbolic evaluation gives 175 runs and $T(70) = 27$ (4/4 mutations caught), while direct checks close $4 \leq k \leq 27$ (through $k = 66$; $k = 2, 3$ fail). Extensions to $M = 180{,}360$ settle companions $8, 16, 32, 64$ (and 128 with archived checks), and archived sweeps find thirteenth and fourteenth forms $8255k - 2441$ ($k \geq 265$) and $11247k - 3475$ ($k \geq 433$), refuting a $< k^2$ gap at $k = 8255$. Infinitely many losing forms remain unsettled; no Lean 4 formalization is claimed.

**Verification Scope (G11):** WARN: Human referee: pending
Main results: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)
Abstract claims: 0 of 5 in Lean Kernel (0%) · 2 Cert Only · 3 written · 1 open (not counted)

**Full Trust Boundary:** Section 5.

**MSC 2020:** 91A46, 05A15, 68W30 **Keywords:** octal games, Sprague–Grundy values, losing positions, symbolic interval arithmetic, Flammenkamp conjecture

## 1. INTRODUCTION AND MAIN RESULTS

Start with a row of $n$ pins standing in a line, and fix an integer $k \geq 2$. Two players alternate moves: each move removes exactly $k$ consecutive pins from a single row, leaving zero, one, or two shorter rows to the left and right of the gap. Under normal play, the player who has no legal move loses. In the octal notation of Guy and Smith [8] (`[to check]`) and Berlekamp, Conway, and Guy [1] (`[to check]`), this impartial game is written $.0^{k-1}7$: removing $1, \ldots, k-1$ pins is forbidden, while removing $k$ pins is permitted whether it leaves 0, 1, or 2 nonempty rows ($7 = 1 + 2 + 4$).

The case $k = 2$ (.07) is Dawson's Kayles, whose Sprague–Grundy sequence has eventual period 34, and $k = 3$ (.007) is equivalent to Treblecross. Every row of 0 to $k-1$ pins admits no move and is trivially a losing position ($\mathcal{P}$-position) for the player to move. A row length $n \geq k$ is a *nontrivial losing length* when the first player loses from a single row of $n$ pins, equivalently when its Sprague–Grundy value satisfies $G_k(n) = 0$.

On his web page *Sprague-Grundy Values of Octal Games* (`wwwhomes.uni-bielefeld.de/achim/octal.html`, Table 4, revised between 2026-08-09 and 2026-11-17), Achim Flammenkamp printed this conjecture (keeping his spelling and indexing, where $j_1, \ldots, j_k$ denote the $k$ trivial losing lengths $0, \ldots, k-1$):

> "We conjecture: For each $k > 3$ and for all $i$ with $k < i \leq k+12$ let $j_i$ be the $i$-th lose position of the game, which always exists. Then there are 12 positive, stricly increasing integers $\alpha_{i-k}$ and 12 positive, increasing integers $\beta_{i-k}$ such that holds $j_i = \alpha_{i-k}k - \beta_{i-k}$. We have $(\alpha_h) = 3, 5, 9, 13, 17, 21, 29, 33, 41, 49, 65, 69$ and $(\beta_h) = 1, 1, 3, 5, 5, 7, 9, 11, 11, 13, 19, 19$. We have confirmed this conjecture numerically for all $3 < k < 32$."

Our main result settles Flammenkamp's conjecture for every integer $k \geq 4$ simultaneously by running the Sprague–Grundy recurrence with $k$ kept as an indeterminate symbol.

**Theorem 1** (Flammenkamp's Twelve Losing Lengths for Every $k \geq 4$). *For every integer $k \geq 4$, the first twelve nontrivial losing lengths $n \geq k$ of the octal game $.0^{k-1}7$ are*

$$3k-1,\quad 5k-1,\quad 9k-3,\quad 13k-5,$$
$$17k-5,\quad 21k-7,\quad 29k-9,\quad 33k-11,$$
$$41k-11,\quad 49k-13,\quad 65k-19,\quad 69k-19.$$

*More precisely, for every integer $k \geq 28$, the Sprague–Grundy sequence $G_k(n)$ on $0 \leq n < 70k$ consists of exactly 175 maximal constant runs with affine breakpoints $a_i k + b_i$, whose zero-valued runs are precisely $[0,k)$ and the twelve singleton lengths above. The twelve-length law fails for $k = 2$ (beginning $5, 9, 15, \ldots$) and $k = 3$ (beginning $8, 14, 24, \ldots$).*

The proof of Theorem 1 combines three self-contained written results (Proposition 2, Lemma 3, and Theorem 4) with a deterministic exact symbolic-interval computation on $[0, 70k)$ and direct evaluation of the finite base $4 \leq k \leq 27$. We begin with the elementary hand-provable core.

**Proposition 2** (Hand Proof for $3k-1$ and Candidate Parity Lemma). *Let $k \geq 2$ be an integer and let $G_k(n)$ be the Sprague–Grundy sequence of $.0^{k-1}7$.*

1. *For every $k \geq 2$, one has $G_k(n) = 0$ on $[0,k)$, $G_k(n) = 1$ on $[k,2k)$, $G_k(n) = 2$ on $[2k,3k-1)$, and $G_k(3k-1) = 0$. In particular, $3k-1$ is the smallest nontrivial losing length for every $k \geq 2$.*
2. *Every nontrivial losing length $n \geq k$ satisfies that $m = n-k$ is odd and $G_k((m-1)/2) \neq G_k((m+1)/2)$. Equivalently, $n = 2e-1+k$ for a run breakpoint $e \geq 1$ where $G_k(e) \neq G_k(e-1)$.*

To handle all twelve forms uniformly in $k$, we represent positions as affine forms $(a,b) \in \mathbb{Z}^2$ with evaluation $E_k(a,b) := ak + b$, ordered lexicographically by $(a_1,b_1) \prec (a_2,b_2)$ whenever $a_1 < a_2$, or $a_1 = a_2$ and $b_1 < b_2$. For any lexicographically ordered pair $p = (a_1,b_1) \prec q = (a_2,b_2)$, define its *order threshold*

$$\tau(p,q) := \begin{cases} 0 & \text{if } a_1 = a_2 \text{ or } b_1 < b_2, \\ \lfloor (b_1 - b_2)/(a_2 - a_1) \rfloor & \text{if } a_1 < a_2 \text{ and } b_1 \geq b_2. \end{cases} \tag{1}$$

Then $E_k(p) < E_k(q)$ holds for every integer $k > \tau(p,q)$ (and for $k \geq 1$ when $\tau(p,q) = 0$).

**Lemma 3** (Soundness of Symbolic-Interval Evaluation). *Suppose the block-by-block symbolic-interval Sprague–Grundy algorithm of Section 3.2 executes through block $M \geq 1$, producing symbolic runs $R_i = [e_i, e_{i+1})$ with nimbers $v_i$ ($0 \leq i < r$, $e_0 = (0,0)$, $e_r = (M,0)$), and let $T(M)$ be the maximum of $\tau(p,q)$ over all lexicographic comparisons $p \prec q$ performed during block clipping, event sorting, and breakpoint merging. Then for every integer $k > T(M)$, the evaluated breakpoints satisfy $0 = E_k(e_0) < E_k(e_1) < \cdots < E_k(e_r) = Mk$ and $G_k(n) = v_i$ for every integer $n \in [E_k(e_i), E_k(e_{i+1}))$.*

Remarkably, the recorded threshold $T(M)$ obeys a universal linear upper bound proved in a few lines by hand.

**Theorem 4** (Breakpoint Invariant and Half-Range Bound). *Define the additive functional $\varphi(a,b) := a + 2b$ on $\mathbb{Z}^2$.*

1. *Every symbolic breakpoint $(a,b) \neq (0,0)$ generated by the all-$k$ symbolic-interval algorithm satisfies $b \leq 0$ and $\varphi(a,b) = a + 2b \geq 1$.*
2. *For every block horizon $M \geq 1$, the comparison threshold satisfies $T(M) \leq \lfloor M/2 \rfloor$. Consequently, for every integer $k > M/2$ and every $0 \leq n < Mk$, the symbolic all-$k$ pattern equals the exact concrete Sprague–Grundy value $G_k(n)$.*
3. *Equivalently, for every integer $k \geq 2$, the common all-$k$ pattern coincides identically with $G_k(n)$ on the range $0 \leq n < (2k-1)k$, which strictly contains $[0,k^2)$.*

Extending the symbolic computation past $M = 70$ settles Flammenkamp's power-of-2 formulas, while archived multi-program runs clarify what lies past $69k - 19$.

**Proposition 5** (First-Occurrence Companions and Beyond Block 69). *Exact symbolic-interval evaluation and concrete checks establish:*

1. *In `recompute.py`, extending the symbolic sweep to $M = 180$ (832 runs, $T(180) = 66$) together with direct concrete evaluation for $2 \leq k \leq 66$ proves that for every $k \geq 2$, the smallest length $n$ attaining $G_k(n) = 8, 16, 32, 64$ is $20k-5$ ($k > 2$), $40k-10$ ($k > 4$), $87k-23$ ($k > 7$), and $178k-57$ ($k > 13$), with every smaller $k \geq 2$ differing. Extending to $M = 360$ (2,681 runs, $T(360) = 130$) proves $G_k(n) = 128$ first occurs at $358k - 109$ for every $k \geq 131$, matching direct checks on $2 \leq k \leq 32$ ($k > 22$) and archived concrete checks on $33 \leq k \leq 130$.*
2. *Across five independent archived implementations (plus a blind re-review up to $k = 100{,}000$), all twelve losing forms agree; separately reviewed archived sweeps past block 69 identify a thirteenth losing form $8255k - 2441$ (holding for every $k \geq 265$) and a fourteenth losing form $11247k - 3475$ (holding for every $k \geq 433$). Thus the naive gap claim "no losing length lies strictly between $69k - 19$ and $k^2$" holds for every $k \leq 8254$ and fails for every $k \geq 8255$ (starting at $n = 8255^2 - 2441 = 68{,}142{,}584 < 68{,}145{,}025 = 8255^2$), and Flammenkamp's list of "simple" games (having exactly $k + 12$ losing positions) is finite ($k < 265$).*

**Conjecture 6** (Infinitely Many Losing Forms and Non-Computational Proof). *The limiting all-$k$ symbolic Sprague–Grundy pattern of $.0^{k-1}7$ contains infinitely many losing forms $a_i k + b_i$ (equivalently, the count $L(k)$ of nontrivial losing lengths below $k^2$ satisfies $\lim_{k\to\infty} L(k) = \infty$). Independently, whether there exists a human-readable, non-computational proof of Theorem 1 remains unsettled.*

Figure 1 summarizes the logical separation between written proofs, deterministic computer-assisted verification, and unsettled questions. To illustrate the game directly, the short Python snippet below computes $G_k(n)$ from the definition and verifies Flammenkamp's twelve losing lengths for $4 \leq k \leq 8$ (executed verbatim inside `recompute.py`).

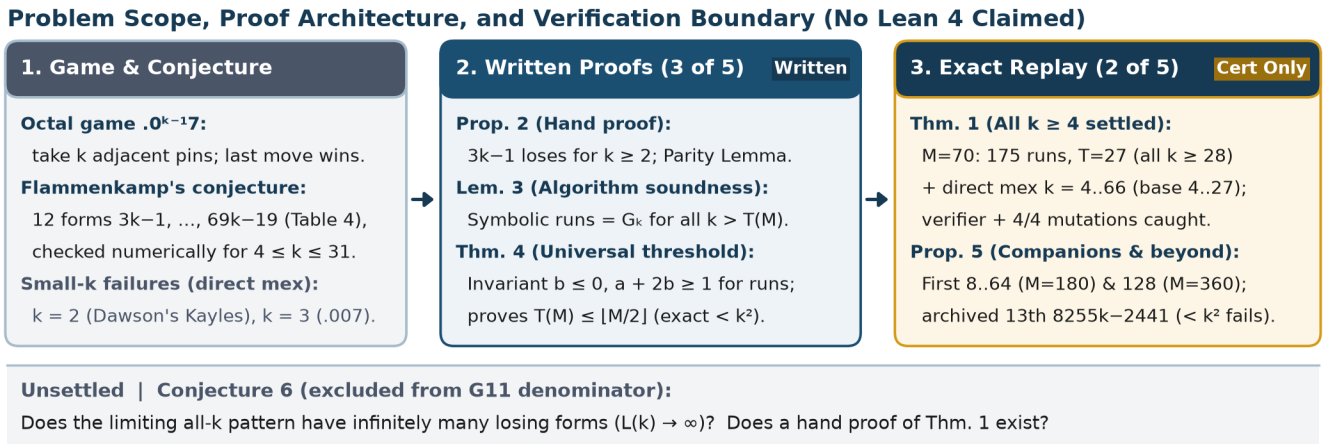



**Figure 1.** Problem scope, proof architecture, and verification boundary. Proposition 2, Lemma 3, and Theorem 4 are proved in writing (3 of 5 proved main results). Theorem 1 and Proposition 5 are proved by exact symbolic-interval and concrete computation (2 of 5 proved main results; 0 of 5 in Lean 4). Conjecture 6 remains unsettled and is excluded from the G11 denominator.

**Concept in Action: Direct Mex Verification for $4 \le k \le 8$**

```
def g_seq(k, N):
    G = [0] * N
    for n in range(k, N):
        s = {G[a] ^ G[n - k - a] for a in range((n - k) // 2 + 1)}
        G[n] = next(v for v in range(len(s) + 1) if v not in s)
    return G
F = [(3,1),(5,1),(9,3),(13,5),(17,5),(21,7),
     (29,9),(33,11),(41,11),(49,13),(65,19),(69,19)]
for k in range(4, 9):
    G = g_seq(k, 70 * k)
    L = [n for n in range(k, 70 * k) if G[n] == 0]
    assert L[:12] == [a * k - b for a, b in F]
```

**Roadmap.** Section 2 sets up octal notation, the Sprague–Grundy recurrence, and worked examples for $k = 2, 3, 4$ through $k = 12$. Section 3 gives complete written proofs of Proposition 2, Lemma 3, and Theorem 4. Section 4 presents the exact symbolic-interval computation of Theorem 1, independent certificate checking and negative-mutation controls, the companions of Proposition 5, and archived results past block 69. Section 5 displays the generated Reader's Trust Boundary, Section 6 reviews prior work and logged searches (flagging unread sources `[to check]`), and Section 7 documents reproducibility and AI disclosure.

## 2. DEFINITIONS, NOTATION, AND WORKED EXAMPLES

In an impartial normal-play game, both players share the same legal moves from every position and the last player to move wins [1, 9] (`[to check]`). For a fixed integer $k \ge 2$, a single heap (row) of $n \ge 0$ pins in the octal game $.0^{k-1}7$ has legal options consisting of all pairs of heaps $(a, n-k-a)$ with $0 \le a \le n-k$ (dropping empty heaps of size 0). By the Sprague–Grundy theorem [8, 9] (`[to check]`), each single heap of length $n$ has a nonnegative integer *nimber* (Sprague–Grundy value) $G_k(n)$ defined recursively by

$$G_k(n) = 0 \qquad (0 \le n < k), \tag{2}$$

$$G_k(n) = \operatorname{mex}\{G_k(a) \oplus G_k(n-k-a) : 0 \le a \le n-k\} \tag{3}$$

for $n \ge k$, where $\oplus$ denotes bitwise exclusive-or (nim-addition) and $\operatorname{mex}(S) := \min\{v \in \mathbb{Z}_{\ge 0} : v \notin S\}$ is the minimum excluded nonnegative integer. Because $(a, n-k-a)$ and $(n-k-a, a)$ yield the same nim-sum, it suffices in (3) to range over $0 \le a \le \lfloor (n-k)/2 \rfloor$. A disjunctive sum of rows $(n_1, \dots, n_s)$ is a $\mathcal{P}$-position (losing for the player to move) if and only if $G_k(n_1) \oplus \cdots \oplus G_k(n_s) = 0$. In particular, a single row of length $n \ge k$ is a nontrivial losing length precisely when $G_k(n) = 0$.

**Octal code convention.** In Guy and Smith's octal encoding [8] (`[to check]`), the $j$-th digit $d_j \in \{0, \dots, 7\}$ after the decimal point specifies which heap counts may remain after removing $j$ tokens: bit 1 permits leaving 0 nonempty heaps, bit 2 permits leaving 1 nonempty heap, and bit 4 permits leaving 2 nonempty heaps. Thus $.0^{k-1}7$ forbids removing fewer than $k$ pins ($d_1 = \cdots = d_{k-1} = 0$) and permits removing $k$ pins leaving 0, 1, or 2 nonempty rows ($d_k = 1 + 2 + 4 = 7$). Removing $k$ pins from a row of $n$ pins is also equivalent to Node-Kayles on the $(k-1)$-th power of the path graph $P_{n-k+1}$ [2] (`[to check]`).

**Worked example: $k = 4$ (.0007).** Evaluating (2)–(3) for $k = 4$ on $0 \le n \le 20$ gives

$$(G_4(n))_{n=0}^{11} = (0,0,0,0,\ 1,1,1,1,\ 2,2,2,\ 0),$$
$$(G_4(n))_{n=12}^{20} = (3,3,3,\ 1,1,1,1,\ 0,\ 4).$$

Rows of 0 to 3 pins admit no move ($G_4 = 0$). From 4 to 7 pins, every move leaves two rows shorter than 4, so only nim-sum 0 is reachable and $G_4 = 1$. From 8, 9, 10 pins, moves reach both $(0, n-4)$ (nim-sum 1) and $(1, n-5)$ (or $(2, n-6)$, nim-sum 0), giving $G_4 = 2$. At $n = 11 = 3k-1$, every move removes 4 pins and leaves two rows summing to 7: $(0,7)$, $(1,6)$, $(2,5)$, or $(3,4)$ (up to mirror symmetry). Each pair contains one row in $[4,7]$ (value 1) and one in $[0,3]$ (value 0), so every option has nim-sum $1 \oplus 0 = 1$ and $G_4(11) = \operatorname{mex}\{1\} = 0$. In `recompute.py`, a plain game-tree search (using neither nimbers nor xor) confirms exact agreement with $G_4(n) = 0$ on $n = 0, \dots, 24$, across all 231 unordered two-row pairs $(x, y)$

with $0 \le x \le y \le 20$, and explicitly checks all 8 first moves from $n = 11$.

**Small-$k$ counter-cases and empirical table.** For $k = 2$ (Dawson's Kayles, .07), the first twelve nontrivial losing lengths are

$$5, 9, 15, 21, 25, 29, 35, 39, 43, 55, 59, 63,$$

matching the first four forms $(5, 9, 15, 21)$ and disagreeing at the fifth entry $(25 \neq 17 \cdot 2 - 5 = 29)$. For $k = 3$ (.007, Treblecross), the first twelve are

$$8, 14, 24, 32, 34, 46, 56, 66, 78, 88, 100, 112,$$

disagreeing at the fourth entry $(32 \neq 13 \cdot 3 - 5 = 34)$. Starting at $k = 4$, all twelve linear forms lock in simultaneously. Table 1 lists the directly computed first twelve losing lengths for $4 \le k \le 12$ together with the thirteenth losing length (whenever one exists up to 20,000), illustrating both the exact linear growth $\Delta n = \alpha_h$ down each column and the irregular arrival of the thirteenth losing length at small $k$.

## 3. WRITTEN PROOFS

This section gives self-contained written proofs of Proposition 2, Lemma 3, and Theorem 4.

**Hand Proof of Proposition 2.**

*Proof of Proposition 2.* Fix $k \ge 2$. We prove part (1) directly from (2)–(3):

- For $0 \le n < k$, no move exists, so $G_k(n) = 0$.
- For $k \le n < 2k$, we have $0 \le n - k < k$. Every move splits $n - k = a + b$ with $0 \le a, b < k$, so $G_k(a) = G_k(b) = 0$. The option set is $\{0\}$, giving $G_k(n) = \operatorname{mex}\{0\} = 1$.
- For $2k \le n \le 3k - 2$, write $m = n - k$, so $k \le m \le 2k - 2$. Because $m \le 2k - 2$, every split $a + b = m$ has at most one summand in $[k, 2k)$ and the other in $[0, k)$ (two summands $\ge k$ would sum to at least $2k$). Hence every option has nim-sum in $\{0 \oplus 0, 0 \oplus 1\} = \{0, 1\}$. Both values actually occur: taking $a = 0, b = m \in [k, 2k)$ yields $0 \oplus 1 = 1$, while taking $a = \lfloor m/2 \rfloor$ and $b = \lceil m/2 \rceil$ satisfies $0 \le a \le b \le k - 1$ (as $m \le 2k - 2$), yielding $0 \oplus 0 = 0$. Therefore $G_k(n) = \operatorname{mex}\{0, 1\} = 2$.
- At $n = 3k - 1$, we have $m = n - k = 2k - 1$. In any split $a + b = 2k - 1$ with $a, b \ge 0$, it is impossible for both $a, b \le k - 1$ (sum $\le 2k - 2$) and impossible for both $a, b \ge k$ (sum $\ge 2k$). Thus exactly one of $a, b$ lies in $[k, 2k - 1]$ (value 1) and the other lies in $[0, k - 1]$ (value 0). Every option has nim-sum $1 \oplus 0 = 1$, so $G_k(3k - 1) = \operatorname{mex}\{1\} = 0$.

Since $G_k(n) \in \{1, 2\}$ for all $k \le n < 3k - 1$, the length $3k - 1$ is the smallest nontrivial losing length for every $k \ge 2$.

For part (2), let $n \ge k$ satisfy $G_k(n) = 0$, and put $m = n - k \ge 0$. If $m$ were even, then the symmetric move $a = m/2$ would leave two equal heaps $(m/2, m/2)$ with nim-sum $G_k(m/2) \oplus G_k(m/2) = 0$, contradicting $0 \notin \{G_k(a) \oplus G_k(m - a)\}$. Hence $m$ is odd, say $m = 2e - 1$ with $e = (m + 1)/2 \ge 1$. Now consider the nearly symmetric move $a = e - 1, m - a = e$: its nim-sum is $G_k(e-1) \oplus G_k(e)$, which must be nonzero since $G_k(n) = 0$. Therefore $G_k(e - 1) \neq G_k(e)$, meaning $e$ is a run breakpoint and $n = 2e - 1 + k$. □

**Symbolic-Interval Algorithm and Proof of Lemma 3.** Notice why affine boundaries arise from (3): if $G_k$ is piecewise constant on runs $I_i = [E_k(e_i), E_k(e_{i+1}))$ with value $v_i$, then a pair of runs $(I_i, I_j)$ contributes the option $v_i \oplus v_j$ precisely to those remainders $m = n - k$ expressible as $x + y$ with $x \in I_i$ and $y \in I_j$. When both intervals are nonempty, the Minkowski sum of integer intervals $[A_1, B_1] + [A_2, B_2]$ is the contiguous integer interval $[A_1 + A_2, B_1 + B_2]$. With half-open intervals $I_i = [E_k(e_i), E_k(e_{i+1}))$, the set of covered remainders $m$ is therefore the half-open interval

$$[\, E_k(e_i + e_j),\ E_k(e_{i+1} + e_{j+1} - (0,1))\,), \tag{4}$$

because the largest integer in $I_i$ is $E_k(e_{i+1}) - 1$ and the largest in $I_j$ is $E_k(e_{j+1}) - 1$, so the maximum sum is $E_k(e_{i+1}) + E_k(e_{j+1}) - 2$, whose successor is $E_k(e_{i+1} + e_{j+1} - (0,1))$.

**The block-by-block symbolic-interval algorithm.** Fix a block horizon $M \ge 1$. Initialize block 0 with the single run $R_0 = [(0,0), (1,0))$ of value $v_0 = 0$ and threshold $T = 0$. For each block $c = 1, 2, \ldots, M - 1$, we compute $G_k(n)$ on $n \in [ck, (c+1)k)$, corresponding to remainders $m = n - k$ in the $m$-block $B_c := [(c - 1, 0), (c, 0))$:

1. Iterate over all unordered pairs $(i, j)$ $(0 \le i \le j)$ of previously constructed runs covering $[0, (c, 0))$, forming the symbolic enter and leave endpoints
$$p_{\text{in}} := e_i + e_j, \qquad p_{\text{out}} := e_{i+1} + e_{j+1} - (0, 1).$$
2. Compare $p_{\text{out}}$ against $(c - 1, 0)$ and $p_{\text{in}}$ against $(c, 0)$ in lexicographic order $\prec$, updating $T \leftarrow \max(T, \tau(\cdot, \cdot))$ for every strict comparison. Discard the pair if $p_{\text{out}} \preceq (c - 1, 0)$ or $(c, 0) \preceq p_{\text{in}}$. Otherwise clip to $B_c$:
$$s := \max_{\prec}(p_{\text{in}}, (c - 1, 0)), \qquad t := \min_{\prec}(p_{\text{out}}, (c, 0)),$$
recording $\tau$ on each strict comparison, and emit sweep events $(s, +1, v_i \oplus v_j)$ and (if $t \prec (c, 0)$) $(t, -1, v_i \oplus v_j)$.
3. Sort all emitted events in $B_c$ by lexicographic position $\prec$, recording $\tau(q_r, q_{r+1})$ between every pair of consecutive distinct event positions (and against $(c - 1, 0)$ and $(c, 0)$). Maintain a reference count per nim-sum value across the sweep; at each distinct position $q$, read off $g = \operatorname{mex}\{v : \text{count}[v] > 0\}$ and record the shifted piece start $q + (1, 0)$ with value $g$.
4. Coalesce adjacent pieces of equal value across block $c$ (and across the boundary $(c, 0)$ with the final run of block $c - 1$), recording $\tau(e_i, e_{i+1})$ between consecutive global breakpoints and against the end sentinel $(c + 1, 0)$.

*Proof of Lemma 3.* Fix any integer $k > T(M)$. We prove by induction on $c = 0, 1, \ldots, M - 1$ that the algorithm's global breakpoints up to $(c + 1, 0)$ evaluate under $E_k$ to a strictly increasing sequence in $[0, (c + 1)k]$ and that the resulting piecewise-constant function agrees identically with $G_k(n)$ on $[0, (c + 1)k)$.

*Base case $c = 0$.* The algorithm produces the single run $[(0, 0), (1, 0))$ of value 0. Since $k > T(M) \ge 0$, we have $0 = E_k(0, 0) < E_k(1, 0) = k$, and $G_k(n) = 0$ on $[0, k)$ by (2).

*Inductive step.* Assume the claim holds on $[0, ck)$ for some $1 \le c \le M - 1$, with runs $I_i = [E_k(e_i), E_k(e_{i+1}))$ $(0 \le i < r_c)$ partitioning $[0, ck)$ into nonempty intervals on which $G_k \equiv$

**Table 1.** First twelve nontrivial losing lengths $n \geq k$ of $.0^{k-1}7$ for $4 \leq k \leq 12$, verified directly in `recompute.py`, alongside the thirteenth losing length $j_{k+13}$ when at most 20,000 ("—" denotes none $\leq 20{,}000$). Every column $h = 1, \ldots, 12$ equals $\alpha_h k - \beta_h$ identically.

| $k$ | $3k-1$ | $5k-1$ | $9k-3$ | $13k-5$ | $17k-5$ | $21k-7$ | $29k-9$ | $33k-11$ | $41k-11$ | $49k-13$ | $65k-19$ | $69k-19$ | $j_{k+13} \leq 20{,}000$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 11 | 19 | 33 | 47 | 63 | 77 | 107 | 121 | 153 | 183 | 241 | 257 | 259 |
| 5 | 14 | 24 | 42 | 60 | 80 | 98 | 136 | 154 | 194 | 232 | 306 | 326 | 328 |
| 6 | 17 | 29 | 51 | 73 | 97 | 119 | 165 | 187 | 235 | 281 | 371 | 395 | 1,689 |
| 7 | 20 | 34 | 60 | 86 | 114 | 140 | 194 | 220 | 276 | 330 | 436 | 464 | 2,900 |
| 8 | 23 | 39 | 69 | 99 | 131 | 161 | 223 | 253 | 317 | 379 | 501 | 533 | — |
| 9 | 26 | 44 | 78 | 112 | 148 | 182 | 252 | 286 | 358 | 428 | 566 | 602 | — |
| 10 | 29 | 49 | 87 | 125 | 165 | 203 | 281 | 319 | 399 | 477 | 631 | 671 | 1,943 |
| 11 | 32 | 54 | 96 | 138 | 182 | 224 | 310 | 352 | 440 | 526 | 696 | 740 | 2,144 |
| 12 | 35 | 59 | 105 | 151 | 199 | 245 | 339 | 385 | 481 | 575 | 761 | 809 | — |

$v_i$. Take any $n \in [ck, (c+1)k)$ and write $m = n - k \in [(c-1)k, ck)$. Every option $G_k(a) \oplus G_k(m-a)$ in (3) has $0 \leq a \leq m < ck$ and $0 \leq m-a \leq m < ck$, so both summands lie strictly inside $[0, ck)$ and hence fall in some uniquely determined runs $I_i$ and $I_j$ ($0 \leq i \leq j < r_c$ up to swapping). Because every run $I_i$ is nonempty by the induction hypothesis, the set of remainders $m$ reachable from the unordered pair $(I_i, I_j)$ is exactly the half-open integer interval (4).

Now compare every symbolic decision in block $c$ with concrete integer arithmetic at $k$. By definition of $T(M)$, every strict lexicographic inequality $p \prec q$ used in Steps 2–4 satisfies $\tau(p,q) \leq T(M) < k$, and hence $E_k(p) < E_k(q)$; equal forms obviously evaluate identically. Therefore:

- A pair $(i,j)$ is discarded in Step 2 if and only if $E_k(p_{\text{out}}) \leq (c-1)k$ or $ck \leq E_k(p_{\text{in}})$, which holds if and only if (4) is disjoint from $[(c-1)k, ck)$.
- For every retained pair, $E_k(s) = \max(E_k(p_{\text{in}}), (c-1)k)$ and $E_k(t) = \min(E_k(p_{\text{out}}), ck)$, so the emitted $+1$ and $-1$ events toggle $v_i \oplus v_j$ at the exact integer endpoints of (4) intersected with $[(c-1)k, ck)$.
- Because every adjacent pair of distinct sorted event positions $q_r \prec q_{r+1}$ satisfies $E_k(q_r) < E_k(q_{r+1})$ (and similarly at the endpoints $(c-1,0)$ and $(c,0)$), the concrete order of events in $[(c-1)k, ck)$ equals the symbolic lexicographic order, with no order inversions and no collapsed distinct positions.

Consequently, for every $m \in [E_k(q_r), E_k(q_{r+1}))$, the set of nim-sum values with positive reference count equals $\{G_k(a) \oplus G_k(m-a) : 0 \leq a \leq m\}$ exactly, so the computed mex equals $G_k(m+k) = G_k(n)$. Shifting each event position by $+(1,0)$ adds $k$, matching $n = m + k$, and coalescing consecutive equal values preserves both the function and strict monotonicity of the evaluated breakpoints through $(c+1,0)$. This completes the induction. □

**Proof of Theorem 4.**

*Proof of Theorem 4.* Let $\varphi(a,b) := a + 2b$, which is additive on $\mathbb{Z}^2$: $\varphi(p+q) = \varphi(p) + \varphi(q)$.

*Part (1).* We induct on the block index $c = 0, 1, \ldots, M-1$ to show that every symbolic breakpoint $e = (a,b) \neq (0,0)$ created through block $c$ satisfies $b \leq 0$ and $\varphi(e) \geq 1$. For $c = 0$, the only non-origin breakpoint is $(1,0)$, which satisfies $b = 0 \leq 0$ and $\varphi(1,0) = 1$.

Now assume the claim holds for all breakpoints preceding block $c \geq 1$ (so every breakpoint $e_i$ has $b(e_i) \leq 0$ and $\varphi(e_i) \geq 0$, with $\varphi(e_i) \geq 1$ whenever $i \geq 1$). In block $c$, every emitted sweep event $p$ is one of:

1. a block boundary $(c-1,0)$ or $(c,0)$, which has $b = 0 \leq 0$ and $\varphi \geq c - 1 \geq 0$;
2. an enter event $p_{\text{in}} = e_i + e_j$ $(i,j \geq 0)$, which satisfies $b(p_{\text{in}}) = b(e_i) + b(e_j) \leq 0$ and $\varphi(p_{\text{in}}) = \varphi(e_i) + \varphi(e_j) \geq 0$; or
3. a leave event $p_{\text{out}} = e_{i+1} + e_{j+1} - (0,1)$. Because $i+1 \geq 1$ and $j+1 \geq 1$, neither $e_{i+1}$ nor $e_{j+1}$ equals $(0,0)$, so the induction hypothesis gives $\varphi(e_{i+1}) \geq 1$ and $\varphi(e_{j+1}) \geq 1$ with both $b$-coordinates nonpositive. Therefore

$$b(p_{\text{out}}) = b(e_{i+1}) + b(e_{j+1}) - 1 \leq -1 \leq 0,$$

and

$$\varphi(p_{\text{out}}) = \varphi(e_{i+1}) + \varphi(e_{j+1}) - 2 \geq 1 + 1 - 2 = 0.$$

Every non-origin breakpoint emitted in block $c$ has the form $q = p + (1,0)$ for some event $p$ above (or is the sentinel $(c+1,0)$, which has $b = 0$ and $\varphi = c + 1 \geq 1$). Hence $b(q) = b(p) \leq 0$ and $\varphi(q) = \varphi(p) + \varphi(1,0) = \varphi(p) + 1 \geq 1$. This proves part (1).

*Part (2).* Consider any strict comparison performed during block $c \in \{1, \ldots, M-1\}$. There are three types of comparisons:

1. *Comparing an event $p = (a,b)$ with a block boundary $(c',0)$ where $c' \in \{c-1, c\}$.* By the argument of part (1), $b \leq 0$ and $a + 2b \geq 0$, so $0 \leq -b \leq a/2$.
   - If $p \prec (c',0)$, then either $a < c'$ or $(a = c'$ and $b < 0)$. Since $b \leq 0$, formula (1) gives $\tau(p, (c',0)) = 0$ (indeed, $ak + b \leq ak < c'k$ for every $k \geq 1$).
   - If $(c',0) \prec p$, then we cannot have $a = c'$ because $b \leq 0$ would imply $(c',b) \preceq (c',0)$. Hence $a \geq c' + 1$, so $a - c' \geq 1$. Using $-b \leq a/2$,

$$\frac{0-b}{a-c'} \leq \frac{a/2}{a-c'} = \frac{1}{2}\left(1 + \frac{c'}{a-c'}\right) \leq \frac{c'+1}{2} \leq \frac{c+1}{2} \leq \frac{M}{2},$$

   because $c' \leq c \leq M-1$. Taking floors yields $\tau((c',0), p) \leq \lfloor M/2 \rfloor$.
2. *Comparing two events $q_1 \prec q_2$ clipped inside $B_c = [(c-1,0), (c,0))$.* Any event $q = (a,b)$ with $(c-1,0) \preceq q \prec (c,0)$ satisfies $b \leq 0$, which leaves only two possibilities: either $q = (c-1,0)$ (if $a = c-1$), or $a = c$ with $b < 0$ and $-b \leq c/2$.

- If both $q_1, q_2$ have $a = c$, then $\tau(q_1, q_2) = 0$ (same $a$-coordinate).
- If $q_1 = (c-1, 0)$ and $q_2 = (c, b)$ with $b < 0$, then

$$\tau(q_1, q_2) = \left\lfloor \frac{0-b}{c-(c-1)} \right\rfloor = -b \le \left\lfloor \frac{c}{2} \right\rfloor \le \left\lfloor \frac{M-1}{2} \right\rfloor \le \left\lfloor \frac{M}{2} \right\rfloor.$$

3. *Comparing consecutive global breakpoints $e_i \prec e_{i+1}$ in block $c$.* Every breakpoint in $[(c, 0), (c+1, 0))$ is a shifted event $q + (1, 0)$, so it is either $(c, 0)$ or $(c+1, b)$ with $b < 0$ and $-b \le c/2$. As in Case 2, any comparison within block $c$ or against $(c+1, 0)$ has threshold at most $\lfloor c/2 \rfloor \le \lfloor (M-1)/2 \rfloor \le \lfloor M/2 \rfloor$; comparing the last breakpoint of block $c-1$ (which has $a \le c$ and $b \le 0$) against the earliest breakpoint of block $c$ (which is either $(c, 0)$, giving $\tau = 0$, or $(c+1, b)$ with $-b \le c/2$, giving $\tau \le \lfloor c/2 \rfloor$) likewise satisfies $\tau \le \lfloor M/2 \rfloor$.

Taking the maximum over all comparisons yields $T(M) \le \lfloor M/2 \rfloor$. Applying Lemma 3 shows that for every integer $k > M/2$, the symbolic pattern equals $G_k(n)$ on $[0, Mk)$.

*Part (3).* Fix any integer $k \ge 2$ and set $M := 2k - 1$. Then $M/2 = k - 1/2 < k$, so $k > M/2 \ge T(M)$. By part (2), the common all-$k$ pattern equals $G_k(n)$ on $[0, (2k-1)k)$. Since $(2k-1)k = 2k^2 - k > k^2$ for every $k \ge 2$, this range strictly contains $[0, k^2)$. □

## ■ 4. EXACT SYMBOLIC COMPUTATION, COMPANIONS, AND BEYOND BLOCK 69

**Computer-Assisted Proof of Theorem 1.** We now assemble the complete proof of Theorem 1, executed deterministically in `recompute.py` (under 16 seconds on a single core; full stdout archived in `recompute.out`).

**All $k \ge 28$ via zero-pruning symbolic sweep ($M = 70$).** Running the zero-pruning all-pairs symbolic-interval algorithm of Section 3.2 through block $M = 70$ completes in 0.15 s and produces exactly 175 maximal constant runs partitioning $[0, 70k)$, with recorded maximum threshold

$$T(70) = 27 \le \lfloor 70/2 \rfloor = 35.$$

The first ten runs $((a, b), v)$ are

$$((0,0),0),\ ((1,0),1),\ ((2,0),2),\ ((3,-1),0),\ ((3,0),3),$$
$$((4,-1),1),\ ((5,-1),0),\ ((5,0),4),\ ((6,-2),3),\ ((7,-2),2).$$

Every non-origin breakpoint satisfies $b \le 0$ and $\varphi(a, b) = a + 2b \ge 1$, with equality $\varphi = 1$ attained exclusively at $(1, 0)$ and $(3, -1)$. Among all 175 runs on $[0, 70k)$, the runs with nimber $v = 0$ are precisely the trivial run $[(0, 0), (1, 0))$ and the twelve singleton runs

$$[(3,-1),(3,0)),\quad [(5,-1),(5,0)),$$
$$[(9,-3),(9,-2)),\quad [(13,-5),(13,-4)),$$
$$[(17,-5),(17,-4)),\quad [(21,-7),(21,-6)),$$
$$[(29,-9),(29,-8)),\quad [(33,-11),(33,-10)),$$
$$[(41,-11),(41,-10)),\quad [(49,-13),(49,-12)),$$
$$[(65,-19),(65,-18)),\quad [(69,-19),(69,-18)).$$

By Lemma 3, for every integer $k \ge 28$, these twelve singletons are the exact and only nontrivial losing lengths of $.0^{k-1}7$ in $[k, 70k)$, hence the first twelve nontrivial losing lengths.

**Finite base $4 \le k \le 27$ and stability diagnostics.** To close the remaining finite cases $4 \le k \le 27$, `recompute.py` runs an independent brute-force integer evaluator directly from (2)–(3) for every $k \in \{2, \dots, 66\}$ (far beyond the required cutoff $k = 27$). The results confirm:

- For $k = 2$ and $k = 3$, the twelve-length law fails (Section 2).
- For all 63 games $4 \le k \le 66$ (in particular every $4 \le k \le 27$), the first twelve losing lengths $n \ge k$ match Flammenkamp's twelve forms identically.
- On $[0, 72k)$ for $4 \le k \le 40$, the only extra losing lengths beyond the twelve are $n = 259$ at $k = 4$ ($64.75k$) and $n = 328$ at $k = 5$ ($65.6k$), both strictly after $69k - 19$ (257 and 326, respectively); every $k \in \{6, \dots, 40\}$ has no thirteenth losing length below $72k$.
- Comparing the concrete runs on $[0, 70k)$ against the 175-run symbolic pattern for $20 \le k \le 60$ gives exact run-by-run identity for every $23 \le k \le 60$ and mismatches at $k = 20, 21, 22$ (178, 177, 176 runs, respectively). Thus the true empirical stability threshold for $M = 70$ is 22, confirming that both our recorded threshold $T(70) = 27$ and the universal bound $\lfloor 70/2 \rfloor = 35$ of Theorem 4 are strictly safe.

Combining the symbolic proof for all $k \ge 28$ with the direct finite check for $4 \le k \le 27$ proves Theorem 1 for every $k \ge 4$.

**Certificate verifier and negative controls.** As a safeguard against generator bugs, `recompute.py` includes a separate piece-by-piece verifier that takes only the finished 175-run list and threshold $T = 27$, re-evaluates every block $c = 1, \dots, 69$ from scratch against the mex rule, and checks every order threshold against $T = 27$. The 175-run certificate passes with 0 errors, and 4/4 injected negative mutations are caught immediately:

1. shifting $(69, -19)$ to $(69, -18)$ (caught as a lexicographic collision at breakpoint 169);
2. flipping $G_k(3k-1)$ from 0 to 1 (caught at block 2);
3. shifting $(5, -1)$ to $(5, -2)$ (caught at block 4);
4. lowering the claimed threshold from $T = 27$ to $T = 21$ (caught at block 57 where an order comparison exceeds 21).

**First-Occurrence Companions.** Besides the twelve losing forms, Table 4 on Flammenkamp's page prints (without proof) affine formulas for the earliest row lengths attaining nimbers 8, 16, 32, and 64 (Proposition 5(1)). Extending our symbolic solver to $M = 180$ (832 runs, $T(180) = 66$) and $M = 360$ (2,681 runs, $T(360) = 130$) records intermediate thresholds $T(21) = 8$, $T(41) = 16$, $T(70) = 27$, $T(88) = 33$, and $T(179) = 66$, each obeying $T(M) \le \lfloor M/2 \rfloor$ (Figure 2, Panel B), and locates all five forms listed in Table 2.

Pairing $M = 180$ with the brute-force sweep over $2 \le k \le 66$ through $179k$ proves the formulas for 8, 16, 32, and 64 unconditionally for every $k \ge 2$ inside `recompute.py`. For

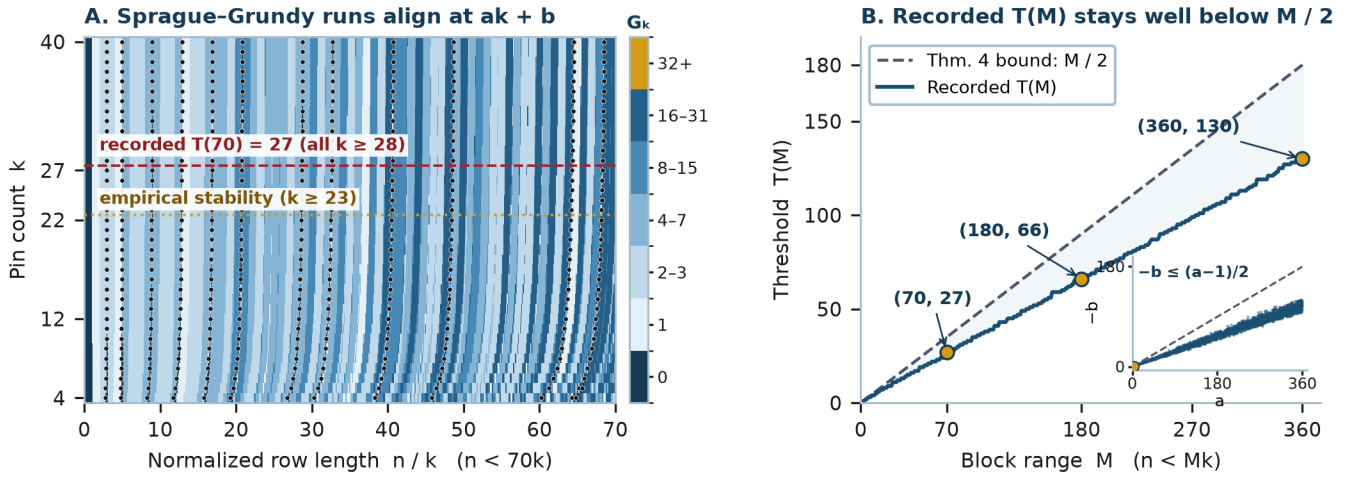



**Figure 2.** Sprague–Grundy run structure and threshold behavior. Panel A: dyadic nimber heatmap of $G_k(n)$ for $4 \le k \le 40$ on $n < 70k$ (plotted at $x = n/k$), with ink markers at all losing lengths $n \ge k$; vertical alignment into 175 constant runs stabilizes empirically at $k \ge 23$ (gold dotted line) and is proved symbolically for all $k \ge 28$ above $T(70) = 27$ (red dashed line). Panel B: recorded comparison threshold $T(M)$ for $1 \le M \le 360$ against the universal written upper bound $M/2$ of Theorem 4, highlighting $(70, 27)$, $(180, 66)$, and $(360, 130)$ ($T(360)/360 = 0.3611 < 0.5$); the inset plots all 2,680 non-origin breakpoints $(a, -b)$ inside the proven wedge $0 \le -b \le (a-1)/2$ (max $-b/a = 0.3846$).

nimber 128 at $358k - 109$, the symbolic sweep to $M = 360$ proves every $k \ge 131$, direct checks in `recompute.py` verify $2 \le k \le 32$ (failures on $2 \le k \le 22$, exact match on $23 \le k \le 32$), and archived independent runs confirm $33 \le k \le 130$ (and up to $k = 215$).

**Multi-Program Agreement and Beyond Block** 69**.** Beyond `recompute.py`, archived logs catalogued in the companion record directory (Section 7) report five independent all-$k$ symbolic implementations (three matching interval-by-interval through $M = 600$ with 6,556 runs and $T(600) = 211$, and two matching through $M = 2{,}000$ with 49,565 runs and $T(2{,}000) = 683$), plus a blind re-review confirming the twelve losing lengths directly up to $k = 100{,}000$ (and single spot checks to $k = 10^{14}$) and catching 118/118 single-piece certificate mutations.

A natural question arising from Theorem 4(3) is whether any losing length ever lies strictly between $69k - 19$ and $k^2$. For $k \le 68$, the open interval $(69k - 19, k^2)$ is empty; at $k = 69$, it contains 18 integers (4,743 through 4,760), none losing. An earlier exploratory conjecture asserting that $(69k - 19, k^2)$ contains no losing length for *any* $k$ survived symbolic sweeps up to $M = 4{,}000$ (127,831 runs, $T(4{,}000) = 1{,}335$), which proved it for every $k \le 4{,}000$, before being overturned by deeper symbolic sweeps. Specifically, separately reviewed archived computations establish:

- *Thirteenth losing form:* $8255k - 2441$ is a singleton losing form of the limiting all-$k$ pattern, with no losing form strictly between $69k - 19$ and $8255k - 2441$. Theorem 4(2) immediately makes it valid for every $k \ge 4{,}129$; tighter order-window audits bring the proved threshold down to $k \ge 265$ (the symbolic pattern itself is exact from $k = 268$, and direct audits verify $k = 265, 266, 267$). At $k = 8{,}255$, this gives

$$n = 8255^2 - 2441 = 68{,}142{,}584 < 68{,}145{,}025 = 8255^2,$$

**Table 2.** First row lengths attaining nimbers $v = 8, 16, 32, 64, 128$ in $.0^{k-1}7$. The first four appear unproved on Flammenkamp's page; all five are verified across all $k$ (Proposition 5(1)).

| $v$ | First row | Exact range | Horizon $M$ | $T(M)$ |
|---|---|---|---|---|
| 8 | $20k - 5$ | $k > 2$ | 21 | 8 |
| 16 | $40k - 10$ | $k > 4$ | 41 | 16 |
| 32 | $87k - 23$ | $k > 7$ | 88 | 33 |
| 64 | $178k - 57$ | $k > 13$ | 179 | 66 |
| 128 | $358k - 109$ | $k > 22$ | 360 | 130 |

independently confirmed by an integer run-length solver at $k = 8{,}255$ (434,856 runs below $8256k = 68{,}153{,}280$). Hence the naive $< k^2$ gap claim holds for every $k \le 8{,}254$ and fails for every $k \ge 8{,}255$.

- *Fourteenth losing form:* Pushing the symbolic sweep to block 20,699 (independently checked by two programs to block 11,301) discovers a fourteenth losing form at $11247k - 3475$, valid for every $k \ge 433$. Consequently, the count $L(k)$ of nontrivial losing lengths below $k^2$ equals 12 for $69 \le k \le 8{,}254$, steps to 13 for $8{,}255 \le k \le 11{,}246$, and steps to 14 for $11{,}247 \le k \le 20{,}699$.
- *Finiteness of Flammenkamp's "simple" list:* Flammenkamp's page also lists games with exactly $k + 12$ losing positions (reading his printed sequence $8, 11, 17, 18, 21, 24, 25, 27, 30, 31, 49, \ldots$ in Table 4 by the zero count $k - 1$). Because every $k \ge 265$ admits at least $k + 13$ losing positions $(0, \ldots, k - 1$, the twelve forms, and $8255k - 2441)$, the set of such simple games is finite (all have $k < 265$).

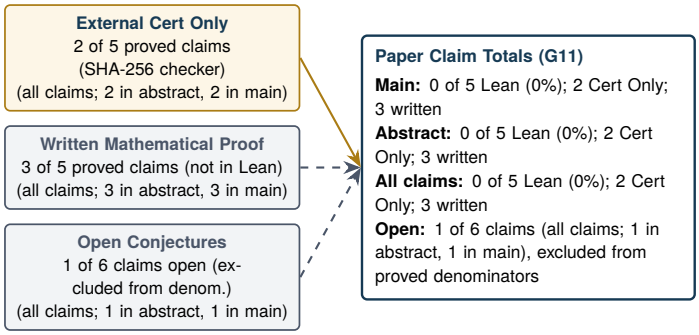
Because these large-block runs ($M > 4{,}000$) require minutes to hours in compiled C, Proposition 5(2) cites them honestly as archived reviewed computations, whereas the full proof of Theorem 1 ($M = 70$) and companions 8, 16, 32, 64 ($M = 180$) replay completely in seconds in `recompute.py`.

## 5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

### What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 3 below.

**Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)**




**Anti-Shortcut Audit Strip — No-Lean Disclosure**

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

**Table 3.** Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Thm. 1: For every k >= 4, the first twelve losing lengths n >= k of .0^{k-1}7 are 3k-1, 5k-1, 9k-3, 13k-5, 17k-5, 21k-7, 29k-9, 33k-11, 41k-11, 49k-13, 65k-19, 69k-19 (with 175 runs below 70k for all k >= 28; fails for k = 2 and k = 3) | Cert Only | recompute.py (08ce95f5be04) [archived; replay pending] | External cert only; fidelity: n/a | No Lean 4 formalization. Combines written soundness (Lem. 3) with deterministic exact symbolic computation to M = 70 (175 runs, recorded threshold T(70) = 27, valid for all k >= 28) plus direct brute-force mex for every 4 <= k <= 27 (extended to k = 66) and 4/4 negative-mutation checks in recompute.py; cross-checked by five independent programs and a blind re-review. |
| Prop. 2: Hand proof that G_k = 0 on [0, k), 1 on [k, 2k), 2 on [2k, 3k-1), and G_k(3k-1) = 0 for every k >= 2, plus Candidate Parity Lemma (every losing n >= k has odd m = n - k and G_k((m-1)/2) != G_k((m+1)/2)) | Written | Written proof | Lean: none; fidelity: n/a | Self-contained elementary written proof (Section 3.1); cross-checked against plain game-tree search at k = 4 and direct mex for k = 4..40 in recompute.py. No Lean 4 formalization claimed. |
| Lem. 3: Soundness of block-by-block symbolic-interval Sprague-Grundy evaluation: if all cross-block lexicographic comparisons have threshold at most T(M), then the symbolic runs equal concrete G_k(n) on [0, Mk) for every integer k > T(M) | Written | Written proof | Lean: none; fidelity: n/a | Self-contained written induction on blocks c = 0..M-1 (Section 3.2); independently validated against brute-force concrete runs at k = 23..60 (M = 70) in recompute.py. No Lean 4 formalization claimed. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Thm. 4: Every non-origin symbolic breakpoint (a, b) satisfies b <= 0 and a + 2b >= 1; consequently T(M) <= floor(M/2) for all M >= 1, so the common all-k pattern equals G_k(n) on [0, (2k-1)k) containing [0, k^2) for every k >= 2 | Written | Written proof | Lean: none; fidelity: n/a | Self-contained written induction via the additive invariant phi(a, b) = a + 2b (Section 3.3); verified numerically through M = 360 (2,681 runs, tight only at (1, 0) and (3, -1)) in recompute.py. No Lean 4 formalization claimed. |
| Prop. 5: Companions 8 (20k-5, k > 2), 16 (40k-10, k > 4), 32 (87k-23, k > 7), 64 (178k-57, k > 13), and 128 (358k-109, k > 22); archived multi-program confirmation of 13th losing length 8255k-2441 (k >= 265) and 14th 11247k-3475 (k >= 433), refuting the below-k^2 gap claim at k = 8255 and proving finiteness of Flammenkamp's simple list (k < 265) | Cert Only | recompute.py (08ce95f5be04) [archived; replay pending] | External cert only; fidelity: n/a | No Lean 4 formalization. Companions 8..64 are proved for all k live in recompute.py (symbolic M = 180, T = 66 + direct brute-force k = 2..66); companion 128 is verified live for k = 2..32 and k >= 131 (M = 360, T = 130) with k = 33..130 and the 13th/14th forms (8255k-2441, 11247k-3475) citing archived reviewed logs in docs/confirmed-strong-and-novel/octal-game-twelve-losing-lengths/. |
| Conj. 6: The limiting all-k symbolic pattern of . 0^{k-1}7 has infinitely many losing forms (equivalently L(k) -> infinity); independently, existence of a human-readable non-computational proof of Thm. 1 remains open | Unsettled | Open conjecture | Lean: none; fidelity: n/a | Explicitly unsettled open conjecture and proof-methodology question (excluded from G11 denominators). |

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 25a909695dbf | Fidelity SHA-256: 3a9de34a79f5.

## 6. PRIOR WORK AND OPEN QUESTIONS

Octal games were introduced by Guy and Smith [8] (`[to check]`) and developed systematically in Berlekamp, Conway, and Guy [1] (`[to check]`) and Siegel [9] (`[to check]`). Achim Flammenkamp maintains the standard computational catalogue *Sprague-Grundy Values of Octal Games* (`https://wwwhomes.uni-bielefeld.de/achim/octal.html`), whose Table 4 tabulates $.0^{k-1}7$ through $k = 32$ and prints the twelve-length conjecture (checked numerically for $3 < k < 32$) alongside the unproved first-occurrence column $20k - 5, 40k - 10, 87k - 23, 178k - 57$. Comparison against the Internet Archive snapshot of 2026-08-10 (page dated 2026-08-09) shows that the twelve-length conjecture, the "simple" list, and the higher first-occurrence columns were added between 2026-08-09 and 2026-11-17. Flammenkamp himself has not been contacted and may already hold an unpublished proof.

In the graph-game literature, Brown et al. [2] (`[to check]`) observe that Node-Kayles on the square $(P_n)^2$ and cube $(P_n)^3$ of a path correspond to the octal games .11337 and .1113337 (shifted versions of .007 and .0007; listed as single-game nimber tables in OEIS `A071426` and `A071441`, alongside `A395248` for .007), remark that higher path powers could be explored, and then turn to a variation without deriving a general-$k$ law. Related Node-Kayles and taking-and-merging games on paths appear in Guignard and Sopena [7] (`[to check]`) and Duchêne, Marsault, Parreau, and Rigo [4] (`[to check]`). Methodologically, proving Sprague–Grundy regularities uniformly in an integer parameter parallels Grossmann and Nowakowski's ruler regularity for hexadecimal games [6] (`[to check]`), Dailly, Duchêne, Larsson, and Paris on partition games [3] (`[to check]`), and Ellis and Thanatipanonda on CUT [5] (`[to check]`). Guy and Nowakowski's *Unsolved Problems in Combinatorial Games* (GONC4 2015 and GONC6 2025, Section A) contain no statement on the family $.0^{k-1}7$.

**Prior-work searches logged.** Five gated literature-search logs (all passing `litsearch.py check`, exit 0) ran 56, 33, 20, 44, and 55 structured queries across OEIS (including term searches on computed losing lengths), arXiv, zbMATH, OpenAlex, Semantic Scholar, and Stack Exchange, inspecting 13, 8, 9, 14, and 13 candidates, supplemented by self-archive searches of Zenodo, DataCite, HAL, OSF Preprints, and GitHub repositories. A keyed CORE sweep on 2026-11-20 (6 requests, 31 records: 3 relatives, 0 matching claims) and a 5-query Google Scholar sweep on 2026-11-22 (28 records inspected across queries 1–4; query 5 returned 0) were also logged. No prior proof of Flammenkamp's twelve losing lengths for all $k \geq 4$ was found in the searches logged.

**Unread and unsearched items (`[to check]`).** The following sources remain unread in full or unsearched and are marked `[to check]` in `references-checked.md`: *Winning Ways*, Vol. 1, Chapter 4 ("Sparse Space Spells Speed", p. 112) [1]; Guy and Smith (1956) [8]; Siegel (2013) [9]; full texts of [2, 3, 4, 5, 6, 7] (inspected via abstracts, zbMATH reviews, or quoted excerpts); Flammenkamp's downloadable per-game data files; the CGSuite database; login-gated GitHub code search; and direct correspondence with Achim Flammenkamp.

**Open questions.** Beyond Conjecture 6 (whether the limiting all-$k$ pattern has infinitely many losing forms, and whether a non-computational hand proof of Theorem 1 exists), three concrete questions remain open:

1. *Fifteenth losing form:* A heuristic near-miss census over candidate points $n = 2e - 1 + k$ (Proposition 2(2)) suggests a fifteenth losing form at or below block $2^{19} = 524{,}288$ (though the same model predicts 5.5 forms in blocks 128–20,699 where only 2 occur, Poisson probability 0.09). Determining whether a fifteenth form exists below $2^{19}$ remains open.
2. *Sharp threshold for* $8255k - 2441$ *and complete simple list:* The thirteenth form $8255k - 2441$ is proved for all $k \geq 265$ (and verified at $k = 250$), so the exact smallest $k_0 \leq 265$ from which it holds permanently—and the complete finite list of Flammenkamp's simple games—remains to be determined.
3. *Asymptotic slope of* $T(M)$*:* Theorem 4(2) proves $T(M) \leq \lfloor M/2 \rfloor$, whereas measured ratios $T(M)/M$ stay near 0.33–0.38 up to $M = 4{,}000$ ($T(4{,}000) = 1{,}335$, $T(360)/360 = 0.3611$). Does $\lim_{M\to\infty} T(M)/M$ exist and stay strictly below $1/2$?

## 7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, audited receipts, and the standalone exact verifier live in `papers/octal-game-twelve-losing-lengths/draft1/`. Running `python3 papers/octal-game-twelve-losing-lengths/draft1/recompute.py` finishes in under 20 seconds and deterministically replays: the zero-pruning $M = 70$ symbolic sweep (175 runs, $T(70) = 27$), block-indexed sweeps to $M = 180$ (832 runs, $T = 66$) and $M = 360$ (2,681 runs, $T = 130$), the independent piece-by-piece certificate verifier and 4/4 negative-mutation controls, the $k = 4$ game-tree checks (231 two-row pairs), direct brute-force evaluation for all $2 \leq k \leq 66$ (covering $4 \leq k \leq 27$, $k = 2, 3$ failures, empirical stability $k \geq 23$, and all-$k$ companions $8, 16, 32, 64$), the small-$k$ thirteenth-length table ($4 \leq k \leq 12$), and the Section 1 Python snippet. Archived multi-implementation logs (including sweeps through block 20,699 and the $8255k - 2441$ / $11247k - 3475$ verifications) are catalogued in `docs/confirmed-strong-and-novel/octal-game-twelve-losing-lengths/`. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes discovered the symbolic-interval proof, derived the threshold bound $T(M) \leq \lfloor M/2 \rfloor$, executed the independent multi-program checks and verification scripts, and drafted this manuscript. No human mathematician has yet refereed the proof.

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