MATHDISCOVERIES
Mathematical Preprint

Working Draft

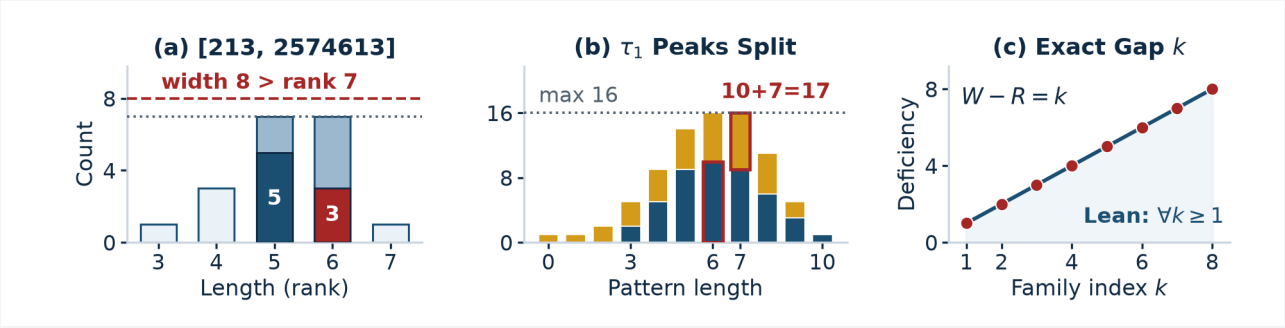
Non-Sperner Intervals and Unbounded Exact Deficiency in the Classical Permutation Pattern Poset

Tyler B. Hughes*,†

†Independent Researcher, United States

Supporting Information: Reproducible exact verifier (`recompute.py`), Lean 4 library (`lean/PatternPoset/`), and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: McNamara and Steingrímsson (2015) asked whether every closed interval in the classical permutation pattern poset $\mathcal{P}$ is Sperner (and if so, strongly Sperner). We refute both parts in Lean 4 (byte-for-byte identical to `PatternPoset`). First, the 19-element interval $[213, 2574613]$ has rank sizes $(1,3,7,7,1)$, largest rank 7, and exact width 8 (8-antichain and 8-chain cover). Second, for every $k \geq 1$, the length-$(6k+4)$ two-leader permutation τ_k has principal downset $D(\tau_k) = [\emptyset, \tau_k]$ splitting into three pattern classes peaking at ranks $3k+3$ and $4k+3$; via a 3D symmetric chain decomposition of $[0, 2k+1]^2 \times [0, 2k]$ ($3k^2+5k+2$ chains), we prove in Lean 4 exact width $3k^2+9k+5$, largest rank $3k^2+8k+5$, and Sperner deficiency k. An external census finds 0/16,912 non-Sperner intervals for $|\tau| \leq 6$, 16/189,524 at $|\tau| = 7$, and 766/2,767,428 at $|\tau| = 8$; rank-unimodality remains open.

Verification Scope (G11): WARN: Human referee: pending
Main results: 8 of 9 in Lean Kernel (89%) · 1 Cert Only · 1 open (not counted)
Abstract claims: 8 of 9 in Lean Kernel (89%) · 1 Cert Only · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 05A05, 06A07 **Keywords:** permutation patterns, pattern poset, Sperner property, antichains, symmetric chain decomposition, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{P} = \bigsqcup_{n \geq 0} S_n$ be the classical permutation pattern poset, ranked by length $|\pi| = n$, where $\sigma \leq \tau$ holds when τ contains a subsequence in the same relative value order as σ (with the unique empty permutation $\emptyset \in S_0$ as minimum element). For any finite subset $S \subseteq \mathcal{P}$, write $R_r(S) = \{\pi \in S : |\pi| = r\}$ for the rank-r level, $R(S) = \max_r |R_r(S)|$ for the largest rank size, and $w(S)$ for the width (maximum cardinality of an antichain $A \subseteq S$). Every rank level is an antichain, so $w(S) \geq R(S)$; the poset S is *Sperner* when $w(S) = R(S)$, and its *Sperner deficiency* is $w(S) - R(S)$.

In their study of the topology of $\mathcal{P}$, McNamara and Steingrímsson [4] posed Question 9.5: *"Are all intervals in $\mathcal{P}$ Sperner? If so, are they strongly Sperner?"* Shortly afterward, Elizalde and McNamara [3] proved that every interval of the *consecutive* pattern poset is strongly Sperner while noting that both parts remained open for the classical order $\mathcal{P}$. We settle Question 9.5 unconditionally in the negative, first with a minimal-length counterexample checkable by hand and then with an infinite principal-downset family attaining every positive integer deficiency $k \geq 1$.

Corollary 1 (Refutation of McNamara–Steingrímsson Question 9.5). *There exist permutations $\sigma \leq \tau$ in the classical permutation pattern poset $\mathcal{P}$ such that the closed interval $[\sigma, \tau]$ is not Sperner.*

Explicitly, $\sigma = 213$ and $\tau = 2574613$ witness Corollary 1 (and because every strongly Sperner poset is Sperner, this refutes both parts of Question 9.5). The standalone Python check below constructs $I = [213, 2574613]$ directly by subsequence standardization and verifies its 19 elements, rank vector $(1,3,7,7,1)$, and 8-element mixed-rank antichain verbatim (executed by `recompute.py`).

Instant Counterexample Check (11 lines, verified by recompute.py)

```
from itertools import combinations
std = lambda s: tuple(sorted(s).index(v)+1 for v in s)
pats = lambda t: {std(c) for r in range(len(t)+1)
                  for c in combinations(t, r)}
sig, tau = (2,1,3), (2,5,7,4,6,1,3)
I = {p for p in pats(tau) if sig in pats(p)}
A = {(1,3,5,2,4),(1,4,3,5,2),(2,4,3,5,1),(3,5,2,4,1),
     (4,3,5,1,2),(2,4,6,5,1,3),(2,5,6,4,1,3),(2,6,4,5,1,3)}
assert len(I) == 19 and A <= I and len(A) == 8
assert [sum(len(p)==r for p in I) for r in range(3,8)] == [1,3,7,7,1]
assert all(x not in pats(y) for x in A for y in A if x != y)
```

To show that Sperner failure persists even for full principal downsets $D(\tau) = [\emptyset, \tau]$ with arbitrarily large deficiency, define for each integer $k \geq 1$ the length-$(6k+4)$ two-leader permutation

$$\tau_k = (4k+3,\, 2k+1,\, t_1, \ldots, t_{6k+2}),$$

where $t_1 > \cdots > t_{6k+2}$ are the remaining entries of $\{1, \ldots, 6k+4\}$ in strictly decreasing order. At $k = 1$, this is the 10-entry permutation $\tau_1 = 7\,3\,10\,9\,8\,6\,5\,4\,2\,1$. Every proved mathematical result in this manuscript (Corollary 1, Propositions 2, 5, 6, 7, and Theorems 3, 4, 8) is proved unconditionally inside the pure Lean 4 kernel (`lean/PatternPoset/`, Section 5), with zero `sorry` and zero `native_decide`.

2. THE SEVEN-ENTRY COUNTEREXAMPLE [213, 2574613]

Proposition 2 (Rank Profile of [213, 2574613]). *For $\sigma_7 = 213$ and $\tau_7 = 2574613$, the closed interval $I = [\sigma_7, \tau_7]$ has rank-level sizes $|R_3(I)| = 1$, $|R_4(I)| = 3$, $|R_5(I)| = 7$, $|R_6(I)| = 7$, $|R_7(I)| = 1$, and $|R_r(I)| = 0$ for every $r \notin \{3,4,5,6,7\}$.*

Proof. Every pattern of $\tau_7 = 2574613$ arises by standardizing one of the $2^7 = 128$ subsequences of τ_7. Filtering to those subsequences containing $\sigma_7 = 213$ yields exactly 19 distinct permutations:

- Rank 3 (1): 213.
- Rank 4 (3): 1324, 2413, 3241.
- Rank 5 (7): 13524, 14352, 24351, 24513, 25413, 35241, 43512.
- Rank 6 (7): 146352, 246351, 246513, 254613, 256413, 264513, 463512.
- Rank 7 (1): 2574613.

No subsequence of τ_7 has length exceeding 7, and every member contains 213 of length 3, so all remaining ranks are empty. In particular, $|I| = 19$ and $R(I) = \max(1,3,7,7,1) = 7$. □

Theorem 3 (Exact Width of [213, 2574613]). *The closed interval $I = [213, 2574613]$ has exact width $w(I) = 8$.*

Proof. Let $A = A_5 \cup A_6$ with $A_5 = \{13524, 14352, 24351, 35241, 43512\}$ and $A_6 = \{246513, 256413, 264513\}$, so $|A| = 8$. No two elements of equal length are comparable. Deleting any single entry from 246513 produces $\{35412, 25413, 24513, 13542, 23541\}$; of these, only 24513 and 25413 contain 213, and neither lies in A_5. The identical check for 256413 and 264513 confirms that all six downward cover edges from A_6 inside I land in $\{24513, 25413\}$ (Figure 1a). Therefore A is an antichain, giving $w(I) \geq 8$.

For the upper bound $w(I) \leq 8$, the eight chains

$$C_1 = \{2413 < 13524\},$$
$$C_2 = \{3241 < 14352 < 146352\},$$
$$C_3 = \{35241 < 246351\},$$
$$C_4 = \{25413 < 246513\},$$
$$C_5 = \{213 < 1324 < 24351 < 254613\},$$
$$C_6 = \{24513 < 256413\},$$
$$C_7 = \{264513\},$$
$$C_8 = \{43512 < 463512 < 2574613\}$$

cover all 19 elements of I (Figure 1b). By the pigeonhole principle (the elementary direction of Dilworth's theorem [2]), any antichain meets each chain at most once, so $w(I) \leq 8$. Thus $w(I) = 8 > 7 = R(I)$, proving Theorem 3 and Corollary 1. □

3. TWO-LEADER STRUCTURE AND 3D BOX SYMMETRIC CHAINS

We now analyze the principal downset $D(\tau_k) = [\emptyset, \tau_k]$ for arbitrary $k \geq 1$. Write $\tau_k = (a, b, t_1, \ldots, t_{6k+2})$ with $a = 4k+3$, $b = 2k+1$, and $t_1 > \cdots > t_{6k+2}$. The decreasing tail splits into three horizontal value bands (Figure 2): $2k+1$ entries above a, $2k+1$ entries strictly between b and a, and $2k$ entries below b.

A permutation π has a *valley* if there exist indices $i < j < m$ with $\pi(j) < \pi(i)$ and $\pi(j) < \pi(m)$ (equivalently, π contains 213 or 312). Because order embeddings preserve both index order and value comparisons, having a valley is inherited upward: if $\sigma \leq \tau$ and σ has a valley, then τ has a valley.

Define three canonical families of permutations:

1. $P(x,y,z)$ (length $x+y+z+2$): first entry $y+z+2$, second entry $z+1$, followed by x entries above $y+z+2$, y entries between $z+1$ and $y+z+2$, and z entries below $z+1$, all strictly decreasing.
2. $Q(p,q)$ (length $p+q+1$): first entry $q+1$, followed by p entries above $q+1$ and q entries below $q+1$, all strictly decreasing.
3. $D(\ell) = (\ell, \ell-1, \ldots, 1)$ (length ℓ, with $D(0) = \emptyset$): strictly decreasing.

Theorem 4 (Classification of $D(\tau_k)$). *For every integer $k \geq 0$, every pattern $\pi \leq \tau_k$ belongs to one of three canonical families:*

1. *valley patterns $P(x,y,z)$ with $0 \leq x, y \leq 2k+1$, $0 \leq z \leq 2k$, and $x + y \geq 1$;*
2. *one-leader valley-free patterns $Q(p,q)$ with $p \geq 1$ and either $(p \leq 2k+1, q \leq 4k+1)$ or $(p \leq 4k+2, q \leq 2k)$;*
3. *strictly decreasing patterns $D(\ell)$ with $0 \leq \ell \leq 6k+2$.*

Moreover, $P(x,y,z)$ has a valley whenever $x + y \geq 1$, whereas every $Q(p,q)$ and $D(\ell)$ is valley-free, so the three families are pairwise disjoint and no valley pattern is contained in any valley-free pattern (verified by `patP_hasValley`, `patQ_not_hasValley`, `patD_not_hasValley`, and `hasValley_mono` in `lean/PatternPoset/`).

Proof. A subsequence of τ_k is determined up to relative order by which leaders it keeps and how many tail entries it selects from each of the three bands. Keeping both leaders a, b

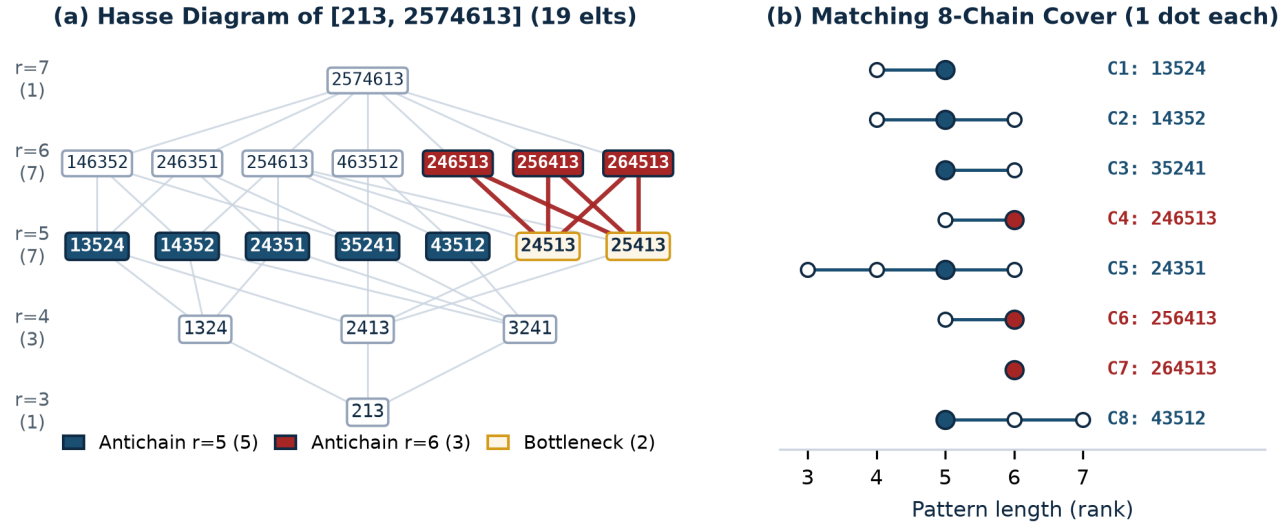


Figure 1. Anatomy of the minimal-length counterexample $I = [213, 2574613]$ (Propositions 2, Theorem 3, Corollary 1). (a) Complete 19-element Hasse diagram across ranks 3–7 with level sizes $(1, 3, 7, 7, 1)$. The three dark-crimson rank-6 permutations $246513, 256413, 264513$ cover only two rank-5 members ($24513, 25413$, highlighted in gold; all six downward cover edges shown in crimson). Replacing those two rank-5 nodes by the three rank-6 nodes yields the 8-element antichain (5 dark-navy + 3 crimson). (b) Explicit cover of all 19 elements by 8 chains $C_1, \ldots, C_8$, each meeting the antichain at a unique filled dot and certifying $w(I) \le 8$.

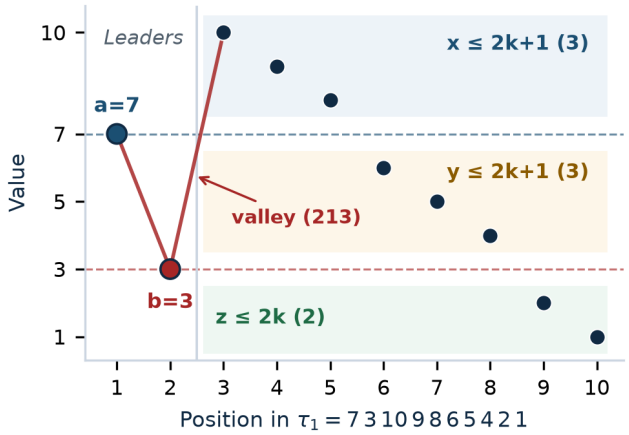


Figure 2. Geometric structure of $\tau_1 = 7\,3\,10\,9\,8\,6\,5\,4\,2\,1$ ($k = 1, n = 10$). Two leading entries $a = 4k + 3 = 7$ and $b = 2k + 1 = 3$ partition the strictly decreasing tail into three bands of sizes $2k + 1$, $2k + 1$, and $2k$. Retaining both leaders and at least one entry from the top two bands creates a valley $P(x, y, z)$ ($x + y \ge 1$); deleting either leader leaves a valley-free pattern $Q(p, q)$ or $D(\ell)$.

with (x, y, z) tail entries in the three bands yields $P(x, y, z)$; if $x + y = 0$, then $P(0, 0, z) = D(z + 2)$ is strictly decreasing, whereas $x + y \ge 1$ puts the first tail entry above b, creating the valley (a, b, t) at indices $(1, 2, 3)$. Keeping only leader a with $p \le 2k + 1$ entries above a and $q \le 4k + 1$ entries below a yields $Q(p, q)$ (or $D(q + 1)$ if $p = 0$). Keeping only leader b with $p \le 4k + 2$ entries above b and $q \le 2k$ entries below b yields $Q(p, q)$ (or $D(q + 1)$ if $p = 0$). Keeping neither leader yields $D(\ell)$ with $\ell \le 6k + 2$. Injectivity on each family follows from length and the first two entries, and disjointness follows from the valley invariant together with the leading entry of $Q(p, q)$ for $p \ge 1$. □

To bound both the rank levels and the width of the valley family, consider the discrete 3D box

$$\mathcal{B}_k = \{0, \ldots, 2k + 1\} \times \{0, \ldots, 2k + 1\} \times \{0, \ldots, 2k\},$$

with slices $B_k(s) = \{(x, y, z) \in \mathcal{B}_k : x + y + z = s\}$.

Proposition 5 (3D Box Slice Formula). *For every $k \ge 0$ and every $0 \le t \le k$, the slice $B_k(3k + 1 + t)$ has exact cardinality*

$$|B_k(3k + 1 + t)| = 3k^2 + 5k + 2 - t^2.$$

Setting $t = 0$ gives $|B_k(3k + 1)| = 3k^2 + 5k + 2$, and the iterated hook decomposition of $\mathcal{B}_k$ partitions $\mathcal{B}_k$ into $3k^2 + 5k + 2$ coordinatewise-increasing symmetric chains crossing $B_k(3k + 1)$ once each, proving that $B_k(3k + 1)$ is the unique middle-slice maximum (`boxSlice_mid_card`, `boxSlice_card_le_mid`, `boxId_image_card_eq` in `BoxChains.lean`).

Proof. Recall the classical symmetric chain decomposition of a product of chains [1]. A 2D rectangle $\{0, \ldots, A\} \times \{0, \ldots, B\}$ decomposes into $\min(A, B) + 1$ nested "hook" chains centered at $(A + B)/2$. Applying this hook rule first to $\{0, \ldots, 2k + 1\}^2$ yields $2k + 2$ symmetric chains of lengths $4k + 3, 4k + 1, \ldots, 1$; taking the product of each chain with $\{0, \ldots, 2k\}$ and applying the hook rule a second time partitions $\mathcal{B}_k$ into symmetric chains centered at $s = 3k + 1$, each intersecting $B_k(3k + 1)$ exactly once.

To evaluate $|B_k(3k + 1 + t)|$ for $0 \le t \le k$, sum over $z \in \{0, \ldots, 2k\}$ the number of pairs $(x, y) \in \{0, \ldots, 2k + 1\}^2$ with $x + y = 3k + 1 + t - z$: this fiber size equals $k + 2 - t + z$ for $0 \le z < k + t$ and $3k + 2 + t - z$ for $k + t \le z \le 2k$. Summing the two arithmetic progressions yields

$$\sum_{z=0}^{k+t-1} (k + 2 - t + z) + \sum_{u=0}^{k-t} (2k + 2 - u) = 3k^2 + 5k + 2 - t^2. \quad \square$$

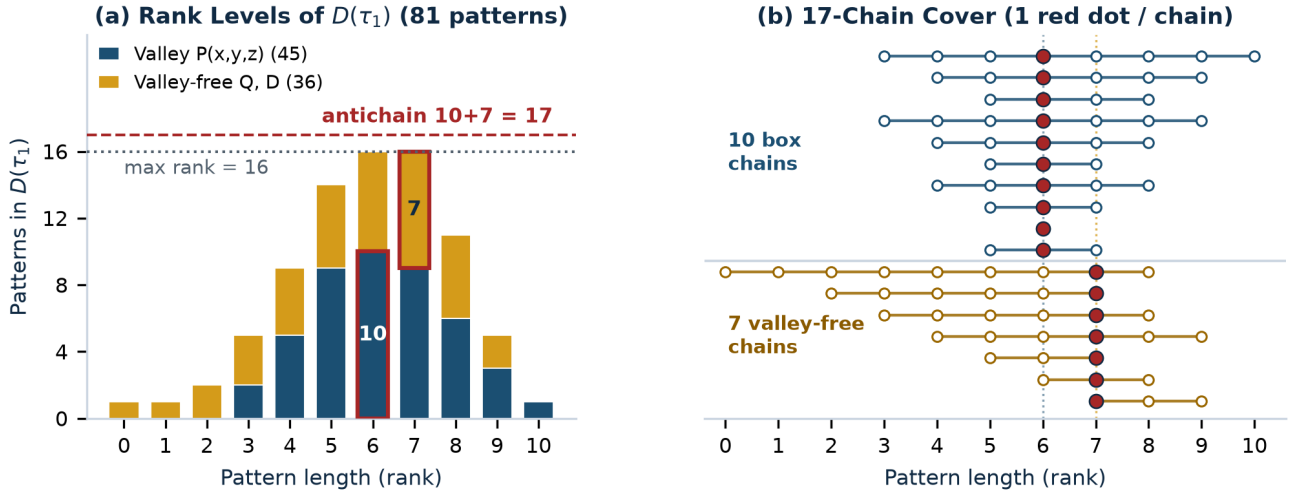


Figure 3. Exact separation of peaks and matching chain cover for $k = 1$ ($\tau_1 = 7\,3\,10\,9\,8\,6\,5\,4\,2\,1$, 81 patterns). (a) Every rank level split into valley patterns $P(x,y,z)$ (navy, 45 total) and valley-free patterns Q,D (gold, 36 total); no single rank exceeds 16, whereas combining the 10 valley patterns of length 6 with the 7 valley-free patterns of length 7 gives the 17-element antichain. (b) The matching 17-chain cover (10 symmetric 3D-box chains plus 7 valley-free chains), with each chain intersecting the antichain at exactly one crimson dot.

4. EXACT SPERNER DEFICIENCY K AND MINIMALITY CENSUS

Proposition 6 (Exact Width of $D(\tau_k)$). *For every integer $k \geq 1$, the principal downset $D(\tau_k)$ has exact width $w(D(\tau_k)) = 3k^2 + 9k + 5$.*

Proof. Lower bound. Let $A_V = \{P(x,y,z) : (x,y,z) \in B_k(3k+1)\}$ be the middle slice of valley patterns (all at length $3k+3$; note $x+y = 3k+1-z \geq k+1 \geq 2$). By Proposition 5, $|A_V| = 3k^2+5k+2$. Let $A_{VF} = \{D(4k+3)\} \cup \{Q(p,4k+2-p) : 1 \leq p \leq 4k+2\}$ be all valley-free patterns of length $4k+3$, so $|A_{VF}| = 4k+3$. Both A_V and A_{VF} are single-rank antichains; since $3k+3 < 4k+3$ and no valley pattern lies below a valley-free pattern (Theorem 4), $A_V \cup A_{VF}$ is an antichain of cardinality $3k^2+9k+5$, giving $w(D(\tau_k)) \geq 3k^2+9k+5$.

Upper bound. Mapping the $3k^2+5k+2$ coordinatewise-increasing symmetric chains of Proposition 5 through P (dropping the three valley-free boundary points $x = y = 0$) covers every valley pattern in $D(\tau_k)$. For the valley-free patterns, take 1 decreasing chain $\{D(\ell) : 0 \leq \ell \leq 6k+2\}$, $2k+1$ fixed-p chains $\{Q(p,q) : 0 \leq q \leq 4k+1\}$ $(1 \leq p \leq 2k+1)$, and $2k+1$ fixed-q chains $\{Q(p,q) : 2k+2 \leq p \leq 4k+2\}$ $(0 \leq q \leq 2k)$. Together these $3k^2+9k+5$ chains cover $D(\tau_k)$ (Figure 3), proving $w(D(\tau_k)) \leq 3k^2+9k+5$. □

Proposition 7 (Exact Largest Rank of $D(\tau_k)$). *For every integer $k \geq 1$, the principal downset $D(\tau_k)$ has exact largest rank $R(D(\tau_k)) = 3k^2+8k+5$.*

Proof. At any length $\ell \geq 2$, Theorem 4 gives $|R_\ell(D(\tau_k))| \leq |B_k(\ell-2)| + \min(\ell, 4k+3)$ (and $|R_0| = |R_1| = 1$). Split into three regimes:

1. $\ell \leq 3k+3$: $|B_k(\ell-2)| \leq 3k^2+5k+2$, so $|R_\ell| \leq (3k^2+5k+2) + (3k+3) = 3k^2+8k+5$.
2. $3k+4 \leq \ell \leq 4k+3$: write $\ell = 3k+3+t$ with $1 \leq t \leq k$. By Proposition 5, $|R_\ell| \leq (3k^2+5k+2-t^2) + (3k+3+t) = 3k^2+8k+5-(t^2-t) \leq 3k^2+8k+5$.
3. $\ell > 4k+3$: monotonicity of $|B_k(s)|$ above the center gives $|B_k(\ell-2)| \leq |B_k(4k+1)| = 2k^2+5k+2$, so $|R_\ell| \leq (2k^2+5k+2)+(4k+3) = 3k^2+8k+5-(k^2-k) \leq 3k^2+8k+5$.

At $\ell = 3k+3$, the disjoint union of A_V $(3k^2+5k+2)$, $D(3k+3)$ (1), and $\{Q(p,3k+2-p) : 1 \leq p \leq 3k+2\}$ $(3k+2)$ achieves equality. □

Theorem 8 (Unbounded Exact Sperner Deficiency). *For every integer $k \geq 1$, the principal downset $D(\tau_k) = [\emptyset, \tau_k]$ of the length-$(6k+4)$ permutation τ_k satisfies*

$$w(D(\tau_k)) = R(D(\tau_k)) + k.$$

Proof. Immediate from Propositions 6 and 7: $(3k^2+9k+5) - (3k^2+8k+5) = k \geq 1$; in particular, $D(\tau_k)$ has exact Sperner deficiency k and is never Sperner (`family_not_sperner`). □

Table 1 tabulates the verified values of $|D(\tau_k)|$, $R(D(\tau_k))$, and $w(D(\tau_k))$ for $k = 1, \ldots, 8$ (rechecked independently by Dilworth bipartite matching in `recompute.py`).

Table 1. Exact counts for the two-leader downsets $D(\tau_k)$ $(k = 1, \ldots, 8)$, matching Propositions 6–7, Theorem 8, and independent bipartite matching.

k	$n = 6k+4$	$\lvert D(\tau_k)\rvert$	$R(D(\tau_k))$	$w(D(\tau_k))$	Gap k
1	10	81	16	17	1
2	16	265	33	35	2
3	22	609	56	59	3
4	28	1,161	85	89	4
5	34	1,969	120	125	5
6	40	3,081	161	167	6
7	46	4,545	208	215	7
8	52	6,409	261	269	8

Exhaustive minimality census. While Corollary 1 and Theorem 3 prove unconditionally that $[213, 2574613]$ fails at $|\tau| = 7$, proving that *no shorter* interval fails requires scanning all finite intervals with $|\tau| \leq 6$. This finite census sits outside Lean (see Section 5 for the concrete barrier).

Proposition 9 (Exhaustive Interval Minimality Census through $|\tau| \leq 8$ — Cert Only). *Exhaustive bipartite-matching enumeration over all closed intervals $[\sigma, \tau]$ in $\mathcal{D}$ yields:*

1. 0 *non-Sperner intervals across all* $16{,}912$ *intervals with* $1 \leq |\tau| \leq 6$ $(2, 6, 28, 178, 1{,}474, 15{,}224$ *respectively), so* $|\tau| = 7$ *is minimum possible;*
2. *exactly* 16 *non-Sperner intervals among all* $189{,}524$ *intervals at* $|\tau| = 7$*, all of rank* 4 *(five levels) with largest rank* 7 *and width* 8*;*
3. *exactly* 766 *non-Sperner intervals among all* $2{,}767{,}428$ *intervals at* $|\tau| = 8$ $(48$ *of rank* 3*,* 90 *of rank* 4*,* 628 *of rank* 5*, maximum deficiency* $2)$*; in particular,* $[14352, 14372685]$ *has rank vector* $(1, 6, 6, 1)$ *and width* 7*, giving minimum failing rank* 3*.*

In contrast to Question 9.5, the neighboring unimodality problem stays completely untouched by our examples (every interval in Proposition 9 and every $D(\tau_k)$ is rank-unimodal).

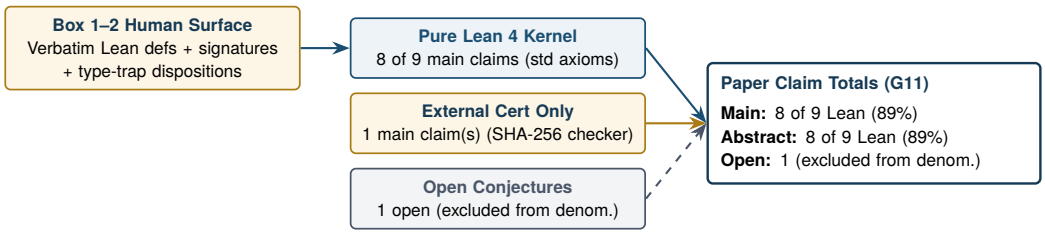
Conjecture 10 (Rank-Unimodality of Pattern Intervals [4, 5]). *Every closed interval $[\sigma, \tau]$ (and in particular every principal downset $D(\tau)$) in the classical permutation pattern poset $\mathcal{D}$ has a unimodal rank sequence.*

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit fcde0f5b1d25 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 2 SHA-256-pinned external artifact(s) plus any claims marked Written or Open in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 99/99 public theorems (source-level approx) | WARN: Fidelity: 8/8 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

PatternPoset.Perm (PatternPoset/Defs.lean:35-37) — Section 1 (finite permutations S_n)

```
structure Perm where
  len : ℕ
  perm : Equiv.Perm (Fin len)
```

PatternPoset.Contains (PatternPoset/Defs.lean:103-105) — Section 1 (classical pattern containment <=)

```
def Contains (σ τ : Perm) : Prop :=
  ∃ f : Fin σ.len ↪o Fin τ.len,
    ∀ i j : Fin σ.len, σ.perm i < σ.perm j ↔ τ.perm (f i) < τ.perm (f j)
```

PatternPoset.IsAntichain (PatternPoset/Defs.lean:167-168) — Section 1 (antichain in Perm)

```
def IsAntichain (A : Finset Perm) : Prop :=
  ∀ x ∈ A, ∀ y ∈ A, x ≤ y → x = y
```

PatternPoset.RankLevel (PatternPoset/Defs.lean:175-176) — Section 1 (rank-r level R_r(S))

```
def RankLevel (S : Finset Perm) (r : ℕ) : Finset Perm :=
  S.filter (fun π => π.len = r)
```

PatternPoset.largestRank (PatternPoset/Defs.lean:185-186) — Section 1 (largest rank size R(S))

```
def largestRank (S : Finset Perm) : ℕ :=
  (S.image Perm.len).sup (fun r => (RankLevel S r).card)
```

PatternPoset.width (PatternPoset/Defs.lean:214-216) — Section 1 (width w(S) via powerset sup)

```
noncomputable def width (S : Finset Perm) : ℕ :=
  letI := Classical.decPred IsAntichain
  (S.powerset.filter IsAntichain).sup Finset.card
```

PatternPoset.IsSperner (PatternPoset/Defs.lean:245-246) — Section 1 (Sperner property w(S) = R(S))

```
def IsSperner (S : Finset Perm) : Prop :=
  width S = largestRank S
```

PatternPoset.Downset (PatternPoset/Defs.lean:386-387) — Section 1 (principal downset D(tau))

```
def Downset (τ : Perm) : Finset Perm :=
  ((Finset.univ : Finset (Fin τ.len)).powerset).image (subseqPerm τ)
```

PatternPoset.Interval (PatternPoset/Defs.lean:409-410) — Section 1 (closed interval [sigma, tau])

```
def Interval (σ τ : Perm) : Finset Perm :=
  (Downset τ).filter (fun π => σ ≤ π)
```

PatternPoset.tau (PatternPoset/Family.lean:78-78) — Section 1 & 3 (two-leader family tau_k)

```
def tau (k : ℕ) : Perm := patP (2 * k + 1) (2 * k + 1) (2 * k)
```

PatternPoset.boxSlice (PatternPoset/BoxChains.lean:85-86) — Section 3 (3D box slice B_k(s))

```
def boxSlice (k s : ℕ) : Finset (ℕ × ℕ × ℕ) :=
  (boxFinset k).filter (fun t => t.1 + t.2.1 + t.2.2 = s)
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): PatternPoset.interval7_card ((Interval σ₇ τ₇).card = 19); PatternPoset.interval7_largestRank (largestRank (Interval σ₇ τ₇) = 7); PatternPoset.anti7_card (anti7.card = 8).

Supporting Semantic Closure Index (non-Box-1 helpers): PatternPoset.boxFinset (PatternPoset/BoxChains.lean), PatternPoset.subseqPerm (PatternPoset/Defs.lean), PatternPoset.patP (PatternPoset/Family.lean), PatternPoset.patQ (PatternPoset/Family.lean), PatternPoset.patD (PatternPoset/Family.lean), PatternPoset.patPVal (PatternPoset/Family.lean), PatternPoset.patPInvVal (PatternPoset/Family.lean), PatternPoset.patPFun (PatternPoset/Family.lean), PatternPoset.patPInv (PatternPoset/Family.lean), PatternPoset.patQVal (PatternPoset/Family.lean), PatternPoset.patQInvVal (PatternPoset/Family.lean), PatternPoset.patQFun (PatternPoset/Family.lean), PatternPoset.patQInv (PatternPoset/Family.lean), PatternPoset.invFin (PatternPoset/SevenPoint.lean), PatternPoset.mkPerm (PatternPoset/SevenPoint.lean), PatternPoset.σ₇ (PatternPoset/SevenPoint.lean), PatternPoset.τ₇ (PatternPoset/SevenPoint.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

PatternPoset.mcnamara_steingrimsson_Q9_5_false (PatternPoset/SevenPoint.lean:593-595) src:43b88524c7c168

```
theorem mcnamara_steingrimsson_Q9_5_false :
    ∃ σ τ : Perm, σ ≤ τ ∧ ¬ IsSperner (Interval σ τ)
```

Assumptions: None beyond Box 1 definitions (pure existential refutation witnessed by [213, 2574613]). Applicability: Unconditional (closed statement; no outer premise witness needed).

PatternPoset.interval7_rank_sizes (PatternPoset/SevenPoint.lean:369-390) src:ad394df6e44c69

```
theorem interval7_rank_sizes :
    (RankLevel (Interval σ₇ τ₇) 3).card = 1 ∧
    (RankLevel (Interval σ₇ τ₇) 4).card = 3 ∧
    (RankLevel (Interval σ₇ τ₇) 5).card = 7 ∧
    (RankLevel (Interval σ₇ τ₇) 6).card = 7 ∧
    (RankLevel (Interval σ₇ τ₇) 7).card = 1 ∧
    ∀ r : ℕ, r ≠ 3 → r ≠ 4 → r ≠ 5 → r ≠ 6 → r ≠ 7 → (RankLevel (Interval σ₇ τ₇)
      ↪ r).card = 0
```

Assumptions: None beyond Box 1 definitions (all 128 subsequences of tau7 checked in-kernel). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

PatternPoset.interval7_width (PatternPoset/SevenPoint.lean:572-583) src:af7f7c62562b77

```
theorem interval7_width : width (Interval σ₇ τ₇) = 8
```

Assumptions: None beyond Box 1 definitions (explicit 8-antichain and 8-chain cover in-kernel). Applicability: Unconditional (closed statement; no outer premise witness needed).

PatternPoset.downset_tau_classify (PatternPoset/Family.lean:548-740) src:1016f6a5caae87

```
theorem downset_tau_classify {k : ℕ} {π : Perm} (h : π ≤ tau k) :
    (∃ x y z : ℕ, x ≤ 2 * k + 1 ∧ y ≤ 2 * k + 1 ∧ z ≤ 2 * k ∧ 1 ≤ x + y ∧ π = patP x
      ↪ y z) ∨
    (∃ p q : ℕ, 1 ≤ p ∧ ((p ≤ 2 * k + 1 ∧ q ≤ 4 * k + 1) ∨ (p ≤ 4 * k + 2 ∧ q ≤ 2 *
      ↪ k)) ∧
      π = patQ p q) ∨
    (∃ l : ℕ, l ≤ 6 * k + 2 ∧ π = patD l)
```

Assumptions: Any integer k >= 0 and any permutation pi <= tau k. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

PatternPoset.boxSlice_card (PatternPoset/BoxChains.lean:332-401) src:39882c8d86e24d

```
theorem boxSlice_card (k t : ℕ) (ht : t ≤ k) :
    (boxSlice k (3 * k + 1 + t)).card = 3 * k ^ 2 + 5 * k + 2 - t ^ 2
```

Assumptions: Any integers k >= 0 and 0 <= t <= k. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

PatternPoset.family_width (PatternPoset/Family.lean:985-997) src:dd21b24e5c54c9

```
theorem family_width (k : ℕ) (hk : 1 ≤ k) :
    width (Downset (tau k)) = 3 * k ^ 2 + 9 * k + 5
```

Assumptions: Any integer k >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

PatternPoset.family_largestRank (PatternPoset/Family.lean:1174-1177) src:e8e90c31f627c0

```
theorem family_largestRank (k : ℕ) (hk : 1 ≤ k) :
    largestRank (Downset (tau k)) = 3 * k ^ 2 + 8 * k + 5
```

Assumptions: Any integer k >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

PatternPoset.family_exact_deficiency (PatternPoset/Family.lean:1182-1185) src:2f9949b0f98d6c

```
theorem family_exact_deficiency (k : ℕ) (hk : 1 ≤ k) :
    width (Downset (tau k)) = largestRank (Downset (tau k)) + k
```

Assumptions: Any integer k >= 1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
`boxFinset`	`finset_range`	`Finset.range`	intended_convention	Half-open range(m+1) yields exact closed integer interval [0, m].
`boxSlice_card`	`nat_sub`	`- t`	intended_convention	Hypothesis t <= k implies t^2 <= k^2 < 3*k^2 + 5*k + 2 (no underflow).
`patD`	`nat_sub`	`- 1`	intended_convention	Fin l is empty at l = 0; for i : Fin l, l >= 1 so l - 1 >= 0.
`patD`	`nat_sub`	`- i.val`	intended_convention	Strict bound i.val < l guarantees i.val <= l - 1 (no Nat underflow).
`patPVal`	`nat_sub`	`- i`	intended_convention	Called via patPFun on i.val < x+y+z+2; branch guards prevent underflow.
`patPInvVal`	`nat_sub`	`- v`	intended_convention	Called via patPInv on v.val < x+y+z+2; branch guards prevent underflow.
`patQVal`	`nat_sub`	`- i`	intended_convention	Called via patQFun on i.val < p+q+1; branch guards prevent underflow.
`patQInvVal`	`nat_sub`	`- v`	intended_convention	Called via patQInv on v.val < p+q+1; branch guards prevent underflow.
`decidableContains`	`local_instance_or_notation`	`decidableContains`	intended_convention	Decidability of <= / Contains proved equivalent to mem_downset_iff.
`decidableLEPerm`	`local_instance_or_notation`	`decidableLEPerm`	intended_convention	Decidability of <= / Contains proved equivalent to mem_downset_iff.

[a]Covers the specified Lean 4 hazard set (`Nat` subtraction/division/mod, totalized real/rational division, `Real.log`/`rpow`, `Finset.range`, `Classical.choose`, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Cor. 1: Refutation of Mc Namara-Steingrimsson Q9.5 (exists non-Sperner interval)	Lean Kernel	`PatternPoset.mcnamara_steingrimsson_Q9_5_false`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (witnessed by [213, 2574613], zero sorry/native_decide).
Prop. 2: Interval [213, 2574613] rank sizes (1, 3, 7, 7, 1) on ranks 3..7 and 0 elsewhere	Lean Kernel	`PatternPoset.interval7_rank_sizes`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (all 128 bitmasks of tau7 checked in-kernel).
Thm. 3: Exact width w([213, 2574613]) = 8 via 8-antichain and 8-chain cover	Lean Kernel	`PatternPoset.interval7_width`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (explicit anti7 and chains7 verified in-kernel).
Thm. 4: Complete classification of D(tau_k) into P(x,y,z), Q(p,q), and D(l)	Lean Kernel	`PatternPoset.downset_tau_classify`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (for every k >= 0 and every pattern pi <= tau_k).
Prop. 5: 3D box slice formula |B_k(3k+1+t)| = 3k^2 + 5k + 2 - t^2 (0 <= t <= k)	Lean Kernel	`PatternPoset.boxSlice_card`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (for all k >= 0 and 0 <= t <= k).
Prop. 6: Exact width w(D(tau_k)) = 3k^2 + 9k + 5 for every k >= 1	Lean Kernel	`PatternPoset.family_width`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (for every integer k >= 1).
Prop. 7: Exact largest rank R(D(tau_k)) = 3k^2 + 8k + 5 for every k >= 1	Lean Kernel	`PatternPoset.family_largestRank`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (for every integer k >= 1).
Thm. 8: Exact Sperner deficiency w(D(tau_k)) = R(D(tau_k)) + k for every k >= 1	Lean Kernel	`PatternPoset.family_exact_deficiency`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (for every integer k >= 1).
Prop. 9: Exhaustive interval census through |tau| <= 8 (0/16,912; 16/189,524; 766/2,767,428)	Cert Only	intervals.py (41392003eb27) [archived; replay pending]; recompute.py (2c4cf4425d20) [archived; replay pending]	External cert only; fidelity: n/a	External exhaustive Python Dilworth census over 2.97M intervals (barrier: scale beyond kernel).
Conj. 10: Rank-unimodality of all classical pattern poset intervals	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open conjecture (Mc Namara-Steingrimsson 2015 / Vatter 2026; excluded from G11).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 36a40bb1c53f | Lean mode: committed_receipt | Closure SHA-256: 83b7fcf67298 | Git anchor: fcde0f5b1d25 | Axioms SHA-256: a22a629db261 | Fidelity SHA-256: a0ced1b39d24.

6. PRIOR WORK AND OPEN QUESTIONS

McNamara and Steingrímsson [4] posed Question 9.5 alongside Conjecture 9.4 (rank-unimodality, verified through $|\tau| \leq 8$), and Elizalde and McNamara [3] proved the consecutive pattern poset strongly Sperner while noting the classical question remained open. Vatter's 2026 problem list [5] records the principal-downset rank-unimodality conjecture (Conjecture 2.1). The symmetric chain decomposition used in Proposition 5 goes back to de Bruijn, van Ebbenhorst Tengbergen, and Kruyswijk [1], and the chain-cover upper bound is the elementary direction of Dilworth's theorem [2].

Our logged prior-work searches across arXiv, CORE, zbMATH, OEIS, Zenodo, DataCite, HAL, and OSF found no prior non-Sperner interval of the classical pattern poset. However, Pouzet–Rosenberg (1986), Burosch–Franke–Roehl (1990), Burosch–Gronau–Laborde–Warnke (1996), print monographs, and Google Scholar (quota-capped during the swarm check) were not exhaustively read, so we make no exhaustive priority claim across unread sources.

Two open problems stand out beyond Conjecture 10: determining the true maximum Sperner deficiency over all S_n (heuristic search already finds deficiency 12 at $n = 16$ and 28 at $n = 19$, far above the linear $k \approx n/6$ of τ_k), and determining whether the ratio $w([\sigma, \tau])/R([\sigma, \tau])$ is uniformly bounded over all intervals of $\mathcal{D}$.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript files, figure generators, and the exact verifier live in `papers/pattern-poset-not-sperner/draft1/`, the Lean 4 library lives in `lean/PatternPoset/` (byte-for-byte identical to the standalone Palomar package at `https://github.com/mathdiscoveries/pattern-poset-not-sperner`), and the archived exhaustive census scripts live in `docs/confirmed-strong-and-novel/pattern-poset-not-sperner/`. Running `python3 papers/pattern-poset-not-sperner/draft1/recompute.py` runs the Section 1 Python box verbatim, verifies the 8-chain cover, checks Theorem 8 by independent bipartite matching for $k = 1, \ldots, 8$, and audits the rank-3 witness [14352, 14372685]. AI research agents built and directed by Tyler B. Hughes discovered the counterexample and family, formalized all eight proved claims in Lean 4, generated the figures, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 23, 2026 **Updated:** November 23, 2026

REFERENCES

(1) de Bruijn, N. G.; van Ebbenhorst Tengbergen, C.; Kruyswijk, D. On the set of divisors of a number. *Nieuw Arch. Wiskunde* **1951**, *23*, 191–193.

(2) Dilworth, R. P. A decomposition theorem for partially ordered sets. *Ann. of Math.* **1950**, *51* (1), 161–166. DOI: 10.2307/1969503.

(3) Elizalde, S.; McNamara, P. R. W. On intervals of the consecutive pattern poset. *arXiv:1508.05963* **2015**.

(4) McNamara, P. R. W.; Steingrímsson, E. On the topology of the permutation pattern poset. *J. Combin. Theory Ser. A* **2015**, *134*, 1–35 (*arXiv:1305.5569*). DOI: 10.1016/j.jcta.2015.01.008.

(5) Vatter, V. Permutation patterns problem list. *arXiv:2602.16355* **2026**.