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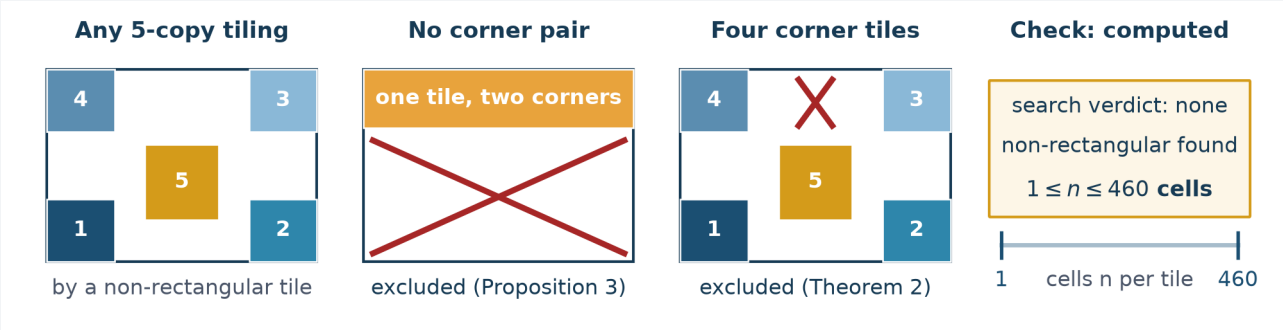
Working Draft

# No Rectangle Is Tiled by Five Congruent Copies of a Non-Rectangular Polyomino

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*Supporting Information:* `recompute.py`, `data/` *(verdict and log digests), and trust-boundary ledgers*

**Disclosure:** AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.




**ABSTRACT:** A polyomino is a finite non-empty edge-connected set of unit grid cells. If congruent copies of a polyomino tile a rectangle, its order is the least number of copies in such a tiling. Stewart and Wormstein proved that no polyomino has order 3, and Winslow's survey asks whether some polyomino has order 5. We prove that no rectangle is tiled by exactly five congruent copies of a polyomino that is not a rectangle (rotations and reflections allowed); hence no polyomino has order 5. In such a tiling no tile could contain two corner cells of the rectangle; we prove this, and then exclude the remaining configuration, with the four corner cells in four distinct tiles, by cases on the corners of the bounding box that the polyomino contains. Separately, the recorded search of a finite reduction reports no such tiling by a polyomino with $n$ cells for $1 \le n \le 460$; no proof here uses it. Lean 4 checks 3 of the 4 main results; the fourth is the computation. AI research agents built and directed by Tyler B. Hughes produced this work. No human mathematician has yet refereed the proof.

**Verification Scope (G11):** WARN: Human referee: pending
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**Full Trust Boundary:** Section 7.

**MSC 2020:** 05B45, 05B50, 52C20 **Keywords:** polyomino, rectangle tiling, congruent copies, order of a polyomino, odd order

## 1. INTRODUCTION AND MAIN RESULT

A *polyomino* is a finite, non-empty set of unit squares of the grid joined through shared edges. Suppose a rectangle is cut into several congruent copies of one polyomino, turned and flipped as needed. If the polyomino is itself a rectangle this is easy to do with any number of copies. Can it be done with exactly five copies of a polyomino that is not a rectangle? We prove that it cannot. The *order* of a polyomino is the least positive number of congruent copies of it that tile some rectangle, when such a tiling exists; Corollary 67 draws the consequence for order 5. Stewart and Wormstein proved that no polyomino has order 3 [1]; Winslow's survey lists whether some polyomino has order 5 as Open Problem 11 in the author's version [3], and Talon's thesis asks the same [6] [to check]. Separately, the recorded search of a finite reduction, a computation and not a proof, reports no such tiling by a polyomino with at most 460 cells; no proof here uses it.

Throughout, a *grid isometry* is a translation composed with one of the eight symmetries of the square, two sets of cells are *congruent* if a grid isometry carries one to the other, and, for integers $a$ and $b$, $R(a,b)$ is the set of cells $(x,y)$ with $0 \le x < a$ and $0 \le y < b$, where $(x,y)$ names the cell by its lower-left corner; it is a rectangle when $a,b \ge 1$ and empty otherwise. A *$k$-copy tiling* of $R(a,b)$ by a polyomino $P$ is a partition of $R(a,b)$ into $k$ sets $T_1, \dots, T_k$, the *tiles*, each congruent to $P$. The smallest rectangle of cells containing a finite non-empty set $S$ is $\mathrm{box}(S)$, and $P$ is *rectangular* if $P = \mathrm{box}(P)$. A polyomino with one or two cells is rectangular, so a non-rectangular one has at least three cells. Every rectangle of cells is a translate of some $R(a,b)$, and translations are grid isometries.

**Theorem 1** (Five copies). *Let $P$ be a polyomino that is not a rectangle. No rectangle is tiled by exactly five congruent copies of $P$ (grid isometries, reflections included): for all integers $a$ and $b$ there is no 5-copy tiling of $R(a,b)$ by $P$.*

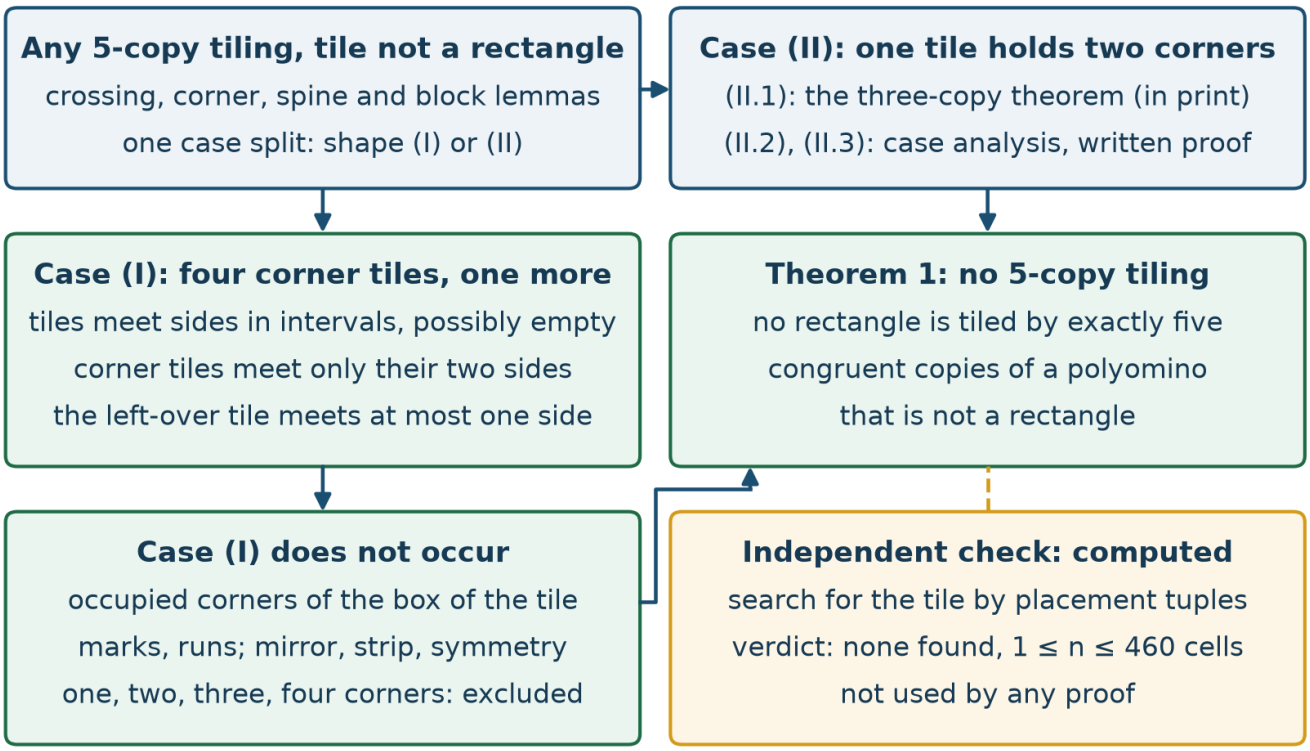



**Figure 1. Discovery and proof architecture.** Where each part of the paper comes from and how it is supported. Lines of the diagram show logical use; the dashed line marks the computation as a check that no proof uses. The written proofs were found and checked by AI research agents, with no human referee; the statements also proved in Lean 4 are listed at the end of this section.

**Theorem 2** (Case (I) does not occur). *Let $P$ be a polyomino that is not a rectangle, and let $a$ and $b$ be integers. There is no 5-copy tiling of $R(a,b)$ by $P$ in which the four corner cells of $R(a,b)$ lie in four distinct tiles.*

**Steps of the proof.** The proof of Theorem 1 first excludes a tile holding two corner cells, and the proof of Theorem 2 uses three facts about the configuration that remains.

**Proposition 3** (Corners). *Let $P$ be a non-rectangular polyomino and let $T_1, \dots, T_5$ be a 5-copy tiling of a rectangle $R(a,b)$ by $P$. Then no tile contains two of the four corner cells of $R(a,b)$. So the four corner cells lie in four distinct tiles.*

**Proposition 4** (Intervals on the sides). *In the setting of Proposition 3, with the four corner cells in four distinct tiles, every tile meets every side of $R(a,b)$ in an interval of cells, possibly empty.*

**Proposition 5** (Corner tiles). *In the setting of Proposition 3, with the four corner cells in four distinct tiles, a tile containing a corner cell of $R(a,b)$ meets only the two sides of $R(a,b)$ at that corner.*

**Proposition 6** (The left-over tile). *In the setting of Proposition 3, with the four corner cells in four distinct tiles, let $T_5$ be the tile containing no corner cell. Then $T_5$ does not meet two different sides of $R(a,b)$. So it meets no side or exactly one.*

Proposition 3 is proved in Section 3, Propositions 4 to 6 in Section 4, and Theorem 2 in Section 5, by cases on the corners of box$(P)$ that belong to $P$. Theorem 1 follows: by Proposition 3 the four corner cells of a 5-copy tiling lie in four distinct tiles, and by Theorem 2 that does not happen. In the Lean library the four propositions are the internal theorems `corners_in_distinct_tiles`, `sides_are_intervals` (stated there for every $k$-copy tiling by a polyomino), `corner_tile_sides` and `free_tile_one_side`.

The next statement reduces the question, one cell count at a time, to a finite search, whose outcome is stated after it. No proof in this paper uses the search. The converse does not hold: reading the outcome of the search as a statement about every tiling uses Theorem 7. Its written proof uses Theorem 16 and Lemmas 14 and 20, and its Lean proof uses Theorem 9 and Proposition 3, all of which the proof of Theorem 1 also uses. So for small tiles the search checks the rest of the argument, Sections 4 and 5, and not the parts it shares.

**Theorem 7** (Reduction to five search families). *Fix $n \geq 1$. Suppose that, over all rectangles and boxes allowed in Section 6, no tuple of the families $I(5)$, $H_1(5)$, $H_2(5)$, $H_1(3)$, $H_2(3)$ has a proper solution that is not a rectangle. Then no rectangle is tiled by exactly 3, and none by exactly 5, congruent copies of a non-rectangular polyomino with $n$ cells.*

**Theorem 8** (Computed, not proved). *The registered verdict of the run of record of the search of Section 6 reports that the run covered every tuple of the families $I(5)$, $H_1(5)$, $H_2(5)$, $H_1(3)$ and $H_2(3)$ for every $n$ with $1 \le n \le 460$, and that no tuple has a proper solution that is not a rectangle.*

Theorem 8 is a computation, not a proof, and no proof in this paper uses it. What the package checks of it is the recorded verdict and the digest list of the logs (Section 6); the logs, the program and the reader are kept with the project, so the coverage itself is not checked here. If the search is correct and the verdict is accurate, Theorem 7 turns it into the statement that no rectangle is tiled by exactly 5, or exactly 3, congruent copies of a non-rectangular polyomino with at most 460 cells, which agrees with Theorem 1; the 3-copy part is contained in a published theorem (Maltby, 1994). The verdict of an earlier run, with the first version of the search program, reports the same for $n \le 319$.

**Theorem 9** (Three copies; Maltby 1994). *No rectangle is tiled by exactly three congruent non-rectangular polyominoes (grid isometries, reflections included).*

Theorem 9 is in print: it is a special case of the theorem of Maltby [2], and nothing is claimed here beyond print. It is used in Sections 3, 4 and 5; Section 3 also gives a self-contained proof for polyominoes.

**What Lean checks.** Lean 4 proves seven statements of this paper: Theorems 1 and 2, Theorems 7 and 9, Corollary 11, and Lemmas 12 and 13; Section 7 lists them with their Lean names. So of the main results, 3 of 4 are in Lean, and the fourth, Theorem 8, is a computation. No other statement is counted in Lean.

The library also has internal theorems for the statements listed next. They were not separately audited as statements of this paper, so these statements are counted as written. The same statement, or a stronger one, is there for Propositions 3 to 6, which were frozen statements of an earlier, audited version of the library. AI readers of two model families compared the following with this paper. The same statement, or a stronger one, is there for Lemmas 10, 18, 40 and 28, Corollaries 21, 23 and 36, and Propositions 19, 22, 27, 29, 31, 32, 35, 42, 43, 45, 33 and 44. For Theorem 16, Propositions 25, 38 and 48, and Lemmas 14, 15, 20, 24, 26, 30, 34, 37, 39 and 41, the internal theorems state only a part; Lemma 47 is there in a different form. The lemmas and propositions of Section 5 are steps of the Lean proof of Theorem 2 in other forms and were not compared one by one with this paper. The rows of these statements in Section 7 name the theorems and what differs. The proofs in this paper do not rely on Lean.

**How to read the rest.** Section 2 sets the conventions and proves the structure results common to all tilings. Section 3 excludes the case in which some tile holds two corner cells (case (II)), and Section 4 proves the facts on the sides of the rectangle in the remaining case (I), in which the four corner cells lie in four distinct tiles. Section 5 excludes case (I) and proves Theorem 1. Section 6 proves the reduction and reports the computation. Section 7 gives the reader's trust boundary, Section 8 the prior work, and Section 9 the data and the use of AI.

**Concept in Action: Connectedness Cannot Be Dropped**

```
U, V, E = (0, 3), (1, 2, 4), (1, -1)
Z = zip([U,U,V,V,V], [(0,1,3),(2,4,5),(0,3),(1,4),(2,5)])
parts = [[(x, y) for x in X for y in Y] for X, Y in Z]
assert len(set(sum(parts, []))) == 30 == sum(map(len, parts))
def canon(S):  # least image under the 8 symmetries
    J = [[(s*x, t*y)[::r] for x, y in S]
         for s in E for t in E for r in E]
    return min(sorted((x - min(j)[0], y - min(q for _, q in j))
                      for x, y in j) for j in J)
assert all(canon(p) == canon(parts[0]) for p in parts)
```

The five sets in the box partition the rectangle $R(5,6)$, they are congruent under grid isometries, none is connected, and the four corner cells lie in four different sets. So for cell sets that need not be connected, five congruent copies of a set that is not a rectangle can tile a rectangle with the four corner cells in four different copies, and any proof that this cannot happen for polyominoes must use that the tiles are connected. The script `recompute.py` runs the box verbatim and checks the corner owners and the failure of connectedness.

## 2. DEFINITIONS AND THE STRUCTURE OF A TILING

**Conventions.** *Connected* always means connected through shared edges. $R = R(a,b)$ has $a$ columns and $b$ rows; row 0 is the bottom row, row $b-1$ the top row, column 0 the left column and column $a-1$ the right column. The corner cells are $(0,0)$, $(a-1,0)$, $(a-1,b-1)$ and $(0,b-1)$. A *boundary cell* of $R$ has an edge on the boundary of the region. If $a = 1$ or $b = 1$ then every non-empty connected subset of $R$ is a rectangle, so in a tiling of $R(a,b)$ by copies of a non-rectangular $P$ we have $a \ge 2$ and $b \ge 2$. Congruent tiles have boxes with the same two side lengths, up to order. If $a, b \ge 2$, a boundary cell $c$ is given an *exit point* $e(c)$: the corner point of $[0,a] \times [0,b]$ in $c$ if $c$ is a corner cell, and otherwise the midpoint of the edge of $c$ on the boundary.

The only topological input is the standard fact that in a closed rectangle, a path joining two boundary points and a path joining two others that separate them on the boundary always meet. It is a consequence of the Jordan curve theorem [11] [to check].

**Lemma 10** (Interlacing). *Let $a, b \ge 2$, and let $p, q, p', q'$ be four distinct boundary cells of $R(a,b)$ whose exit points occur in the cyclic order $e(p), e(q), e(p'), e(q')$ on the boundary. If $S$ and $S'$ are connected subsets of $R(a,b)$ with $p, p' \in S$ and $q, q' \in S'$, then $S$ and $S'$ share a cell.*

*Proof.* Join $e(p)$ to the centre of $p$, then centre to centre along a chain of cells of $S$ from $p$ to $p'$, then the centre of $p'$ to $e(p')$; this is a path in the rectangle. Do the same for $S'$ from $q$ to $q'$. By the topological fact the two paths meet. A point of the first path is inside a cell of $S$, at the midpoint of an edge shared by two cells of $S$, or one of $e(p), e(p')$; the same holds for the second path and $S'$. The first path meets the boundary of the rectangle only at $e(p)$ and $e(p')$, and the second only at $e(q)$ and $e(q')$, and these four points are distinct; so the meeting point is an inner point of both paths, hence inside a cell of $S$ or on an edge shared by two cells of $S$, and

likewise for $S'$. If it is inside a cell, that cell is in both sets; if it is on a shared edge, both cells of that edge are in both sets. In every case $S$ and $S'$ share a cell. □

**Corollary 11** (Crossing). *Let $a, b \geq 2$. If $S$ is a connected subset of $R(a,b)$ meeting both the left and the right column, and $S'$ is a connected subset of $R(a,b)$ meeting both the bottom and the top row, then $S$ and $S'$ share a cell.*

*Proof.* Take $p, p' \in S$ in the left and the right column and $q, q' \in S'$ in the bottom and the top row. If two of the four cells coincide, the common cell is one of $p, p'$ and one of $q, q'$, so it is in both sets. Otherwise the exit points of $p$ and $p'$ lie on the two arcs into which $e(q)$ and $e(q')$ cut the boundary, and Lemma 10 applies. □

**Lemma 12** (No tile holds two opposite corners). *Let $P$ be a polyomino, $k \geq 2$, and let $T_1, \dots, T_k$ be a $k$-copy tiling of $R(a,b)$ by $P$. No tile contains two opposite corner cells of $R(a,b)$.*

*Proof.* If $a = 1$ or $b = 1$, a connected set containing both ends of the strip is the whole strip, so $k = 1$. Let $a, b \geq 2$ and suppose $T$ contains $(0,0)$ and $(a-1, b-1)$. Then $\mathrm{box}(T) = R(a,b)$, so every tile has a box equal to $R(a,b)$. Take another tile $T'$. It is connected and meets the left and the right column, and $T$ meets the bottom and the top row; by Corollary 11 they share a cell, a contradiction. The other pair of opposite corners is the mirror image. □

**Lemma 13** (Two adjacent corners force the whole side). *Let $P$ be a polyomino, $k \geq 2$, and let $T_1, \dots, T_k$ be a $k$-copy tiling of $R(a,b)$ by $P$. If a tile $T$ contains the corner cells $(0,0)$ and $(a-1,0)$, then $T$ contains every cell of row 0.*

*Proof.* If $b = 1$ then $T = R(a,b)$, so $k = 1$. If $a \leq 2$ there is nothing to show. Let $a \geq 3$ and $b \geq 2$, let $\mathrm{box}(T) = [0,a) \times [0,d)$, and suppose a cell $(x,0)$ with $0 < x < a-1$ is not in $T$. Let $T'$ be the tile containing $q = (x,0)$, and put $p = (0,0)$, $p' = (a-1,0)$. If $T'$ contains a cell $q' = (0,y)$ of the left column, then $y \geq 1$ and the exit points of $p, q, p', q'$ occur in this cyclic order, so Lemma 10 gives a common cell of $T$ and $T'$; the right column is the same. If $T'$ has a cell in a row $y \geq d$ (so $b > d$), then $T'$ together with all rows $d, \dots, b-1$ is connected and disjoint from $T$ (no cell of $T$ is in a row $\geq d$), and it contains $q$ and $(0, b-1)$; again Lemma 10 gives a contradiction. So $\mathrm{box}(T')$ lies inside $[1, a-1) \times [0,d)$: its horizontal extent is at most $a-2$ and its vertical extent at most $d$. But $\mathrm{box}(T')$ has the same two side lengths as $\mathrm{box}(T)$, which are $a$ and $d$. The first is impossible because $a > a-2$; the second needs $d \leq a-2$ and $a \leq d$, impossible. □

**Lemma 14** (Spines). *Let $P$ be a non-rectangular polyomino, let $T_1, \dots, T_k$ be a $k$-copy tiling of $R(a,b)$ by $P$, and let a tile $T$ contain row 0, with $\mathrm{box}(T) = [0,a) \times [0,d)$. Then $d \geq 2$. Choose an isometry $g$ with $T = g(P)$, and use it for $T$; let $s_0 = g^{-1}(\text{row } 0)$, a full side of $\mathrm{box}(P)$ of $a$ cells contained in $P$. For each tile $T_i = g_i(P)$ its* spine *is $g_i(s_0)$, a full side of $\mathrm{box}(T_i)$ of $a$ cells contained in $T_i$. Call $T_i$* horizontal *if its spine is horizontal and* vertical *otherwise. Then:*

(a) *the spine of a horizontal tile is a full row $r$ of $R$, distinct for distinct horizontal tiles; the tile lies in rows $r, \dots, r+d-1$ (up-type) or in rows $r-d+1, \dots, r$ (down-type), and has a cell outside row $r$;*

(b) *the box of a vertical tile has dimensions $(d,a)$; if a vertical tile exists then $d \leq a \leq b-1$;*

(c) *if a horizontal tile has spine row $r$, every other tile lies entirely in rows above $r$ or entirely in rows below $r$.*

*Proof.* If $d = 1$ then $T$ is row 0, a rectangle, so $P$ is rectangular; hence $d \geq 2$. (a) A horizontal segment of $a$ cells inside $R$ is a full row, and the tiles are disjoint. Its box has horizontal side $a$, so it lies within $d-1$ rows of the spine on one side, and since the box is the smallest rectangle containing the tile and $d \geq 2$, it reaches the row at distance $d-1$. (b) The box of a vertical tile has its side of $a$ cells vertical, so its dimensions are $(d,a)$, and $d \leq a$ because it lies in $R$; its spine avoids row 0, which belongs to $T$, so $a \leq b-1$. (c) A connected tile with no cell in row $r$ lies entirely above or entirely below it. □

**Lemma 15** (Blocks). *Under the hypotheses of Lemma 14, call the line $y = c$* clean *if no tile has cells both in rows below $c$ and in rows at or above $c$.*

(a) *If a horizontal tile with spine row $r$ is up-type, the line $y = r$ is clean; if it is down-type, $y = r+1$ is clean.*

(b) *Let $0 = c_0 < c_1 < \cdots < c_t = b$ be the distinct values among 0, $b$ and these clean lines. Each* block, *the rows $c_{j-1}, \dots, c_j - 1$, is a rectangle with $a$ columns that is a union of at least 2 tiles. It contains at most one up-type horizontal tile, with its spine on the bottom row of the block, and at most one down-type horizontal tile, with its spine on the top row; all other tiles of the block are vertical.*

(c) *So $k \geq 2t$, and a block with an odd number of tiles contains a vertical tile.*

*Proof.* (a) By Lemma 14(c) every other tile lies entirely above or entirely below row $r$, and an up-type tile lies in rows at least $r$; so no tile has cells on both sides of $y = r$. The down-type case is the mirror image. (b) Each $c_j$ is clean, so each tile lies in one block; a block that is a single tile would be a rectangle and so would be $P$. A horizontal tile in a block with spine row $r$ has its own clean line in the block, next to row $r$; the line must be the bottom or top of the block, so $r$ is its bottom row (up-type) or its top row (down-type). Two up-type tiles in one block would share a spine row, against Lemma 14(a); likewise two down-type. (c) is immediate from (b): a block without vertical tiles has at most 2, hence exactly 2, tiles. □

**Theorem 16** (Shape). *Let $P$ be a non-rectangular polyomino and let $T_1, \dots, T_5$ be a 5-copy tiling of a rectangle by $P$. Then either* (I) *the four corner cells lie in four distinct tiles, or* (II) *after a grid isometry the rectangle is $R(a,b)$, a tile $T$ contains all of row 0, the other dimension $d$ of $\mathrm{box}(P)$ satisfies $2 \leq d \leq a \leq b-1$, $P$ contains a full side of $\mathrm{box}(P)$ with $a$ cells, and, for every choice of the isometries in Lemma 14, some tile is vertical and at least one of the following holds:* (II.1) *a horizontal grid line splits $R(a,b)$ into two rectangles, one tiled by 2 and one by 3 of the tiles;* (II.2) *$T$ is the only horizontal tile and the other four tiles are vertical;* (II.3) *there are exactly two horizontal tiles, $T$ (up-type, spine row 0) and*

*a down-type tile with spine row $b-1$, the other three are vertical, and $a \le b-2$.*

*Proof.* Suppose (I) fails: some tile contains two corner cells. By Lemma 12 they are adjacent; after an isometry moving their side to row 0, the tile contains row 0 by Lemma 13, and Lemmas 14 and 15 apply. By Lemma 15(c), $5 \ge 2t$, so the number $t$ of blocks is 1 or 2. If $t = 2$ the blocks have at least two tiles each, hence 2 and 3 tiles, which is (II.1). If $t = 1$ the only block is $R(a,b)$, and by Lemma 15(b) the horizontal tiles are $T$ (its spine is row 0) and at most one down-type tile with spine row $b-1$: this is (II.2) or (II.3), and in (II.3) the spine of a vertical tile avoids rows 0 and $b-1$, so $a \le b-2$. Five is odd, so some block has an odd number of tiles, and by Lemma 15(c) it contains a vertical tile, for every choice of the isometries; so $2 \le d \le a \le b-1$ by Lemma 14. □

**Corollary 17** (Corner tiles are placed by their orientation). *In case (I) of Theorem 16, name the tiles so that $T_1$ contains $(0,0)$ and $T_2, T_3, T_4$ contain the corner cells $(a-1,0)$, $(a-1,b-1)$, $(0,b-1)$, and replace $P$ by the congruent polyomino $T_1$. Then the tiling is determined by $(a,b)$, by $P$, and by three elements $h_2, h_3, h_4$ of the symmetry group of the square: $T_i$ is the only translate of $h_i(P)$ inside $R$ containing its corner cell, and $T_5$ is the complement of the other four tiles.*

*Proof.* If $Q'$ is a translate of a finite cell set $Q$ inside $R$ that contains $(0,0)$, then the lower-left cell of $\mathrm{box}(Q')$ is $(0,0)$, because the box lies in $R$; this fixes the translation. The other corners are the same. □

This replacement of $P$ is needed. For the polyomino $\{(1,0),(0,1),(1,1)\}$, $R(3,4)$ has a four-copy tiling with four distinct corner tiles, its rows from the bottom being $ABB$, $AAB$, $CDD$, $CCD$, and no translate of the polyomino contains the lower-left cell of its box. The search does this by taking the tile on $(0,0)$ as the unknown.

## 3. CASE (II): A TILE HOLDING TWO CORNER CELLS

Throughout this section a tile $T$ contains row 0, $\mathrm{box}(T) = [0,a) \times [0,d)$, the isometries of Lemma 14 are chosen, and $h = b-a$. There are at least two tiles, because a single tile would be $R$, a rectangle. Let $u[x]$ be the number of cells of $T$ in column $x$ and $v[y]$ the number in row $y$; $u_0 = u[0]$, $u_1 = u[a-1]$, $t = v[d-1]$, and Top is the set of columns $x$ with $(x, d-1) \in T$. A vertical tile $V_i$ has the box $[x_i, x_i+d) \times [y_i, y_i+a)$, and its spine is column $x_i$ or column $x_i+d-1$ of this box, in the rows $[y_i, y_i+a)$.

**Lemma 18** (Cross-sections travel with the spine). *Let $V$ be a vertical tile with spine $(c,y),\dots,(c,y+a-1)$. There is a map $\rho$ from columns to rows, either $\rho(x) = y+x$ for all $x$ (the order is kept) or $\rho(x) = y+a-1-x$ for all $x$ (reversed), such that $V$ has exactly $u[x]$ cells in row $\rho(x)$. The cells of $T$ in row $j$ go to the column of $\mathrm{box}(V)$ at distance $j$ from the spine.*

*Proof.* The isometry $g$ carrying $T$ to $V$ maps row 0 onto the spine, so it exchanges the axis directions and maps columns to rows and rows to columns. Column $x$ contains $(x,0)$, so it goes to the row of $g(x,0) = (c, \rho(x))$, and $\rho$ keeps adjacency, so it has one of the two forms. Then $V \cap (\text{row } \rho(x)) = g(T \cap$ column $x)$. Row $j$ goes to a column at distance $j$ from the spine, inside the box. □

### 3.1 One horizontal tile: a square box.

**Proposition 19.** *Suppose every tile other than $T$ is vertical. If $d = a$ then the number $k$ of tiles is even.*

*Proof. Step 1.* There is a vertical tile, so $a \le b-1$ by Lemma 14(b), and $T$ has no cell in row $b-1$. Let $V$ and $V'$ be the tiles on $(0,b-1)$ and $(a-1,b-1)$. Their boxes are $a$ by $a$ and contain a cell of row $b-1$, so both are $S = [0,a) \times [b-a,b)$. $V$ meets the left and the right column of $S$, and $V'$ the bottom and the top row, so they share a cell by Corollary 11 applied to $S$, and $V = V'$. By Lemma 13, $V$ contains row $b-1$. *Step 2.* Let $g$ be the isometry with $g(T) = V$ and row 0 going to the spine; it exchanges the axis directions. Row $b-1$ is a side of $\mathrm{box}(V)$ contained in $V$, so $g^{-1}(\text{row } b-1)$ is a vertical side of $\mathrm{box}(T)$ contained in $T$. So $T$ contains two perpendicular full sides of its box. *Step 3.* Let $W = f(T)$ be any tile, $W = T$ allowed. The images of the two sides are perpendicular full sides of $\mathrm{box}(W)$ with $a$ cells, contained in $W$; one is horizontal, and a horizontal segment of $a$ cells in $R$ is a full row $r$. It is the bottom row of $\mathrm{box}(W)$ ($W$ up-type) or the top row (down-type); if both occur, fix one. Every other tile has no cell in row $r$, so it lies above it or below it. Hence no tile crosses the line $y = r$ (up-type) or $y = r+1$ (down-type). Take the lines of all tiles and $0, b$ as the block boundaries. A block holds at most one up-type tile, on its bottom row, and at most one down-type tile, on its top row, so at most 2 tiles; a block that is one tile is a rectangle. So every block has exactly 2 tiles and $k$ is even. □

**3.2 The top tiles.** In this subsection every tile other than $T$ is vertical and $d \le a$. A vertical tile with a cell in row $b-1$ is a *top tile*; its box is $[x, x+d) \times [b-a, b)$, inside $S = [0,a) \times [b-a,b)$, and its spine is column $x$ of $S$ (*right-type*) or column $x+d-1$ (*left-type*).

**Lemma 20** (Top row). (a) *Row $b-1$ is covered by the top tiles; distinct top tiles have distinct spine columns, and a top tile lies entirely left or entirely right of the spine column of any other.* (b) *Attach $c$ to a right-type top tile with spine column $c$ and $c+1$ to a left-type one, and let $0 = e_0 < \cdots < e_m = a$ be the distinct values among $0, a$ and these numbers. Each top tile lies in one block (columns $e_{l-1}, \dots, e_l - 1$). A block holds at most one right-type top tile, whose spine is its first column, and at most one left-type one, whose spine is its last column; it holds at least one and has at least $d$ columns.* (c) *A block with exactly one top tile $W$ has exactly $d$ columns, $W$ contains the top row of the block, and $T$ contains column 0 or column $a-1$ of its box in full.* (d) *A block with two top tiles has $w$ columns with $d+1 \le w \le 2d$.*

*Proof.* (a) $T$ lies in rows at most $d-1 \le a-1 \le b-2$, so a cell of row $b-1$ is in a vertical tile. A top tile lies in $S$ with spine a full column of $S$; tiles are disjoint and connected. (b) As in Lemma 15: for a right-type tile with spine column $c = e_l$ every other top tile lies left of column $c$ or right of it, so none has cells on both sides of the line at $e_l$, and likewise for left-type tiles. If a right-type tile lies in a block, its number $c$ is one of the $e$'s and $e_{l-1} \le c < e_l$, so $c = e_{l-1}$; if a left-type tile lies in a block, $c+1 = e_l$. Two tiles of one type have the same

spine column. The cell $(e_{l-1}, b-1)$ is in a top tile of block $l$, whose box has $d$ columns inside the block. (c) The top row of the block is in $W$, which has at most $d$ cells in a row, so the block has exactly $d$ columns. The top row is a horizontal side of $\mathrm{box}(W)$ in $W$, and the isometry carrying $T$ to $W$ exchanges the axis directions, so the matching vertical side of $\mathrm{box}(T)$ is in $T$. (d) The two tiles cover the top row with at most $d$ cells each, so $w \le 2d$. If $w = d$, both boxes are the block's part of $S$; the left-type tile has a cell in the first column of its box, which is the spine of the right-type tile. □

**Corollary 21.** *Let $k = 5$, so there are four vertical tiles, $d < a$, and let $p$ be the number of top tiles. Then $2 \le p \le 4$ and $a \le pd$. If $h > d$, then $p = 2$, $a \le 2d$, and the two other vertical tiles both have cells in every row $r$ with $d \le r < h$.*

*Proof.* A top tile has at most $d$ cells in row $b-1$, so $a \le pd$, and $d < a$ gives $p \ge 2$. A row $r$ with $d \le r < h$ has no cell of $T$ and none of a top tile, so its $a$ cells are in the $4-p$ other vertical tiles, at most $d$ each; $a > d$ gives $4 - p \ge 2$, and then both are needed. □

**Proposition 22.** *If $d < a$, not every vertical tile is a top tile. In particular, for $k = 3$ no rectangle has a tiling in which a tile contains row 0 and the other two tiles are vertical, when $P$ is non-rectangular.*

*Proof.* Suppose every vertical tile is a top tile, $h \ge 1$. (0) No vertical tile has a cell in rows $0, \dots, h-1$, so $T$ contains these rows; as $T$ is not its box, $h < d$. (1) Suppose a block of Lemma 20 holds one top tile $W$. It has $d$ columns. A cell of the block in a row at least $d$ is not in $T$, so it is in $W$; so the top $b - d = a + h - d$ rows of $\mathrm{box}(W)$ are full rows of $W$. By Lemma 18, $T$ contains every cell of its box in an interval $I$ of $a+h-d$ columns, $I = [0, a+h-d)$ or $[d-h, a)$. If $c \in I$ is the spine column of a top tile, the cell $(c, h)$ is in $T$ ($h < d$) and in that spine (rows $h, \dots, b-1$), a contradiction. So $I$ contains no spine column. Take $I = [0, a+h-d)$; the other case is the mirror image. The first block holds a top tile whose spine is column 0, which is in $I$, or column $e_1 - 1$. If there is a second block then $e_1 \le a - d$ by Lemma 20(b), so $e_1 - 1 \in I$. So there is one block, of $a > d$ columns, against the block of $d$ columns. (2) So every block holds two top tiles; the first block has a right-type tile with spine column 0 and the last a left-type one with spine column $a-1$. Column 0 is rows $0, \dots, h-1$, in $T$, and a spine, so $u_0 = h$; likewise $u_1 = h$. A top tile has $u_0$ or $u_1$ cells in row $b-1$, that is $h$. With $m = k-1$ top tiles covering row $b-1$, $a = mh$ and $b = a + h = kh$, so $n = ab/k = ah$. $T$ contains the $ah$ cells of rows $0, \dots, h-1$, so $T$ is that rectangle. For $k = 3$ Proposition 19 excludes $d = a$ and a top tile has at most $d < a$ cells in row $b-1$, so both vertical tiles are top tiles. □

**Corollary 23.** *Let $k = 5$ and let every tile other than $T$ be vertical. Then $d < a$, and the number of top tiles is 2 or 3.*

*Proof.* Proposition 19 gives $d < a$, Corollary 21 gives $2 \le p \le 4$ and Proposition 22 excludes $p = 4$. □

### 3.3 Case (II.2) with $h > d$.

**Lemma 24** (Bottom tiles). *Let every tile other than $T$ be vertical, and let $s$ be the number of full rows at the bottom of $T$: rows $0, \dots, s-1$ lie in $T$ and some cell of row $s$ is not in $T$. Then $1 \le s \le d-1$ and: (a) every vertical tile has $y_i \ge s$, those with a cell in row $s$ are exactly those with $y_i = s$ (bottom tiles), and there is at least one; (b) every vertical tile contains in full the $s$ columns of its box nearest its spine, the spine included (its thick spine); (c) if $c$ is a thick-spine column of a bottom tile, then $T$ meets column $c$ in rows $0, \dots, s-1$ only, so $u[c] = s$; (d) for such $c$, every vertical tile has exactly $s$ cells in the row $\rho(c)$ of Lemma 18.*

*Proof.* $s \ge 1$ because row 0 is in $T$, and $s \le d-1$ because $T$ is not its box. (a) A vertical tile has a cell in row $y_i$, and rows below $s$ are in $T$. A cell of row $s$ outside $T$ is in a vertical tile, which then has $y_i = s$; conversely. (b) Row $j$ of $T$ goes to the column at distance $j$ from the spine (Lemma 18), and rows $0, \dots, s-1$ are in $T$. (c) Rows $s, \dots, s+a-1$ of column $c$ are in the bottom tile, and $T$ lies in rows below $d \le a < s + a$. (d) Lemma 18. □

**Proposition 25** (Layout). *In case (II.2) with $k = 5$, two top tiles and $h > d$, name the top tiles $W_1, W_2$ and the other vertical tiles $Z_1, Z_2$. After renaming: (i) $d < a \le 2d$ and $a + h \le 5d - 1$; (ii) $\mathrm{box}(Z_1) = [0, d) \times [y_1, y_1 + a)$, $\mathrm{box}(Z_2) = [a-d, a) \times [y_2, y_2 + a)$ with $y_i \ge 1$ and $h - a \le y_i \le d$, $\mathrm{box}(W_1) = [0, d) \times [h, b)$, $\mathrm{box}(W_2) = [a-d, a) \times [h, b)$; (iii) $Z_1$ contains $[0, a-d) \times [d, h)$ and $Z_2$ contains $[d, a) \times [d, h)$; $W_1$ contains $[0, a-d) \times [y_1 + a, b)$ and $W_2$ contains $[d, a) \times [y_2 + a, b)$; (iv) for $d \le r < h$ the cells of $Z_1$ and $Z_2$ in row $r$ are a in all; (v) column 0: $u_0 = h - a$ if $Z_1$ and $W_1$ both have spine column 0 (and then $y_1 = h - a$), $u_0 = h - t$ if exactly one has, $u_0 = a + h - 2t$ if neither; the same for column $a-1$ with $u_1$, $Z_2$, $W_2$; the two top tiles have $u_0$ or $u_1$ cells each in row $b-1$, adding to $a$; (vi) not both $y_1 = h - a$ and $y_2 = h - a$.*

*Proof.* (i) Corollary 21, and $n \le ad - 1$ with $5n = a(a+h)$. (ii), (iii) By Corollary 21 both $Z_i$ meet every row $r$ with $d \le r < h$, and they alone cover it with at most $d$ cells each, so their boxes cover all columns, one starting at column 0 and the other ending at column $a-1$; the rows give $y_i \le d$ and $y_i + a \ge h$. A cell of $[0, a-d) \times [d, h)$ is in no box but that of $Z_1$. The top corners give the boxes of $W_1$ and $W_2$, and a cell of $[0, a-d)$ in a row at least $y_1 + a > d$ is in no box but that of $W_1$. (iv) is the cover of row $r$. (v) Column 0 meets only $T$, $Z_1$ and $W_1$; a vertical tile with $x_i = 0$ has $a$ cells there (its spine) or $t$ (the image of the top row of $T$); two spines in one column are disjoint, so $y_1 + a \le h$ then. (vi) Otherwise the line $y = h$ is clean, and $T, Z_1, Z_2$ tile $R(a, h)$ with $T$ on row 0 and two vertical tiles, against Proposition 22. □

**Lemma 26** (Walls). *In the layout of Proposition 25, call a vertical tile* outer *if its spine is in column 0 or $a-1$ and* inner *otherwise (spine in column $d-1$ for $Z_1, W_1$, in column $a-d$ for $Z_2, W_2$). (a) $T$ meets column 0 in rows $0, \dots, u_0 - 1$ exactly, and column $a-1$ in rows $0, \dots, u_1 - 1$. (b) Column 0 reads, from the bottom: $T$ on rows $[0, u_0)$, $Z_1$ on $[u_0, \rho)$, $W_1$ on $[\rho, b)$, for some $\rho \ge h$; so $y_1 \le u_0$. Likewise at column $a-1$ with $u_1$, $Z_2$, $W_2$. (c) A vertical tile meets the bottom row of its box, and the top row, in an interval starting at its spine cell, of $u_0$ or $u_1$ cells. So $W_1$ outer has $[0, u)$ in row $b-1$ with $u \in \{u_0, u_1\}$, and $W_1$ inner has all of $[0, d)$*

*there and the matching end column of $T$ is full ($u = d$). Likewise $W_2$.* (d) *At column 0 one of four cases holds.* A ($Z_1$, $W_1$ *outer*): $u_0 = y_1 = h - a$. B ($Z_1$ *outer,* $W_1$ *inner*): $u_0 = y_1$, $t = h - y_1$, *and* Top *is an end interval of $t$ columns.* C ($Z_1$ *inner,* $W_1$ *outer*): $t = h - u_0$, *and* Top *is the set of columns $x$ with $u_0 \le \rho(x) < h$, $\rho$ the map of Lemma 18 for $Z_1$ (the interval $[u_0 - y_1, h - y_1)$ if $Z_1$ keeps the order, its reversal in $[0, a)$ otherwise).* D ($Z_1$, $W_1$ *inner*): $y_1 = h - a$, $t = a$, $u_0 = u_1 = d$ *and* $h = a + d$. *At column $a - 1$ the cases* A′, B′, C′, D′ *are the same with $u_1, y_2, Z_2, W_2$.* (e) *If* Top *contains 0 then $u_0 = d$; if it contains $a - 1$ then $u_1 = d$.*

*Proof.* (a) $(0,0) \in T$ and $(0,d) \in Z_1$ by Proposition 25(iii). A cell $(0,j)$, $j < d$, outside $T$ is in $Z_1$, the only other tile with a cell in column 0 below row $h$. If $(0,j) \in Z_1$ and $(0,i) \in T$ with $j < i < d$, then $(0,0), (0,j), (0,i), (0,d)$ lie in $T, Z_1, T, Z_1$ in this order along the left side, against Lemma 10. (b) Rows $[u_0, h)$ of column 0 are in $Z_1$ by (a) and (iii); above row $h$ only $Z_1$ and $W_1$ occur, $(0, b-1) \in W_1$, and $Z_1, W_1, Z_1, W_1$ going up is excluded in the same way. (c) The isometry carrying $T$ to a vertical tile $V$ maps column 0 or $a - 1$ of box($T$) onto an end row of box($V$), row 0 going to the spine; apply (a). The corner cell $(0, b-1)$ is in $W_1$. (d) $Z_1$ outer: its cells in column 0 are its spine, rows $[y_1, y_1 + a)$, so $u_0 = y_1$ and $\rho = y_1 + a$; then $W_1$ has $h - y_1$ cells there, which is $a$ if $W_1$ is outer and $t$ if inner, and in the second case they are the top $t$ rows of box($W_1$), whose preimage is an end interval. $Z_1$ inner: it has $t$ cells in column 0, so $\rho = u_0 + t$. $W_1$ outer gives $\rho = h$. $W_1$ inner: two spines in column $d - 1$ are disjoint, so $y_1 + a \le h$ and $y_1 = h - a$; then $Z_1$ lies below row $h$ and the $a$ cells of column 0 in rows $[h, b)$ are in $W_1$, in its far column, so $t = a$, $u_0 + 2a = a + h$, and by (e) $u_0 = u_1 = d$, so $h = a + d$. (e) By (a). □

**Proposition 27.** *In the layout of Proposition 25 the following pairs (left case, right case) do not occur, nor their mirror images:* AA′*; every pair with* D *or* D′*;* BB′*; and* CC′ *when $a - d \ge d - 1$.*

*Proof.* A gives $y_1 = h - a$ and A′ gives $y_2 = h - a$, against Proposition 25(vi). D′ gives $y_2 = h - a$, $t = a$ and $u_0 = u_1 = d = h - a$. On the left, A and D give $y_1 = h - a$; B gives $y_1 = u_0 = d$; in C the interval Top has $a$ columns, so $y_1 = u_0 = d$. In each case $y_1 = y_2 = h - a$, against (vi). BB′: both top tiles are inner, so by Lemma 26(c) row $b - 1$ is $[0, d)$ and $[a - d, a)$, $a = 2d$, and an end column of $T$ is full; B and B′ give $u_0 = y_1$, $u_1 = y_2$ with $t = h - y_1 = h - y_2$, so $u_0 = u_1 = y_1 = y_2 = d$, no vertical tile is below row $d$, and $T$ is its box. CC′: $t = h - u_0 = h - u_1$, and row $b - 1$ gives $u_0 = u_1 = a/2$. If $a - d = d - 1$, the spines of $Z_1$ and $Z_2$ are the same column in the rows $d, \dots, h - 1$. If $a = 2d$ then $u_0 = u_1 = d$, Top is an interval containing 0 and $a - 1$, so $t = a$, $h = a + d$, against $a + h \le 5d - 1$. □

**Lemma 28.** *In the layout of Proposition 25, if $W_1$ or $W_2$ is inner then $a = 2d$.*

*Proof.* Let $W_1$ be inner (left-type, spine column $d - 1$). If $W_2$ is outer (left-type), the numbers of Lemma 20(b) are $0, d, a$ and the block $[d, a)$ has at least $d$ columns. If $W_2$ is inner (right-type, spine column $a - d$), they are $0, a - d, d, a$; for $a - d < d$ the tile $W_1$ would lie in the block $[a - d, d)$ and not contain $(0, b - 1)$. □

**Proposition 29.** *The layout of Proposition 25 does not occur. So in case (II.2) with $k = 5$: $2 \le d < a$ and $h \le d$.*

*Proof.* Put $e = a - d$ and $f = h - d$, with $s$ as in Lemma 24. By Proposition 27 it remains to exclude AB′, AC′, BC′ and CC′ with $e \le d - 2$, and mirror images. For D′ the tile $W_2$ is inner, so $a = 2d$ by Lemma 28, $y_2 = h - a$, and by Proposition 25(iii) $W_2$ contains its whole box; the same for D. AB′. $W_2$ is inner, so $a = 2d$; $y_1 = h - a$, so $W_1$ contains $[0, d) \times [h, b)$, its whole box. BC′. $a = 2d$ as $W_1$ is inner. B gives $u_0 = y_1$, $t = h - y_1$; C′ gives $t = h - u_1$; so $u_0 = u_1 = y_1$. By Lemma 26(c) an end column of $T$ is full, so $u_0 = u_1 = d$, both 0 and $a - 1$ are in Top, an end interval of $t = f$ columns, so $f = a$, $h = 3d$, against $a + h \le 5d - 1$. CC′, $e \le d - 2$. Here $u_0 = u_1 = a/2$, $t = h - a/2$, and Top is the preimage of $[a/2 - y_i, h - y_i)$ in box($Z_i$) for $i = 1, 2$. The boxes of $Z_1$ and $Z_2$ have the columns $[0, d)$ and $[e, a)$, and the spine of $Z_1$ is column $d - 1$, a column of box($Z_2$). A tile has a cell in every column of its box, so $y_1 = y_2$ would put a cell of $Z_2$ in the spine of $Z_1$; so $y_1 \ne y_2$. If $Z_i$ keeps the order then Top $= [a/2 - y_i, h - y_i)$, and if it reverses it then Top $= [a - h + y_i, a/2 + y_i)$. So the row orders of $Z_1$ and $Z_2$ are opposite and $y_1 + y_2 = h - a/2 = t$. By the mirror image take $y_1 < y_2$, so $s = y_1$ and $Z_1$ is a bottom tile with thick spine columns $[d - s, d)$. (1) If $Z_1$ kept the order, Top $= [a/2 - s, h - s)$ would contain $d - s$, a column with $u = s$ by Lemma 24(c), while $s \le d - 1$. So $Z_1$ reverses the order and $Z_2$ keeps it. (2) Column $e$ holds $T, Z_1, W_1$, the spine of $Z_2$ ($a$ cells) and the far column of $W_2$ ($t$ cells). Above the spine only $W_1$ and $W_2$ occur, so $W_1$ has $h - y_2 - t = a/2 - y_2$ cells there, and this is $v[e]$ because $W_1$ is outer. Below the spine only $T$ and $Z_1$ occur, so $u[e] + v[d - 1 - e] = y_2$. Column $d - 1$ holds $T$, the spine of $Z_1$, the far column of $W_1$ ($t$ cells), $v[d - 1 - e]$ cells of $Z_2$ and $v[e]$ cells of $W_2$; below the spine of $Z_1$ only $T$ occurs, so $u[d - 1] = y_1$. Counting, $v[d - 1 - e] = y_2 - y_1$, and so $u[e] = y_1 = s$. (3) $v[e] = a/2 - y_2 < a$, so row $e$ is not full and $e \ge s$. (4) $W_2$, which is outer, contains $[d, a) \times [y_2 + a, b)$, so $T$ contains $K \times [0, e)$ for an end interval $K$ of $h - y_2 = a/2 + s$ columns. $K$ contains $e$ or $d - 1$, columns with $u = s$, so $s \ge e$. Hence $s = e$. (5) $Z_1$, inner and reversed, contains $[0, e) \times [d, h)$, so $T$ contains $[2e - f, 2e) \times [d - e, d)$, in particular $(2e - 1, d - 1)$. The thick spine of $Z_2$ is $[e, 2e) \times [y_2, y_2 + a)$, and $y_2 \le a/2 \le d - 1$. So $(2e - 1, d - 1)$ is in both, a contradiction.

AC′. Here $u_0 = y_1 = h - a$; by Proposition 25(vi), $y_2 > y_1$, so $s = y_1$ and $Z_1$ is a bottom tile. (1) $W_1$ contains $[0, e) \times [h, b)$, so $T$ contains its bottom $e$ rows and $s \ge e$. The thick spines $[0, s) \times [s, h)$ of $Z_1$ and $[e, e + s) \times [y_2, y_2 + a)$ of $Z_2$ share rows, so $s \le e$. Hence $s = e, f = 2e$, $y_1 = e$. If $e = d$, then $Z_1$ is its box; so $e < d$. (2) $Z_1$ has no cell in $[e, 2e) \times [y_2, h)$ and $W_1$ none in $[e, 2e) \times [h, y_2 + a)$, by the thick spine of $Z_2$. Pulled back, a cell $(x, j)$ of $T$ with $e \le j < \min(2e, d)$ goes to a box row of $Z_1$ below $y_2 - e$ and to a box row of $W_1$ at least $y_2 - e$. If $Z_1$ and $W_1$ had the same order, row $e$ of $T$ would be empty. So their orders differ, and the cells of $T$ in the rows $[e, \min(2e, d))$ lie in an end interval $X$ of $y_2 - e$ columns: $X = [0, y_2 - e)$ if $Z_1$ keeps the order and $X = [a - y_2 + e, a)$ if it reverses it. (3) Suppose $Z_1$ keeps and $W_1$ reverses. The part of row $b - 1$ in $W_1$ is $[0, e)$, the image of column 0, so $W_2$ has $d$ cells there and $u_1 = d$; then $(a - 1, e) \in T$ but $a - 1 \notin X$. (4) Suppose $Z_1$ reverses and $W_1$ keeps. Row $b - 1$ gives $u_1 = d$ or

$u_1 = a/2$. For $e \le r < \min(2e, y_2)$ row $r$ holds only $T$ and $Z_1$, so $v[r] + u[a-1-(r-e)] = a$, so $v[r] \ge e$, and $v[r] \le y_2 - e$ by (2); hence $y_2 \ge 2e$. Then for $e \le r < 2e$ the cells left of $X$ are in $Z_1$: $Z_1$ contains $[0, a-y_2+e) \times [e, 2e)$, and $T$ contains $[a-e, a) \times [0, a-y_2+e)$. So $u_1 \ge d + 2e - y_2$. (4a) $u_1 = a/2$: $y_2 \le u_1 = a/2$ by Lemma 26(b), but $y_2 \ge d + 2e - a/2 > a/2$. (4b) $u_1 = d$: $t = f = 2e$, and Top, an interval of $2e$ columns containing $a-1$, is $[d-e, a)$. By Lemma 26(d) it is the preimage of $[d-y_2, h-y_2)$ in box$(Z_2)$; with the order kept this needs $y_2 = e$, excluded; reversed, $y_2 = d$. Then rows $e, \dots, d-1$ hold only $T$ and $Z_1$, so columns $[d, a)$ of $T$ are full, $v[r] = e$ for $e \le r < 2e$, and $T$ meets row $e$ exactly in $[d, a)$. In the rows $d + e \le r < d + 2e$ the tile $Z_1$ has $u[x]$ cells with $x < e$, that is $e$ cells, because columns $[0, e)$ hold $T$ below row $e$ and then thick spines. So $Z_2$ has $d$ cells there, the images of the columns $[d-e, d)$ of $T$: these are full, and $(d-1, e) \in T$, outside $[d, a)$, a contradiction.

So $h > d$ does not occur, as it would give two top tiles by Corollary 21 and the layout of Proposition 25. □

**3.4 Case (II.2) with $h \le d$.** Let $k = 5$, $d < a$, every tile other than $T$ vertical and $1 \le h \le d$, and use $s$ and the thick spines of Lemma 24. A vertical tile is *outer* if its spine is in column 0 or $a-1$. A *non-top* tile is a vertical tile with $y_i < h$; one exists by Proposition 22.

**Lemma 30** (Walls for $h \le d$). (a) *Every vertical tile has $s \le y_i \le h$, so its spine meets every row from $h$ to $a + s - 1$, at least two rows. The four spines lie in four distinct columns and the four thick spines are disjoint.* (b) *$T$ meets column 0 in the rows $[0, u_0)$ exactly and column $a-1$ in $[0, u_1)$.* (c) *The bottom and the top row of the box of a vertical tile $V$ meet $V$ in intervals starting at the spine cell, one of $u_0$ cells and one of $u_1$; the cells of $V$ in the top row of* box$(V)$ *are the image of the cells of $T$ in column 0 if $V$ reverses the order, and in column $a-1$ if $V$ keeps it.* (d) *At column 0 exactly one of three states holds. $\alpha$: one tile has $x_i = 0$; it is a top tile with spine column 0, and $u_0 = h$. $\beta$: two tiles have $x_i = 0$, a top tile with spine column 0 and a non-top tile $Z$ with spine column $d-1$; $u_0 < h$, $t = h - u_0$, and $Z$ meets column 0 in the rows $[u_0, h)$. $\gamma$: two tiles have $x_i = 0$, a non-top tile $Z$ with spine column 0 and $y_Z = u_0 < h$, and a top tile $W$ with spine column $d-1$; $t = h - u_0$, and $W$ meets column 0 in the rows $[u_0 + a, b)$. The same holds at column $a-1$ ($\alpha', \beta', \gamma'$) with $u_1$ and the tiles with $x_i = a - d$.* (e) *$u_0 \le h$ and $u_1 \le h$. If a top tile has $d$ cells in row $b-1$ (such as $W$ in $\gamma$, or the tile of a one-tile block of Lemma 20), then $h = d$ and an end with $u = d$ is in state $\alpha$ (or $\alpha'$). So $\gamma$ occurs only with $\alpha'$, and $\gamma'$ only with $\alpha$.* (f) *No spine column of a non-top tile is in* Top. *If $h < d$, no thick-spine column of any vertical tile is in* Top.

*Proof.* (a) Boxes lie in $R$; Lemma 24(a); $h \le d < a$. (b) Let $(0, j) \notin T$ and $(0, i) \in T$ with $j < i < d$. The tile $V$ of $(0, j)$ is vertical and has a cell in the top row of its box, a row at least $a > d$, so a chain in $V$ runs from $(0, j)$ to a cell $(x', d)$ inside the rectangle $[0, a) \times [0, d+1)$. The cells $(0,0)$, $(0, j)$, $(0, i)$, $(x', d)$ lie in this order on its boundary and belong to $T, V, T, V$, against Lemma 10. (c) The end columns of box$(T)$ go to the end rows of box$(V)$, row 0 to the spine; apply (b). (d) A box contains column 0 only if $x_i = 0$, and then the spine is column 0 or $d-1$; by (a) at most one tile of each kind. Column 0 above row $u_0$ is covered by these tiles. One tile, spine 0: $u_0 + a = a + h$. Two tiles, $V$ with spine 0 in rows $[y, y+a)$ and $V'$ with $t$ cells in column 0: $u_0 + t = h$, and $V'$ cannot have cells both below and above the spine of $V$, because its box has $a$ rows; so $y = u_0$ ($\gamma$) or $y = h$ ($\beta$). One tile $V$ with spine $d-1$: $V$ holds the rows $[u_0, b)$ of column 0, so $V$ is a top tile, $t = a + h - u_0 \ge a + h - d$, and these are the top $t$ rows of box$(V)$, so Top is an end interval of $t$ columns. If $h = d$ then $t = a$, but the spine column of a non-top tile is not in Top. If $h < d$, the cell $(d-1, d-1)$ is in the spine of $V$, so Top $= [0, t)$ with $t \le d-1$ (the other end would need $t \le a - d$), hence $a \le 2d - 1 - h$; but $V$ is left-type with the number $d$ of Lemma 20(b), and the block starting at column $d$ has at least $d$ columns, so $a \ge 2d$. So this fourth state does not occur. (e) From (d); a full end row of a box is the image of a full end column of $T$, and $u_i \le h \le d$; $\beta$ and $\gamma$ have $u_0 < h$. (f) Such a cell $(x, d-1)$ lies in the vertical tile, whose rows start at $y_i \le d-1$. □

**Proposition 31.** *There are not three top tiles.*

*Proof.* With three top tiles a block of Lemma 20 holds one tile, there are at least two blocks, and $a \ge 2d+1$. By Lemma 30(e), $h = d$; take $u_0 = d$ (mirror image), state $\alpha$. The fourth vertical tile $Z$ has $y_Z = s < d$, and a cell of a row $r$ with $s \le r < d$ outside $T$ is in $Z$. So $T$ contains every cell of its box outside the columns $[x_Z, x_Z + d)$ (the *window*): its columns outside the window are full. If $x_Z = a - d$ (states $\beta'$ or $\gamma'$), then $t = d - u_1 \le d - 1$, but Top contains $[0, a-d)$, at least $d+1$ columns. So the right state is $\alpha'$: $u_1 = d$, every top tile has $d$ cells in row $b-1$, $a = 3d$, the top tiles have the boxes $[0, d)$, $[d, 2d)$, $[2d, 3d)$ in the columns and $[d, 4d)$ in the rows, and $1 \le x_Z \le 2d - 1$. The rows of box$(Z)$ are the images of the columns of $T$, so all rows of box$(Z)$ outside $d$ consecutive ones are full; at least $d + s$ of the full ones are rows $r \ge d$ of $R$. In such a row $Z$ holds exactly the window, so a top tile $W'$ whose box meets the window has fewer than $d$ cells there. $W'$ then has at least $d + s$ rows of its box that are not full, while $T$ has at most $d$ columns that are not full, a contradiction. □

**Proposition 32.** *There are not two top tiles.*

*Proof.* Let $W_1, W_2$ be the top tiles, with $x = 0$ and $x = a - d$, and $Z_1, Z_2$ the non-top tiles.

*Step 1 (both top tiles outer).* Row $b-1$ is $[0, l_1)$ in $W_1$ and $[a - l_2, a)$ in $W_2$, with $l_1 + l_2 = a$; row $h$ holds $[0, m_1)$ of $W_1$, $[a - m_2, a)$ of $W_2$ and the $2s$ thick-spine cells of $Z_1$ and $Z_2$, with $l_i + m_i = u_0 + u_1$ by Lemma 30(c). So $u_0 + u_1 \le a - s$. Then $l_1, l_2$ are not $u_0$ and $u_1$ in some order: both are the same $u_E = a/2$, and the other is at most $a/2 - s$. Take $E = 0$ (mirror image): $u_0 = a/2 > u_1$, and both top tiles reverse the order. State $\alpha'$ would give $u_1 = h \ge u_0$, so the right state is $\beta'$, with $t = h - u_1$; $\beta$ on the left would give $t = h - u_0$, so the left state is $\alpha$, with $u_0 = h$ and $a = 2h$.

*Step 2 ($W_2$ inner).* By Lemma 30(e) the states are $\alpha$ and $\gamma'$, $h = d$ and $u_0 = d$; $W_1$ and $W_2$ are right-type with numbers 0 and $a - d$, so $a = 2d$ by Lemma 20(c); both have $d$ cells in row $b-1$, the image of column 0, so both reverse the order.

So in every case, up to the mirror image, $a = 2h$, $b = 3h$, $u_0 = h$, $W_1$ has spine column 0 and box $[0, d) \times [h, 3h)$, no

other tile has $x_i = 0$, and $W_1$ reverses the order. Put $e = a-d$. Then $1 \le x_{Z_1} \le e-1$; write $x_1 = x_{Z_1}$.

*Tail for $h = d$.* Suppose $a = 2d$, the boxes of $W_2$ and $Z_2$ have the columns $[d, 2d)$, $y_2 = u_1 < d$, Top $= [0, d-u_1)$, $Z_2$ contains $[d, 2d) \times \{u_1\}$, and $Z_1$ has no cell $(d, r)$ with $r \ge u_1$. $Z_1$ has a cell in column $d$ ($1 \le x_1 \le d-1$), so $y_1 < u_1$ and $s = y_1$. Its spine is not column $x_1 + d - 1$, because the cell of row $u_1$ there is in $Z_2$; so it is column $x_1$ and $u[x_1] = s$ (Lemma 24(c)), so $x_1 \notin$ Top. $T$ contains $[0, x_1) \times [0, d)$, because rows $r$ with $s \le r < d$ outside $T$ belong to $Z_1$ and $Z_2$. So $t = x_1 = d - u_1$. Row $r^* = 3d - 1 - x_1$ of $W_1$ is the image of column $x_1$: only its $s$ thick-spine cells. $r^*$ is above the top row $s + 2d - 1$ of $Z_1$, and $T$, $W_2$, $Z_2$ have no cell $(s, r^*)$, so this cell is uncovered.

*Step 3 (states $\alpha, \gamma'$).* $Z_2$ has spine column $2d-1$ and $y_2 = u_1$; $W_2$ has spine column $d$ and meets column $2d-1$ in the top $t = d - u_1$ rows, so Top $= [0, t)$. The $t$ cells of $Z_2$ in column $d$ lie in the rows $[u_1, d)$ (above is the spine of $W_2$), the bottom $t$ rows of its box, so $Z_2$ keeps the order and contains the image $[d, 2d) \times \{u_1\}$ of column 0; $Z_1$ has no cell of column $d$ in those rows or above. The tail applies.

*Step 4 (states $\alpha, \beta'$).* $Z_2$ has box $[e, 2h) \times [y_2, y_2 + 2h)$ and spine column $e$, and meets column $2h-1$ in the rows $[u_1, h)$; $t = h - u_1$. The $t$ cells of $W_2$ in column $e$ lie above the spine of $Z_2$, in the rows $3h - 1 - x$ with $x \in$ Top, so Top lies in $[0, h - y_2)$; hence $Z_2$ keeps the order and Top $= [u_1 - y_2, h - y_2)$. (1) $T$ contains $[0, x_1) \times [0, h)$. (2) Its images in $W_1$ and $W_2$ are $[0, h)$ and $[h, 2h)$ in the columns, in the rows $[3h - x_1, 3h)$, which fill these rows; so $T$ meets column $j < x_1$ in $[0, h)$ exactly. (3) Columns $x < x_1$ hold only $T$ and $W_1$, so $v[x] = 2h$ there and $x_1 \le s$. $h = d$. Then $e = d$, and (1) puts 0 in Top: $y_2 = u_1$, Top $= [0, d - u_1)$, $Z_2$ contains $[d, 2d) \times \{u_1\}$; column $d$ is the spine of $Z_2$ and above it are the $t$ cells of $W_2$. The tail applies. $h < d$. By Lemma 30(f), $u_1 - y_2 \ge s$, and $u_1 \le a/2 - s = h - s$ by Step 1, so $s \le y_2 \le u_1 - s \le h - 2s$. *Spine of $Z_1$ in column $x_1$.* Column $x_1$ holds $T$, $W_1$ and the spine, so $u[x_1] + v[x_1] = h$, so $x_1 = s$, and $T$ meets column $s$ in $[0, y_1)$. Row $r = 3h - 1 - s$ then has $[0, y_1)$ in $W_1$ and $[2h - y_1, 2h)$ in $W_2$, no cell of $T$, and none of $Z_2$ ($y_2 + 2h - 1 < r$). So $Z_1$ has cells there, $y_1 \ge h - s$; and its spine cell $(s, r)$ is not in $W_1$, so $y_1 \le s$. So $h \le 2s$, against $h \ge 3s$. *Spine of $Z_1$ in column $c = x_1 + d - 1$.* Row $h$ gives $c < 2h - u_1$, so $u_1 \le e - x_1$; Top avoids $e$ and starts below it, so $h - y_2 \le e$. Hence $2e \ge h + x_1 + s$, and $\kappa = c - e = x_1 + 2h - 2e - 1 < h$, so the cell $(0, \kappa)$ is in $T$. Its image in $Z_2$ is $(c, y_2)$, outside the spine of $Z_1$, so $y_2 < y_1$ and $s = y_2$. Its image in $Z_1$ is $(e, y_1)$ or $(e, y_1 + 2h - 1)$; the first is in the spine of $Z_2$, so $Z_1$ reverses the order. The cells of $Z_1$ in column $x_1$ are then in the rows $y_1 + 2h - 1 - x$ with $x \in$ Top $= [u_1 - s, h - s)$, all above row $h$; a column $x_1 < s$ would be in the thick spine of $W_1$, so $x_1 = s$. Column $s$ holds $T$, $W_1$, $Z_1$ only, so $T$ contains $\{s\} \times [0, h)$; as in (2), $u[s] = h$, and $v[s] = 3h - h - t$. Row $s$ outside $T$ is the bottom row of $Z_2$ alone, $h$ cells, so $v[s] = h$. So $t = h$, against $t = h - u_1$. □

**Proposition 33.** *Case (II.2) of Theorem 16 does not occur.*

*Proof.* By Corollary 23, $d < a$ and there are two or three top tiles. By Proposition 29, $h \le d$, and $h \ge 1$. Propositions 31 and 32 exclude both numbers. □

**3.5 Case (II.3) with a square box.** In case (II.3) with the isometries fixed, $T'$ is the down-type tile with spine row $b-1$, so box$(T') = [0, a) \times [b-d, b)$; $V_1, V_2, V_3$ are the vertical tiles. Let $F$ be the reflection $y \mapsto b - 1 - y$ of $R$, and let $s$ be the number of full bottom rows of $T$.

**Lemma 34** (Order on the walls). *Let $d = a$.* (a) *Every tile meets column 0 and column $a-1$, each in an interval of rows.* (b) *The five tiles occur in the same order, from the bottom, on both walls: $T$, the vertical tiles (call them $V_1, V_2, V_3$ in this order), $T'$.* (c) *$T' = F(T)$ or $T'$ is the image of $T$ under the half turn of $R$. A vertical tile with spine column 0 has $a$ cells in column 0 and $t$ in column $a-1$; with spine column $a-1$, $t$ and $a$.*

*Proof.* (a) Every box has the columns $[0, a)$. If $(0, r_1), (0, r_3)$ are in a tile $A$ and $(0, r_2)$ in another tile $B$, $r_1 < r_2 < r_3$, take a cell of $B$ in column $a-1$; the four cells are interlaced on the boundary of $R$, against Lemma 10. (b) If $A$ is below $B$ on the left wall and above it on the right, four such cells are interlaced again. $T$ holds $(0, 0)$ and $T'$ holds $(0, b-1)$. (c) The isometry carrying $T$ to $T'$ maps box$(T)$ onto box$(T')$ and row 0 onto row $b-1$; there are two. The far column of a vertical tile is the image of row $d-1$ of $T$. □

**Proposition 35.** *Case (II.3) with $d = a$ does not occur.*

*Proof.* If $T$ contains column 0 or column $a-1$ of its box in full, Step 3 of Proposition 19 applies, since it uses only that $T$ contains row 0 and a perpendicular full side of its square box, and gives an even number of tiles. So $u_0, u_1 \le a - 1$, and then $t \le a - 1$, because $t = a$ puts $(0, a-1)$ in $T$ and by Lemma 34(a) all of column 0. Let $q$ be the number of vertical tiles with spine column 0. If $T'$ is the half turn of $T$, counting the two walls gives $u_0 + u_1 + qa + (3-q)t = b = u_0 + u_1 + (3-q)a + qt$, so $(2q-3)(a-t) = 0$ and $t = a$, excluded. So $T' = F(T)$, and $2u_0 + qa + (3-q)t = b = 2u_1 + (3-q)a + qt$ gives $2(u_0 - u_1) = (3-2q)(a-t)$, which is not zero because $3-2q$ is odd and $t < a$. Take $u_0 < u_1$ (mirror image). (1) $V_1$ meets column 0 from row $u_0$ and column $a-1$ from row $u_1$. If its spine were column $a-1$, its box would start at row $u_1$ and not hold $(0, u_0)$. So the spine of $V_1$ is column 0 and box$(V_1) = [0, a) \times [u_0, u_0 + a)$. (2) Row $u_0$ lies in $T$ and $V_1$: $T'$ and the other vertical tiles meet column 0 above row $u_0 + a - 1$, a tile with spine column 0 then starts at a row at least $u_0 + a$, a tile with spine column $a-1$ starts above row $u_1$, and $T'$ starts at a row at least $2u_0$ (column 0 holds $u_0$ cells of $T$, $a$ of $V_1$ and $u_0$ of $T'$ at the top). (3) $F$ maps the tiling to a tiling with the same $T$ and $T'$, whose tiles in wall order are $T, F(V_3), F(V_2), F(V_1), T'$, with the images of the spines. By (1) and (2) applied to it, $F(V_3)$ has spine column 0, box $[0, a) \times [u_0, u_0 + a)$, and its cells in row $u_0$ are those of row $u_0$ outside $T$, as for $V_1$. (4) Two isometries carry box$(T)$ onto that box with row 0 onto its left column; under one the bottom row of the box receives $u_0$ cells, under the other $u_1$. $V_1$ and $F(V_3)$ have the same cells in row $u_0$ and $u_0 \ne u_1$, so $V_1 = F(V_3)$. (5) So $V_2$, the complement of $T, T', V_1, V_3$, satisfies $F(V_2) = V_2$. Carried back to $T$ by the isometry of $V_2$, $F$ becomes the reflection of box$(T)$ in a vertical line, $x \mapsto a - 1 - x$, which preserves $T$. Then $u_0 = u_1$, a contradiction. □

**3.6 Case (II.3) with $d < a$.** Let $G$ be the isometry with $T' = G(T)$: $G = F$ or the half turn $H$ of $R$, because $T'$ has row $b-1$ as its spine. $T'$ has $s$ full top rows. A *bottom tile* is a vertical tile with $y_i = s$ and a *top tile* one with $y_i = h - s$.

**Corollary 36.** *In case (II.3) with $d < a$: $2 \le d < a \le \min(3d, b-2)$, and every vertical tile has $s \le y_i \le h - s$, so $h \ge 2s$.*

*Proof.* The lower bounds and $a \le b-2$ are in Theorem 16. If $b \ge 2d+1$, row $d$ has no cell of $T$ or $T'$ and at most $d$ cells of each vertical tile, so $a \le 3d$; if $b \le 2d$, $a \le b-2$ gives $a < 2d$. The top $s$ rows are in $T'$ and the bottom $s$ in $T$. □

**Lemma 37.** *Let $d < a$.* (a) *$T$ meets column 0 in the rows $[0, u_0)$ and column $a-1$ in $[0, u_1)$; $T'$ meets them in their top rows, $u_0$ and $u_1$ of them if $G = F$, $u_1$ and $u_0$ if $G = H$.* (b) *$T'$ has no cell in row $s$ and $T$ none in row $b-1-s$. A bottom tile and a top tile exist.* (c) *$h \ge 2s+1$; so no tile is both.*

*Proof.* (a) As in Lemma 30(b): the tile of a cell $(0, j)$ between cells of $T$ is vertical or is $T'$, and has a cell in a row at least $d$. (b) $b - d \le s$ would give $a \le b - 2s \le d - s$. A cell of row $s$ outside $T$ is in a vertical tile, which has $y_i = s$; reflect for the top. (c) Let $h = 2s$. Every vertical tile has $y_i = s$, so the three lie in $M = [0, a) \times [s, s+a)$ with spines full columns of $M$; each lies left or right of the spine column of another, so its box contains no other spine column. Two of the three have the spine on the same side of the box; their boxes are disjoint, so $a \ge 2d$ and $b \ge 2d+2$. For every $r$ with $d \le r < b - d$, row $r$ has no cell of $T$ or $T'$ and lies in $M$. With the numbers of Lemma 20(b), each block holds a tile and has at least $d$ columns (by the row $r$), and with three tiles some block holds one tile $W$. Every cell of that block in row $r$ is in $W$, so the block has $d$ columns and $W$ contains the rows $d-s, \dots, a+s-d-1$ of its box in full. This set of rows is its own reversal, so $T$ contains the columns $[d-s, a+s-d)$ of its box in full. Then every vertical tile has $d$ cells in row $r$, so $a = 3d$ and the boxes have columns $[0, d)$, $[d, 2d)$, $[2d, 3d)$. The spine of the middle tile is column $d$ or $2d-1$, a full column of $T$, and its cell in row $s$ is in both, a contradiction. □

**Proposition 38** (No two vertical tiles are stacked). *In case (II.3) with $d < a$ the row ranges $[y_i, y_i + a)$ of any two vertical tiles overlap. So $2s+1 \le h \le a + 2s - 1$ and each of the three spines meets every row from $h - s$ to $s + a - 1$.*

*Proof.* Suppose two have disjoint ranges, $y' \ge y + a$. The rows $y+a-1, \dots, y'$ are at least $d$ and below $b-d$, so they hold only vertical tiles, at least two each ($d < a$). Hence $y' = y + a$ and the third tile has cells in rows $y+a-1$ and $y+a$. Applied to a bottom and a top tile ($h - 2s \ge a$): $h = a + 2s$, $b = 2a + 2s$, and the tiles are $V_B$ (rows $[s, s+a)$), $V_T$ (rows $[s+a, s+2a)$) and $V_M$ (rows $[y_m, y_m + a)$, $s+1 \le y_m \le s+a-1$). $V_M$ covers every row $r$ with $d \le r < b-d$, so $a + 2s - d \le y_m \le d$. In the rows $[d, s+a)$ and $[s+a, b-d)$ two boxes cover all columns; after a mirror image $V_B$ and $V_T$ have columns $[0, d)$ and $V_M$ has $[a-d, a)$. Rows $r$ with $s \le r < y_m$ outside $T$ belong to $V_B$, so $T$ contains $[d, a) \times [0, y_m)$ and $u_1 \ge y_m > s$. Column 0 meets the boxes of $T, T', V_B, V_T$; column $a-1$ those of $T, T', V_M$. A vertical tile has $a$ cells in such a column (spine) or $t$. $G = H$. Both columns hold $u_0 + u_1$ cells of $T$ and $T'$, so the count of $V_M$ equals the sum for $V_B$ and $V_T$: $a = 2t$, $V_M$ has spine $a-1$, and $V_B$, $V_T$ have spine $d-1$. Then $u_1 = y_m$, $u_0 = a + 2s - y_m$, and in column 0 the cells of $V_B$ (below row $s+a$) and of $V_T$ (from it) fill the rows $[u_0, u_0 + a)$, so $y_m = s + a/2 = u_0 = u_1$, and $V_B$ has there the top $a/2$ rows of its box, so Top is an end interval of $a/2$ columns. It contains 0 or $a-1$, so $u_0 = u_1 = d$, and then both: $t = a$, a contradiction. *$G = F$, one of $V_B, V_T$ with spine column 0, the other with $d-1$.* The spine in column 0 gives $u_0 = s$; the count gives $t = a$, so $(0, d-1) \in T$ and $u_0 = d$. *$G = F$, both spines in column 0.* $u_0 = s < u_1$. *$G = F$, both spines in column $d-1$.* Column 0 gives $2u_0 + 2t = b$, and $V_B$ has there the rows $[u_0, s+a)$, the top $t$ rows of its box, so Top is an end interval of $t = a + s - u_0 > a - d$ columns. It does not contain $d-1$ (the spine of $V_B$), so it is $[0, t)$ and $u_0 = d$. If $u_1 = d$ too, Top contains both ends, so $t = a$ and $s = d$. So $u_0 \ne u_1$. *In the last two cases* compare with the reflected tiling $F(T') = T$, $F(T) = T'$, $F(V_T)$, $F(V_M)$, $F(V_B)$. $F(V_T)$ has the box and the spine column of $V_B$. In the rows $r$ with $s \le r < \min(y_m, a+2s-y_m)$, at least one, both $V_B$ and $F(V_T)$ are the complement of $T$. Two isometries carry box$(T)$ onto that box with row 0 on that spine, and their images of $T$ have $u_0$ and $u_1$ cells in the bottom row. So $V_B = F(V_T)$. Then $V_M$, the complement, has $F(V_M) = V_M$, and $T$ is preserved by $x \mapsto a - 1 - x$, so $u_0 = u_1$, a contradiction. □

**Lemma 39** (Wall states). *Let $d < a$, $f = h - 2s$ (so $1 \le f \le a-1$), and let $u'_0, u'_1$ be the numbers of cells of $T'$ in columns 0 and $a-1$.* (a) *At column 0 exactly one of these holds (and likewise at column $a-1$ with $u_1, u'_1$). o: no vertical tile has $x_i = 0$; $u_0 + u'_0 = b$, so $b \le 2d$. α: one, with spine column 0; its $y_i$ is $u_0$ and $u_0 + u'_0 = h$. β: one, with spine column $d-1$; it meets column 0 in the rows $[u_0, b - u'_0)$, $t = a + h - u_0 - u'_0$, and* Top *is an interval of columns. γ: two, $V$ with spine column 0 and $V'$ with spine column $d-1$; $t = h - u_0 - u'_0$.* (b) *If $G = H$ and $K = u_0 + u_1$, the two walls are in the same state, $(o, o)$ with $K = b$, $(\alpha, \alpha)$ with $K = h$, or $(\beta, \beta)$ with $t = a + h - K$.*

*Proof.* (a) A box contains column 0 only if $x_i = 0$; the spine is then column 0 or $d-1$, and distinct tiles have distinct spine columns by Proposition 38. Between the rows of $T$ and of $T'$ (Lemma 37(a)) column 0 is covered by these tiles: a spine has $a$ cells, a far column $t$, and the box of $V'$ has $a$ rows, so it is not on both sides of a spine. (b) Both walls have $u_0 + u'_0 = u_1 + u'_1 = K$. States $o$ and $\alpha$ would give $b = h$, and $o$ and $\beta$ would give $t = 0$. If one wall is in state $\gamma$, two of the three vertical tiles are on that wall, and no box touches both walls. A wall in state $\alpha$ and one in $\beta$ give $t = a$; the spine column of a bottom tile is not in Top. □

**Lemma 40** (End rows against spines). *Let $c_i$ be the spine column of a vertical tile $V_i$; the bottom row and the top row of $V_i$ are its cells in the bottom and the top row of its box.* (a) *Let $X, Y$ be different vertical tiles with $y_X \le y_Y$. The top row of the box of $X$ is a spine row of $Y$, and the bottom row of the box of $Y$ is a spine row of $X$. So the top row of $X$ does not contain column $c_Y$, and the bottom row of $Y$ does not contain column $c_X$.* (b) *If two different vertical tiles $X$ and $Y$ have the spine on the same side of the box, then $|c_X - c_Y| \ge \min(u_0, u_1)$.* (c) $\min(u_0, u_1) \le a - d$.

*Proof.* (a) By Proposition 38, $y_Y \le y_X + a - 1$, so $y_Y \le y_X + a - 1 < y_Y + a$ and $y_X \le y_Y < y_X + a$. (b) Let both spines be on the left, $Z$ the tile with the smaller spine column and $Z'$ the other. One end row of the box of $Z$ is a spine row of $Z'$ (the top row if $y_Z \le y_{Z'}$, the bottom row otherwise). By Lemma 37(a) and Lemma 18 it is an interval of $u_0$ or $u_1$ columns from $c_Z$ to the right, and by (a) it does not contain $c_{Z'}$. Mirror image for the right. (c) Two of the three tiles have the spine on the same side; their spine columns both lie in $[0, a-d]$ or both in $[d-1, a-1]$. □

**Lemma 41** (The extreme rows). *Let $P_b$ and $Q_t$ be the numbers of bottom and of top tiles, so $P_b, Q_t \ge 1$ and no tile is both.* (a) *The cells of row $s$ outside $T$ are the cells of the bottom rows of the bottom tiles, and the cells of row $b-1-s$ outside $T'$ are the cells of the top rows of the top tiles. $G$ maps the first set onto the second (same columns for $G = F$, columns reversed for $G = H$), and the image of the tiling under $G$ is a tiling with the same two sets $T$ and $T'$, exchanged by $G$, in which the images of the top tiles are the bottom tiles.* (b) *If $u_0 = u_1$ then $P_b = Q_t = 1$. If $P_b = 2$ and $Q_t = 1$, then there are integers $m \ge 1$ and $c$ such that $\min(u_0, u_1) = m$ and $\max(u_0, u_1) = 2m$, the bottom rows of the two bottom tiles are the halves $[c-m, c)$ and $[c, c+m)$ of an interval $I$, $I$ is the image under $G$ of the top row of the top tile $W$, that row has $2m$ cells, and the bottom row of $W$ has $m$.* (c) *If two different vertical tiles $X, Y$ have $y_X = y_Y$, the box of $X$ does not contain the column $c_Y$.*

*Proof.* (a) By Lemma 37(b), $T'$ has no cell in row $s$, and a vertical tile with a cell in row $s$ has $y_i = s$; reflect for the top. $G$ exchanges $T$ and $T'$ and the rows $s$ and $b-1-s$. (b) Every end row has $u_0$ or $u_1$ cells, and by (a) the $P_b$ bottom rows and the $Q_t$ top rows have the same total. If $u_0 = u_1$ this gives $P_b = Q_t$, so both are 1. If $P_b = 2$ and $Q_t = 1$: two of the numbers $u_0, u_1$ add up to a third, so the two are equal to $m = \min$ and the third is $2m = \max$; two disjoint intervals of $m$ columns whose union is an interval of $2m$ columns are its halves; $u_0 \ne u_1$, so the other end row of $W$ has $m$ cells. (c) $T$ is connected with cells in rows 0 and $d-1$, so it has a cell in every row of its box, and $X$ has a cell in every column of its box, in the rows $[y_X, y_X + a)$. In column $c_Y$ these rows are the spine of $Y$. □

**Proposition 42.** *$P_b = Q_t = 1$ does not occur.*

*Proof.* Let $B$ be the bottom tile, $W$ the top tile and $U$ the third. Write $n_X(r)$ for the number of cells of $X$ in row $r$; then $n_{G(X)}(r) = n_X(b-1-r)$. $u_0 \ne u_1$ (*either* $G$). $G(W)$ is a tile of the image tiling with the rows $[s, s+a)$, and by Lemma 41(a) its bottom row is the bottom row of $B$. The number of cells of that row tells which of the columns $0, a-1$ of $T$ it is the image of, so $B$ and $G(W)$ carry the columns of $T$ to their box rows in the same order and have the same number of cells in every row of $R$. So $n_{G(W)} = n_B$, and then $n_{G(B)}(r) = n_B(b-1-r) = n_{G(W)}(b-1-r) = n_W(r)$. Row $r$ has $a$ cells in both tilings, $n_T + n_{T'} + n_B + n_W + n_U = a = n_T + n_{T'} + n_{G(B)} + n_{G(W)} + n_{G(U)}$, so $n_U(r) = n_U(b-1-r)$ for every $r$. The rows of $U$, $[y_U, y_U + a)$, are then exchanged by $r \mapsto b-1-r$, so the bottom and the top row of its box have the same number of cells; they have $u_0$ and $u_1$, a contradiction. $u_0 = u_1 = u$. Both end rows of a vertical tile $V_i$ are then the same interval of columns $J_i$, the $u$ columns from $c_i$ into its box. By Lemma 40(a), $J_i$ contains no spine column of another tile, so the three $J_i$ are different. $G = F$: by Lemma 41(a), $J_B = J_W$, a contradiction. $G = H$: by Lemma 41(a), $J_B$ is the reversal of $J_W$, and by Lemma 39(b) the walls are $(o,o)$, $(\alpha,\alpha)$ or $(\beta,\beta)$. $(o,o)$: $2u = b$, but $u \le d$ and $u \le a-d$ (Lemma 40(c)) give $2u \le a$. $(\alpha,\alpha)$: the two wall tiles have $y = u_0$ and $y = h - u_0$, both $h/2$, so neither is a bottom or a top tile and the third would be both. $(\beta,\beta)$: the wall tiles have the intervals $[d-u, d)$ and $[a-d, a-d+u)$, each the reversal of the other. One of $B, W$ is a wall tile; the interval of the other is the reversal, so it is the other wall tile. Each meets its wall in the rows $[u, b-u)$, the image of Top (Lemma 39). If $B$ and $W$ carried the columns of $T$ in the same order, these two sets of rows would differ by the shift $y_W - y_B = f \ge 1$. So the orders are opposite, and $H(W)$, which has the columns, the spine column and the rows of $B$ and the order opposite to $W$, is $B$. Then $H(B) = W$, $H(U) = U$, $T$ is preserved by the half turn of its box, and $t = a$. But the spine column of $B$ has a cell in row $d-1$ ($s \le d-1 < s+a$), so it is not in Top. □

**Proposition 43.** *$P_b = Q_t$.*

*Proof.* Let $P_b \ne Q_t$. Three vertical tiles and Lemma 37(c) leave $(2,1)$ and $(1,2)$, and after replacing the tiling by its image under $G$ (Lemma 41(a)), $P_b = 2$ and $Q_t = 1$. Notation of Lemma 41(b): bottom tiles $B_1, B_2$ with bottom rows $[c-m, c)$ and $[c, c+m)$, top tile $W$. *The bottom boxes.* The spine of $B_1$ is an end of its bottom row. If its box extended to the right from column $c-m$, it would contain $I$ ($2m \le d$) and with it the spine column of $B_2$, against Lemma 41(c); for $m = 1$ the same holds for the box $[c-1, c-1+d)$. So $B_1$ has spine column $c-1$ and box columns $[c-d, c)$; likewise $B_2$ has spine column $c$ and box columns $[c, c+d)$. In particular $a \ge 2d$. $G = F$. The top row of $W$ is $I$, with the spine of $W$ at one end; the bottom row of $W$ is the $m$ columns from that end, one of the halves of $I$, and contains $c-1$ or $c$. It lies in row $s+f$, a spine row of both bottom tiles ($f \le a-1$), against Lemma 40(a). $G = H$. The top row of $W$ is the reversal of $I$. $(o,o)$ needs $b \le 2d$, but $b > a \ge 2d$; so by Lemma 39(b) exactly one box lies at each wall, and the same holds for the image tiling, which has the same $T$ and $T'$. If $W$ is at a wall, the reversal of $I$ lies in its box, so $I$ lies in the $d$ columns at the other wall. The tile there is $B_1$ or $B_2$, and its box contains $I$ and so the spine column of the other, against Lemma 41(c). If $W$ is at no wall, $B_1$ and $B_2$ are the wall tiles: $c = d$, $a = 2d$, $I = [d-m, d+m)$ is its own reversal, and the argument for $G = F$ applies. □

**Proposition 44.** *Case (II.3) of Theorem 16 does not occur.*

*Proof.* $d = a$: Proposition 35. $d < a$: Corollary 36, Lemma 37 and Proposition 38 give the setting; $(P_b, Q_t)$ is $(1,1)$, $(2,1)$ or $(1,2)$, and Propositions 42 and 43 exclude all three. □

### 3.7 Three copies and case (II.1).

**Proposition 45** (Three tiles, two of them horizontal). *No rectangle $R(a,b)$ has a 3-copy tiling by a non-rectangular polyomino in which $T$ contains row 0, a second tile $T'$ is horizontal with spine row $b-1$, and the third tile $V$ is vertical.*

*Proof.* $T'$ is down-type with box $[0,a)\times[b-d,b)$, and $V$ has box $[x_0,x_0+d)\times[y,y+a)$ with $2\le d\le a\le b-1$ and spine an end column $c$ of it. With $v[j]=0$ outside $[0,d)$: $v[0]=a$, and $v[j]$ is positive for $0\le j<d$, because $T$ is connected and has cells in rows 0 and $d-1$. (1) The isometry carrying $T$ to $T'$ keeps the axis directions, so $T'$ has $v[b-1-r]$ cells in row $r$ and, in all columns $x$, either $u[x]$ cells or, in all, $u[a-1-x]$. By Lemma 18, $V$ has $u[i]$ cells in row $y+i$ for all $i$, or $u[a-1-i]$ for all $i$, and $v[j]$ cells in the column at distance $j$ from $c$. (2) Row $r$ has $a$ cells, so $n_V(r)=n_V(b-1-r)$. $V$ has a cell exactly in the rows $[y,y+a)$, so that interval is symmetric and $n_V(y+i)=n_V(y+a-1-i)$, so $u[i]=u[a-1-i]$ for all $i$, and $T'$ has $u[x]$ cells in column $x$. (3) Column $x$ has $b$ cells, so the number $m_V(x)=b-2u[x]$ of cells of $V$ in it satisfies $m_V(x)=m_V(a-1-x)$. $V$ has a cell exactly in the $d$ columns of its box, so that interval is symmetric, $c$ is one end of it and $a-1-c$ the other, at distance $d-1\ge1$. Then $v[d-1]=m_V(a-1-c)=m_V(c)=v[0]=a$, so $T$ contains row $d-1$ and $T'$ contains row $b-d$. (4) If $b-d<d-1$, then $T'$, connected, with cells in rows $b-d$ and $b-1$, has a cell in row $d-1$, which is in $T$; so $b-d\ge d-1$, and $b-d\ne d-1$ because these rows lie in different tiles. If $b-d=d$, then $V$ has no cell in rows $0,d-1,d,b-1$, and its $a\ge d$ consecutive rows do not fit in the $d-2$ rows strictly between 0 and $d-1$ or in those strictly between $d$ and $b-1$. So $b-d>d$; $V$ has no cell in rows $0,d-1,b-d,b-1$ and has $a\ge d$ consecutive rows, while at most $d-2$ rows lie strictly between 0 and $d-1$, and likewise at the top, so $V$ lies in the rows $[d,b-d)$. Then the rows $[0,d)$ hold no cell of $V$ and none of $T'$, so $T$ is its box. □

*Proof of Theorem 9.* If $a=1$ or $b=1$, every connected set is a rectangle. Otherwise the four corner cells lie in three tiles, so a tile holds two of them, adjacent ones by Lemma 12, and after an isometry it contains row 0 (Lemma 13). Lemmas 14 and 15 apply with $k=3$: $3\ge2t$ gives one block, which is $R$ and holds $T$, at most one down-type horizontal tile, with spine row $b-1$, and at least one vertical tile. Two vertical tiles: Proposition 22. One: Proposition 45. The statement is in print in Maltby [2]. □

**Proposition 46.** *Case (II.1) of Theorem 16 does not occur.*

*Proof.* In case (II.1) a rectangle is tiled by exactly three of the tiles, against Theorem 9. □

*Proof of Proposition 3.* By Theorem 16, a tiling either satisfies (I), which is the statement, or is of type (II), and Propositions 46, 33 and 44 exclude (II.1), (II.2) and (II.3). □

## 4. CASE (I): THE SIDES OF THE RECTANGLE

In case (I) the four corner cells lie in the tiles $T_1,T_2,T_3,T_4$ in the counterclockwise order of Corollary 17, and $T_5$ is the fifth tile. Let $p\ge q$ be the side lengths of box$(P)$.

**Lemma 47** (Enclosure). *Let $T,T'$ be different tiles of a tiling, let a path of cells $\gamma$ in $T$ join two cells on the boundary of $R$, and let $\Omega$ be a union of connected components of $R\setminus\gamma$. Fix a horizontal direction $e$ and a vertical direction $f$, the same for all cells, and suppose (i) every cell of $\Omega$ has a cell of $\gamma$ strictly beyond it in direction $e$ in its row and one strictly beyond it in direction $f$ in its column; (ii) if $\Omega$ has a cell on the side of $R$ opposite to $e$, then $T$ has a cell on that side, and likewise for $f$. If $T'$ has a cell in $\Omega$, then $T'$ lies in $\Omega$, both sides of* box$(T')$ *are shorter than the parallel sides of* box$(T)$*, and $T$ and $T'$ are not congruent.*

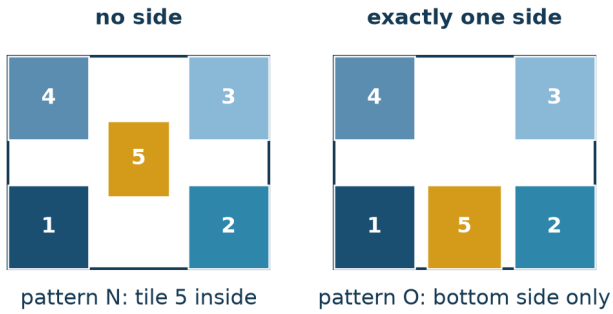



**Figure 2. Case (I) on the sides of $R(a,b)$.** The four corner tiles and the left-over tile $T_5$, drawn schematically (not an actual tiling). Each side reads a corner tile, then $T_5$ or nothing, then a corner tile. By Proposition 6, $T_5$ meets at most one side; the patterns with $T_5$ on no side and with $T_5$ on one side are patterns N and O of Section 5, which excludes both.

*Proof.* $T'$ is connected and disjoint from $\gamma$, so it stays in one component of $R\setminus\gamma$, inside $\Omega$. Take $e$ to the right and $f$ upward. By (i) the largest column of $T'$ is smaller than the largest column of $\gamma$, hence of $T$. Let $c$ be a cell of $\Omega$ in its least column. Its left neighbour is outside $R$, or on $\gamma$, or in the same component as $c$; the last contradicts the choice of $c$. If it is on $\gamma$, then $T$ has a cell in a smaller column than every cell of $\Omega$; if it is outside $R$, then $c$ is on the left side and by (ii) $T$ has a cell in column 0. In both cases the least column of $T$ is at most that of $T'$, so box$(T')$ has fewer columns than box$(T)$. The same argument for rows finishes the proof; congruent tiles have the same box sides. □

In the uses below, $\Omega$ is the component of a boundary cell $q$ not on the path in $R\setminus\gamma$, where $\gamma$ joins two boundary cells $p$ and $p'$. If $A$ is the set of boundary cells whose exit points lie strictly between $e(p)$ and $e(p')$ on the side of $e(q)$, then every boundary cell in $\Omega$ lies in $A$, by Lemma 10; so any row or column of a cell of $\Omega$ that ends at a boundary cell outside $A$ meets $\gamma$ before its end, and $\Omega$ has no cell on a side of $R$ with no cell of $A$.

*Proof of Proposition 4.* Let two cells of a tile $T$ on the bottom side have a cell $(x_2,0)$ of another tile $T'$ between them, where $x_1<x_2<x_3$ and $(x_1,0),(x_3,0)\in T$. Let $\gamma$ be a path in $T$ between the two outer cells and $\Omega$ the component of $(x_2,0)$ in $R\setminus\gamma$. Here the set $A$ of the remark after Lemma 47 consists of the bottom cells strictly between $x_1$ and $x_3$, so every cell of $\Omega$ has a cell of $\gamma$ strictly to its left, to its right and above it, and $\Omega$ has no cell in the left or the right column. Apply Lemma 47 with $e$ to the right and $f$ upward: condition (ii) asks for a cell of $T$ in row 0 only, and $T$ has one. So $T'$ is not congruent to $T$, a contradiction. The other sides are the same after a symmetry of the grid that maps $R$ onto $R(a,b)$ or, exchanging $a$ and $b$, onto $R(b,a)$. □

*Proof of Proposition 5.* After a symmetry of the grid (which may exchange $a$ and $b$), three pairs of a corner tile and a side

are enough. Suppose that the tile $T_3$ on the corner $(a-1,b-1)$ has a cell $(x,0)$ on the bottom side, $x \le a-2$. A path in $T_3$ from it to $(a-1,b-1)$ cuts off the corner $(a-1,0)$ of $T_2$. The component of $(a-1,0)$ has path cells strictly to the left and strictly above each of its cells, and $T_3$ has cells in column $a-1$ and in row 0; by Lemma 47, $T_2$ is not congruent to $T_3$. The same with axes exchanged rules out a cell of $T_1$ in row $b-1$ (it cuts off $(0,b-1)$, the corner of $T_4$), and a cell of $T_1$ in column $a-1$ (it cuts off $(a-1,0)$, the corner of $T_2$). □

**Proposition 48** (The sides $T_5$ can meet). *In case (I), $T_5$ meets no side of $R$, or exactly one, or exactly two opposite sides. If it meets two opposite sides, say the left and the right, then $p = a < b$, $\mathrm{box}(T_5)$ is $a$ wide and $q$ high, and every corner tile has a box $q$ wide and $a$ high with $q \le a-1$.*

*Proof.* Suppose $T_5$ meets the left side in $(0,y)$ and the bottom side in $(x,0)$. A path in $T_5$ between them cuts off the corner cell $(0,0)$, so $T_1$; in that region each cell has path cells strictly to its right and strictly above, and $T_5$ has cells in column 0 and row 0, so Lemma 47 shows that $T_1$ is not congruent to $T_5$. By symmetry $T_5$ meets no two adjacent sides. It does not meet three sides, since three sides contain two adjacent ones, nor four. If it meets the left and the right, $\mathrm{box}(T_5)$ has $a$ columns; by Proposition 5 the box of a corner tile has at most $a-1$ columns and at most $b-1$ rows, so it has its side $p$ vertical, that is, $p=a$ and $a \le b-1$. Then every corner tile has a box $q$ wide and $a$ high with $q \le a-1$. □

*Proof of Proposition 6.* By Proposition 48 it is enough to show that $T_5$ does not meet two opposite sides. Suppose it meets the left and the right side. Put $R = R(a,b)$, $\mathrm{box}(T_5) = [0,a)\times[y_5, y_5+q)$ and $m = b - y_5 - q$. In the frame of $T_5$, let $u[x]$ be the number of cells of $P$ in column $x$, and let the left and the right end columns of $P$ be the row intervals $[l, l+\lambda)$ and $[r, r+\rho)$ (Proposition 4). The corner tiles have boxes $q$ wide and $a$ high, and the following hold. (1) Every cell of $T_1$ or $T_2$ has a cell of $T_5$ strictly above it in its column, and every cell of $T_3$ or $T_4$ one strictly below: otherwise the tile together with the cells of that column above (or below) the cell would be a connected set that avoids $T_5$ and meets the bottom and the top row, against Corollary 11; hence $T_5$ meets neither the bottom nor the top side, the rows below $y_5$ lie in $T_1$ and $T_2$, the rows from $y_5+q$ lie in $T_3$ and $T_4$, and $a \le 2q$. The cell of $T_1$ in row $a-1$ has a cell of $T_5$ above it, so $y_5 \ge a+1-q$, and in the same way $m \ge a+1-q$. (2) The rows below $y_5$ are covered by $T_1$ and $T_2$, whose boxes are $a$ high from row 0; so $y_5 \le a$. If the bottom row of $\mathrm{box}(T_5)$ is in $T_5$ in full, then $T_1$ and $T_2$ tile $[0,a)\times[0,y_5)$, so $y_5 = a$, and $T_5$, $T_3$ and $T_4$ tile the rectangle $[0,a)\times[a,b)$, against Theorem 9. If $y_5 = a$ and the bottom row is not full, row $a$ has no cell of $T_1$, $T_2$ (which lie in rows below $a$) and, by (1), none of $T_3$ or $T_4$, so it lies in $T_5$ in full, which is excluded. So $y_5 \le a-1$ and, by the reflection of $R$, $m \le a-1$, and $P$ contains neither long side of its box in full. (3) Each corner tile $T_i$ puts a corner cell $K_i$ of $\mathrm{box}(P)$ on its corner of $R$, and the tiling is determined by $P$, $y_5$ and $K_1,\dots,K_4$; the tile $T_i$ meets its side of $R$ in an interval of $\beta_i$ cells starting at the corner, where $\beta_1 = y_5 + l$, $\beta_2 = y_5 + r$, $\beta_4 = m+q-l-\lambda$, $\beta_3 = m+q-r-\rho$, because $T_5$ follows the corner tiles on the sides. No two of the $K_i$ are the two ends of one long side of $\mathrm{box}(P)$: the long side through $K_i$ lies in $T_i$ along a vertical side of $R$, so by Proposition 4 its cells in $P$ form an interval, and an interval through both ends is the full side, against (2). (4) For $0 \le j < y_5$, row $j$ of $T_i$ $(i=1,2)$ is the image of column $c_i(j)$ of $P$, where $c_i(j) = j$ if $K_i$ is in the left column of $\mathrm{box}(P)$ and $a-1-j$ if it is in the right one; so $u[c_1(j)] + u[c_2(j)] = a$ for $0 \le j < y_5$; likewise at the top, with rows counted from the top, $u[c_3(j)] + u[c_4(j)] = a$ for $0 \le j < m$. In the bottom row of $\mathrm{box}(T_5)$, the tiles $T_1$ and $T_2$ have a cell, so $T_5$ does not cover $[0,q)$ or $[a-q,a)$ of that row; the top row is the same.

Let $BL, BR, TL, TR$ be the corner cells of $\mathrm{box}(P)$ in the frame of $T_5$; let $x_B$ and $x_T$ be the numbers of cells of the bottom row of $P$ starting at $BL$ and of the top row ending at $TR$, when these corners are among the $K_i$. The set of corners in use contains neither $\{BL, BR\}$ nor $\{TL, TR\}$ by (3); so it has at most two elements, and up to a mirror image of $R$ and its reflection it is $\{BL\}$, $\{BL, TL\}$ or $\{BL, TR\}$. *One corner.* If all $K_i = BL$, then $BL$ is in $P$, so $l = 0$, and $K_1 = K_2$ gives $\beta_1 = \beta_2$, so $r = 0$; then $BR$ is in $P$ and the bottom row, an interval from $BL$, is full, against (2). *Two corners of a column,* $\{BL, TL\}$. Then $l = 0$, $\lambda = q$, all four $K_i$ are in the left column of the box, so the bottom and top interval lengths are $q$ and $a = 2q$; by (4) $u[j] = q$ for $j < y_5$, so these columns are full. Since $y_5 \ge a+1-q = q+1$ by (1), the bottom row of $T_5$ covers $[0,q)$, against (4). *Opposite corners,* $\{BL, TR\}$. Here $BL$ and $TR$ are in $P$, so $l = 0$ and $r+\rho = q$. The cell $BR$ is not in $P$, since the bottom row would be full, so $r \ge 1$; likewise $TL$ is not in $P$, so $\lambda \le q-1$. Then (3) gives $\beta_1 = y_5$, $\beta_2 = y_5 + r$, $\beta_3 = m$ and $\beta_4 = m+q-\lambda$. If $K_1 = K_2$ then $\beta_1 = \beta_2$, so $r = 0$, impossible; if $K_3 = K_4$ then $\beta_3 = \beta_4$, so $\lambda = q$, impossible. So $\{K_1, K_2\} = \{K_3, K_4\} = \{BL, TR\}$, and four assignments remain. When $K_1 = BL$ and $K_2 = TR$, the intervals give $x_B = y_5$ and $x_T = y_5 + r$; when $K_1 = TR$ and $K_2 = BL$, $x_T = y_5$ and $x_B = y_5 + r$; when $K_3 = TR$ and $K_4 = BL$, $x_T = m$ and $x_B = m+q-\lambda$; when $K_3 = BL$ and $K_4 = TR$, $x_B = m$ and $x_T = m+q-\lambda$. The assignment $(BL, TR, TR, BL)$ gives $y_5 = x_B = m+q-\lambda = x_T + q - \lambda = y_5 + r + q - \lambda$, so $\lambda = q + r$, impossible; the assignment $(TR, BL, BL, TR)$ is the same. The remaining assignments $(BL, TR, BL, TR)$ and $(TR, BL, TR, BL)$ give $K_3 = K_1$ and $K_4 = K_2$, so the half turn of $R$ exchanges $T_1$ with $T_3$ and $T_2$ with $T_4$ and fixes $T_5$; then $P$ is preserved by the half turn of its box, so $x_T = x_B$, while they differ by $r \ge 1$. So $T_5$ meets no two opposite sides. □

**Remark 49** (What is left). For $T_5$ on no side or on exactly one side, no tile cuts another off in the sense of Lemma 47: around the boundary of $R$ the tiles occupy arcs of cells, one each when $T_5$ meets exactly one side, and no two tiles have boundary cells that alternate around the boundary. Section 5 excludes these two configurations with other tools: the corners of $\mathrm{box}(P)$ that belong to $P$, and the counts of cells along lines.

## 5. CASE (I) DOES NOT OCCUR

This section proves Theorem 2. Throughout, $P$ is a non-rectangular polyomino and $T_1,\dots,T_5$ is a 5-copy tiling of $R = R(a,b)$ by $P$ in which the corner cells $(0,0)$, $(a-1,0)$, $(a-1,b-1)$, $(0,b-1)$ lie in $T_1$, $T_2$, $T_3$, $T_4$, four distinct tiles (the *corner tiles*); we derive a contradiction. Five facts proved

earlier are used: Proposition 4 (every tile meets every side of $R$ in an interval), Proposition 5 (a corner tile meets only the two sides at its corner), Proposition 6 ($T_5$ meets no side or exactly one), Theorem 9 (three copies), and Lemma 10 with Corollary 11.

For readers of the Lean library: pattern N below is the structure `FrameN`, pattern O is `FrameO`, Lemmas 51 to 55 are the theorems `T1_*` to `T5_*`, and Lemma 10 is `interlace`.

**5.1 Setting.** Figure 3 shows the marks and runs of this subsection and the strip lemma on small examples, and Figure 4 shows one frame for each class of occupied corners excluded in Sections 5.3 to 5.7.

Fix isometries $g_i$ with $T_i = g_i(P)$. Let $p \geq q$ be the side lengths of $\mathrm{box}(P)$; $q \geq 2$, since a connected set of cells in one line is a rectangle, and $a, b \geq 2$. In this section only, $W_i$ and $H_i$ are the numbers of columns and rows of $\mathrm{box}(T_i)$, so $\{W_i, H_i\} = \{p, q\}$. Each $T_i$ with $i \leq 4$ contains its corner cell, so

$$\begin{aligned} \mathrm{box}(T_1) &= [0, W_1) \times [0, H_1), \\ \mathrm{box}(T_2) &= [a - W_2, a) \times [0, H_2), \\ \mathrm{box}(T_3) &= [a - W_3, a) \times [b - H_3, b), \\ \mathrm{box}(T_4) &= [0, W_4) \times [b - H_4, b), \end{aligned}$$

and by Proposition 5, $W_i \leq a - 1$ and $H_i \leq b - 1$ for $i \leq 4$. Write $\mathrm{box}(T_5) = [x_5, x_5 + W_5) \times [y_5, y_5 + H_5)$. For $i \leq 4$ let $u_i$ and $v_i$ be the numbers of cells of $T_i$ on its horizontal and on its vertical side of $R$; by Proposition 4 they are the first $u_i$, $v_i$ cells of that side counted from the corner, and $1 \leq u_i \leq W_i$, $1 \leq v_i \leq H_i$.

By Proposition 6, after a symmetry of the grid (which may exchange $a$ and $b$), one of two patterns holds.

- **Pattern N**: $T_5$ meets no side; $x_5, y_5 \geq 1$, $x_5 + W_5 \leq a - 1$, $y_5 + H_5 \leq b - 1$.
- **Pattern O**: $T_5$ meets the bottom side and no other; $y_5 = 0$, $x_5 \geq 1$, $x_5 + W_5 \leq a - 1$, $H_5 \leq b - 1$. The bottom row reads $T_1$, $T_5$, $T_2$, and $T_5$ has $\mu \geq 1$ cells there, an interval.

A side of $R$ that $T_5$ does not meet is *free*: all four sides in pattern N; the left, top and right side in pattern O. A free side consists of cells of its two corner tiles, so

$$\begin{aligned} v_1 + v_4 = b = v_2 + v_3, \qquad u_3 + u_4 = a, \\ u_1 + u_2 = a \ (\mathrm{N}), \qquad u_1 + \mu + u_2 = a \ (\mathrm{O}). \end{aligned} \tag{1}$$

**Occupied corners, marks and runs.** Let $S$ be the set of corner cells of $\mathrm{box}(P)$ that belong to $P$, the *occupied corners*. For $i \leq 4$ the *mark* $m_i$ of $T_i$ is the corner cell of $\mathrm{box}(P)$ that $g_i$ puts on the corner cell of $R$ in $T_i$. Each mark is in $S$, so $S$ is not empty. A corner of $\mathrm{box}(P)$ is *used* if it is the mark of some $T_i$.

**Lemma 50** (Runs). *Let $m$ be a used corner and $s$ a side of $\mathrm{box}(P)$ at $m$. The cells of $P$ on $s$ form an interval that starts at $m$; their number is the* run *of $m$ along $s$. If both ends of a side of $\mathrm{box}(P)$ are occupied and at least one of them is used, the side lies in $P$ (it is* full*). So the run of a used $m$ along $s$ is the whole side if and only if the other end of $s$ is occupied.*

*Proof.* Let $m = m_i$. The isometry $g_i$ puts $s$ on a side of $R$, since $\mathrm{box}(T_i)$ has its corner at the corner of $R$, and the cells of $T_i$ on that side of $R$ are its cells on the corresponding side of $\mathrm{box}(T_i)$. By Proposition 4 they form an interval, and it contains the corner cell. The other two claims follow. □

**Placed copies.** A copy $g(P)$, $g$ a grid isometry, whose box lies in $R$ and has a corner on a corner cell of $R$ is determined by that cell, the corner of $\mathrm{box}(P)$ that $g$ puts on it, and the direction of $\mathrm{box}(P)$ that $g$ makes horizontal: $g(\mathrm{box}(P))$ lies on the inner side of both sides of $R$ at that cell, so these data fix the linear part of $g$, and the cell fixes its translation. So a corner tile is determined by its corner of $R$, its mark, and which of the two directions of $\mathrm{box}(P)$ is horizontal, and every copy $g(P)$ as above with the same three data is that tile, also when $g(P)$ is the image of another tile under a symmetry of $R$ or of a strip of $R$. For $p > q$ the third datum is the orientation of the box: *wide* ($p$ columns, $q$ rows) or *tall* ($q$ columns, $p$ rows). For $|S| = 4$ all four sides of $\mathrm{box}(P)$ have occupied ends; for $|S| = 3$ two sides, with a common end; for two adjacent occupied corners one; for two opposite occupied corners, or $|S| = 1$, none.

**Lines.** For a side $s$ of $\mathrm{box}(P)$, the line at distance $c$ from $s$ is the row or column of $\mathrm{box}(P)$ parallel to $s$ at distance $c$ ($c = 0$ is $s$ itself). $P$ is connected and meets the two outermost lines in each direction, so it meets every line. For a set $T$ of cells, $\mathrm{col}_T(x)$ and $\mathrm{row}_T(y)$ are the numbers of cells of $T$ in column $x$ and in row $y$. Since the tiles partition $R$,

$$\begin{aligned} \textstyle\sum_i \mathrm{col}_{T_i}(x) = b, \quad 0 \leq x < a, \\ \textstyle\sum_i \mathrm{row}_{T_i}(y) = a, \quad 0 \leq y < b. \end{aligned} \tag{2}$$

**Letters with a local meaning.** Some letters keep a meaning only in the part named and are introduced again where it changes. $s$ is a side of $\mathrm{box}(P)$ or of $R$ in Section 5.1, in Lemmas 55, 59 and 62, and the side length $a = b$ in Propositions 63 and 65. $H$ and $W$ without an index are numbers of rows and columns, among them the height and width of a box (Lemmas 52 and 55, Propositions 58 and 60, Section 5.7), except in Proposition 65, where $H$ and $V$ name two sides of $R$; the upright E, H and V are kinds of tiles in Proposition 60, and the upright h and v are the two types of tiles in Section 5.6. The italic $h$ and $v$ are numbers: a number of rows in Lemma 53, the runs $u, v$ in Proposition 56, and runs in Section 5.7. $e_i$ is a direction of $\mathrm{box}(P)$ in Proposition 56, and $e$ an end of $F$ in Section 5.6; $r$ is a run in Section 5.6, from its setting and Lemma 62 on; $k$, $\sigma$ and $\tau$ are defined anew where they are used.

**5.2 Five tools.**

**Lemma 51** (Mirror). *Let $T, T'$ be the two corner tiles at a free side, with the same mark and with the same direction of $\mathrm{box}(P)$ along the side. Then $T'$ is the mirror image of $T$ in the perpendicular bisector of the side, and both runs along the side are as long as the extent of the box along it, which is half the side.*

*Proof.* Take the bottom side, $T = T_1$, $T' = T_2$. The reflection $\phi(x, y) = (a-1-x, y)$ maps $T_1$ to a copy at the corner $(a-1, 0)$ with the same mark and direction, so to $T_2$ (placed copies, Section 5.1). $T_1$ and $\phi(T_1)$ are disjoint, so $T_1$ has no cell with

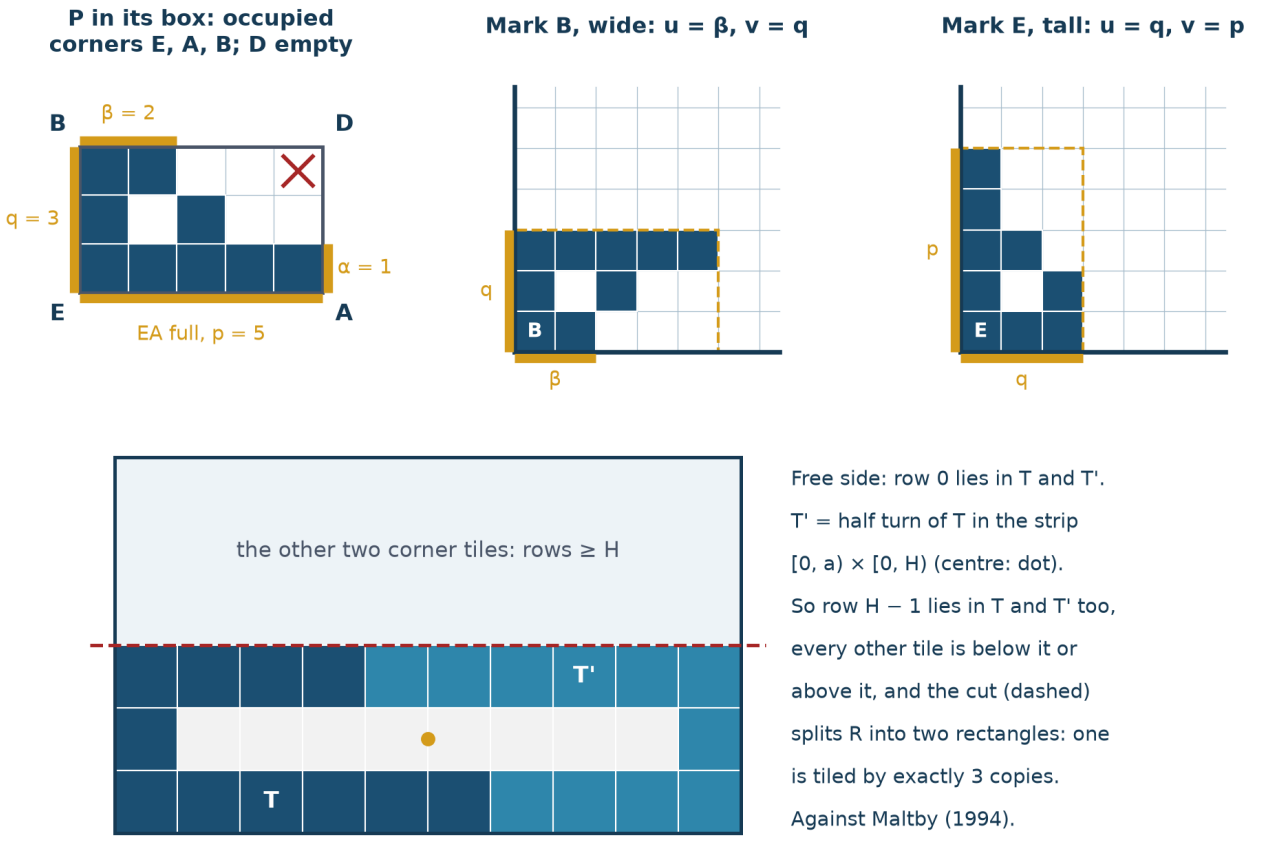



**Figure 3. Marks and runs; the strip.** Left: a polyomino in its box with occupied corners $E, A, B$ and empty corner $D$ (the names of Section 5.5); $EA$ is full, and the runs of $A$ and $B$ along the other two sides are $\alpha = 1$ and $\beta = 2$. Middle and right: a corner tile at the corner $(0,0)$ of $R$ with mark $B$, wide, and with mark $E$, tall; the runs on the two sides of $R$ are $(u,v) = (\beta,q)$ and $(q,p)$. Bottom: Lemma 55 (strip). At a free side, $T'$ is the half turn of $T$ in the strip of $H$ rows. Then row $H-1$ lies in $T$ and $T'$ as well, so the cut between rows $H-1$ and $H$ splits $R$ into two rectangles, one of them tiled by exactly three copies of $P$, against Theorem 9. Small examples, not part of any tiling; `figures/case_one_marks.out` records the cells drawn and the checks they pass.

$2x = a-1$, and no two adjacent cells $(x,y)$, $(x+1,y)$ with $2x+1 = a-1$, since such a pair is exchanged by $\phi$. So $2x-(a-1)$ does not change sign along a path in $T_1$; as $(0,0) \in T_1$, $2x < a-1$ on $T_1$. So $2W_1 \le a$, and the same holds for $T_2$. By (1) the two runs are equal and add up to $a$, so $u_1 = a/2 \ge W_1 \ge u_1$. The top side is the same after the reflection of $R$ in its horizontal midline; a vertical side is the same with $x$ and $y$, and $a$ and $b$, exchanged. □

**Lemma 52** (Row order). *Let $T, T'$ be disjoint connected sets of cells with no cell below row 0. Let $T$ lie in the rows $[0,H)$ and contain $(x_1,0)$ and $(x_2,H-1)$, and let $T'$ contain $(x_1',0)$ and a cell in some row $\ge H-1$. (a) If $x_1 < x_1'$, then $T'$ has a cell $(x',H-1)$ with $x' > x_2$. (b) If moreover $T'$ lies in the rows $[0,H)$, every cell $(x_2',H-1)$ of $T'$ has $x_2' > x_2$.*

*Proof.* Let $H \ge 2$. Take a path in $T$ between its two cells. For (a) take the part of a path in $T'$ from $(x_1',0)$ up to its first cell $(x',H-1)$ in row $H-1$; for (b) a path in $T'$ from $(x_1',0)$ to the cell $(x',H-1)$ in question, which lies in the rows $[0,H)$ by hypothesis. Both paths lie in the rectangle of the rows $[0,H)$ and of all columns from the smallest to the largest column of their cells; it has at least two columns, as $x_1 < x_1'$, and the four end cells lie on its bottom and top rows. $x' \ne x_2$, as $T$ and $T'$ are disjoint. If $x' < x_2$, the pairs $(x_1,0)$, $(x_2,H-1)$ and $(x_1',0)$, $(x',H-1)$ interlace on the boundary of that rectangle, and by Lemma 10 the sets share a cell. So $x' > x_2$. Let $H = 1$. Both claims are clear without Lemma 10: $T$ lies in row 0 and is connected, so it is an interval of row 0 that contains $x_1$ and $x_2$; $T'$ is disjoint from $T$ and contains $(x_1',0)$ with $x_1' > x_1$, so $x_1' > x_2$, which is (a) with $x' = x_1'$; in (b) $T'$ is an interval of row 0 as well, it contains $x_1'$ and no cell of $T$, so all its cells are to the right of $T$. ($H \ge 2$ wherever the lemma is used below.) □

The mirror images of Lemma 52 and its versions for the other sides of $R$ are used under the same name.

**Lemma 53** (Barrier). *In pattern $O$, let $T_5$ contain $(x,y)$ for all $0 \le y < h$. If $H_1 \le h$ then $W_1 \le x$; if $H_2 \le h$ then $W_2 \le a-1-x$.*

*Proof.* $T_1$ lies in the rows $[0,H_1)$, so it has no cell in column $x$; it is connected and contains $(0,0)$, and $x \ge 1$. So all its cells are left of column $x$. The second claim is the mirror image. □

**Lemma 54** (Symmetry). *Let a symmetry $\phi$ of $R$ permute the four sets $T_1, \dots, T_4$. Then $g_5^{-1}\phi g_5$ maps $P$ to itself; it is a reflection in an axis-parallel line, a half turn or a quarter turn if $\phi$ is one, and it maps* box$(P)$ *and $S$ to themselves. If $\phi$ is a reflection in an axis-parallel line or a half turn, $|S|$ is even; a half turn maps each corner of* box$(P)$ *to the opposite one, a reflection to an adjacent one. If $\phi$ is a quarter turn, $p = q$ and $|S|$ is 0 or 4.*

*Proof.* $\phi$ permutes the five tiles and fixes the set $T_1 \cup \dots \cup T_4$, so $\phi(T_5) = T_5$, and $g_5^{-1}\phi g_5$ maps $P$ onto itself; it has the same kind as $\phi$, and it maps box$(P)$ and so the set $S$ to themselves. A reflection in an axis-parallel line and a half turn of a box fix none of its corners and have order two, so $S$ splits into pairs. A quarter turn maps the box onto itself only if $p = q$, and it moves the four corners in one cycle. □

**Lemma 55** (Strip). *The following does not occur: $T, T'$ are the two corner tiles at a free side $s$, both boxes have the same extent $H$ perpendicular to $s$, and $T'$ is the image of $T$ under the half turn of the strip of the $H$ lines of $R$ next to $s$.*

*Proof.* Take the bottom side. Row 0 lies in $T$ and $T'$ together, and the half turn of $[0,a) \times [0,H)$ exchanges $T$ and $T'$ and takes row 0 to row $H-1$; so row $H-1$ lies in $T$ and $T'$ together. Every other tile is connected and misses row $H-1$, so it lies in the rows $< H-1$ or in the rows $\geq H$. The other two corner tiles contain cells of row $b-1$, and $H \leq b-1$, so they lie in the rows $\geq H$. $T$ and $T'$ lie in the rows $[0,H)$. So $[0,a) \times [0,H)$ and $[0,a) \times [H,b)$ are rectangles; one is tiled by $T, T'$ and, if it lies there, $T_5$, and the other by the remaining tiles. One of the two is tiled by exactly three copies of $P$, against Theorem 9. □

The three-copy theorem, Theorem 9, is used in the proof of Theorem 2 only here, through Lemmas 59 and 62.

### 5.3 All marks equal.

**Proposition 56** (All marks equal). *Suppose $m_1 = m_2 = m_3 = m_4$. In pattern N, $|S| = 4$. In pattern O, $P$ has a reflection in an axis-parallel line as a symmetry, so $S$ is a pair of adjacent corners or $|S| = 4$.*

*Proof.* Let $m$ be the common mark, 1 and 2 the two directions of box$(P)$, $u$ and $v$ the runs of $m$ in these directions, and $e_i$ the direction that is horizontal at $T_i$; so $(u_i, v_i) = (u, v)$ if $e_i = 1$ and $(v, u)$ if $e_i = 2$.

*Pattern N.* (i) If $e_i = e_j$ for the two tiles at a side, Lemma 51 shows that the run along this side is the whole extent of the box, so a second corner of box$(P)$ is occupied. (ii) If $e_1 = e_3$ and $e_2 = e_4$ with $e_1 \neq e_2$, then $a = u + v = b$ by (1), and the quarter turn of $R$ takes $T_1$ to the copy at $(a-1, 0)$ with mark $m$ and the other direction horizontal, which is $T_2$, and so on around. By Lemma 54, $|S| = 4$. (iii) If all $e_i$ are equal, the two reflections of $R$ in its midlines permute $T_1, \dots, T_4$; by Lemma 54, $S$ is invariant under both reflections of box$(P)$, so $|S| = 4$. (iv) Otherwise, if $u \neq v$, (1) gives the same number of tiles with $e = 1$ on the bottom and on the top, and on the left and on the right; then $e_1 = e_3$ and $e_2 = e_4$, which is (ii) or (iii). So $u = v$ and $a = b = 2u$. Either three of the $e_i$ are equal, so some corner has the same $e$ as both its neighbours and by (i) both its runs are whole sides of box$(P)$; or the $e_i$ are equal in two pairs at two opposite sides of $R$, with different values, and by (i) again the runs in both directions are whole sides. So box$(P)$ is $u \times u$, the two corners next to $m$ are occupied, and the boxes of $T_1, \dots, T_4$ are the four quadrants $Q_1, \dots, Q_4$ of $R$. Suppose $|S| = 3$, and put $k = u$. $T_i$ misses the corner of $Q_i$ at the centre of $R$, the corner opposite to $m$. So $T_5$ is the union of the sets $Q_i \setminus T_i$, contains the four cells around the centre, and $1 \leq x_5, y_5 \leq k-1$. $T_5$ contains the full horizontal and the full vertical side of its box that meet at a corner of box$(T_5)$; reflect $R$ in none, one or both of its midlines so that this is the lower left corner (the reflected tiling has the same description: quadrant boxes, all marks $m$, $|S| = 3$; the tiles are named by their new corners). So $T_5$ contains $\{x_5\} \times [y_5, y_5+k)$ and $[x_5, x_5+k) \times \{y_5\}$. In $Q_1 = [0,k) \times [0,k)$, $T_1$ is connected, contains $(0,0)$, and to reach a cell with $x \geq x_5$ and $y \geq y_5$ it would have to enter a cell of one of these two segments. So $T_5$ contains the rectangle $D_1 = [x_5, k) \times [y_5, k)$, and $Q_1 \setminus T_1 = D_1$, as $T_5$ is in its box. All the sets $Q_i \setminus T_i$ are congruent to box$(P) \setminus P$, so they are rectangles with side lengths $\{k-x_5, k-y_5\}$, each with a corner at the centre of $R$. $Q_2 \setminus T_2$ has $y_5$ as its lowest row ($T_5$ contains $(k, y_5)$ and no lower cell), so its height is $k - y_5$ and its width is $k - x_5$; but row $y_5$ of $T_5$ is $[x_5, x_5 + k)$, so this width is $x_5$. So $k = 2x_5$, and in the same way, with $Q_4$, $k = 2y_5$. So the four sets are squares of side $k/2$ around the centre and $T_5$ is a $k \times k$ square, so $P$ is a rectangle, a contradiction. So $|S| = 4$.

*Pattern O.* If $u = v$, (1) gives $a = 2u$ on top and $a = 2u + \mu$ on the bottom. So $u \neq v$; let $\sigma < \tau$ be the two runs and $c_i = 1$ if the horizontal run of $T_i$ is $\tau$, else $c_i = 0$. By (1), $c_1 + c_4 = c_2 + c_3$ and $\mu = (\tau - \sigma)(c_3 + c_4 - c_1 - c_2) = 2(\tau - \sigma)(c_4 - c_2)$. So $c_4 = 1, c_2 = 0$ and $c_3 - c_1 = c_4 - c_2 = 1$. So $e_1 = e_2, e_3 = e_4$, and the reflection of $R$ in its vertical midline exchanges $T_1, T_2$ and $T_3, T_4$. Apply Lemma 54. □

**Corollary 57.** *$|S| = 1$ does not occur. If $|S| = 3$, or $|S| = 2$ with opposite occupied corners, or $|S| = 2$ in pattern N, two different corners are used. So for $|S| = 2$ with opposite occupied corners, and for $|S| = 2$ in pattern N, both occupied corners are used.*

*Proof.* If $|S| = 1$ there is one mark, and Proposition 56 gives a contradiction in both patterns. In the other cases one mark for all four tiles contradicts Proposition 56. □

### 5.4 Four occupied corners.

**Proposition 58.** *$|S| = 4$ does not occur.*

*Proof.* By Lemma 50 both runs of every mark are whole sides: $u_i = W_i$ and $v_i = H_i$.

*Pattern O.* By (1), $W_3 + W_4 > W_1 + W_2$ and $H_1 + H_4 = H_2 + H_3$, that is, as $H_i = p + q - W_i$, $W_3 - W_1 = W_4 - W_2$. So both differences are positive: $W_3 = W_4 = p$, $W_1 = W_2 = q$, $p > q$, $a = 2p$ and $b = p + q$. Then $H_1 = H_2 = p$ and $H_5 \leq p$, so the rows $[p, p+q)$ lie in $T_3$ and $T_4$. Their boxes $[p, 2p) \times [p, p+q)$ and $[0, p) \times [p, p+q)$ are disjoint, so $T_3$ fills its box, and $P$ is a rectangle, a contradiction.

*Pattern N.* By (1), $W_1 + W_2 = W_3 + W_4$ and $W_1 + W_4 = W_2 + W_3$, so $W_1 = W_3$ and $W_2 = W_4$.

*Equal boxes, $W_1 = W_2$.* The boxes are the four quadrants $Q_1, \dots, Q_4$ of $R$. Each $T_i$ misses a cell of $Q_i$, and $T_5$ is the union of the sets $Q_i \setminus T_i$. Take a path in $T_5$ from $Q_2$ to $Q_4$.

Between its last cell in $Q_2$ before it first meets $Q_4$ and that first cell in $Q_4$, it stays in $Q_1$ or stays in $Q_3$, because no cell of $Q_1$ is adjacent to one of $Q_3$ and none of $Q_2$ to one of $Q_4$. Say it stays in $Q_1 = [0,W)\times[0,H)$. Then $T_5$ has a connected set in $Q_1$ from the right column to the top row of $Q_1$. $T_1$ is connected, lies in $Q_1$, and contains $(0,0)$ and $(W-1,H-1)$, as all corners are occupied. That cell is not in $T_5$, so the ends of the set of $T_5$ lie on the right column below it and on the top row left of it: the ends interlace, and Lemma 10 gives a common cell, a contradiction. $Q_3$ is the same after a half turn of $R$.

*Unequal boxes, $W_1 \neq W_2$.* Then $p > q$; reflect $R$ so that $W_1 = W_3 = p$ and $W_2 = W_4 = q$. Then $a = b = p+q$ and

$$\begin{aligned}\mathrm{box}(T_1) &= [0,p)\times[0,q),\\ \mathrm{box}(T_2) &= [p,p+q)\times[0,p),\\ \mathrm{box}(T_3) &= [q,p+q)\times[p,p+q),\\ \mathrm{box}(T_4) &= [0,q)\times[q,p+q).\end{aligned}$$

The four boxes are disjoint and leave out $[q,p)\times[q,p)$. $T_5$ contains this square and meets each of the four boxes, since $T_i$ is not a rectangle. If all marks are equal, the quarter turn of $R$ permutes $T_1,\dots,T_4$ (mark and orientation agree), so $p = q$ by Lemma 54, a contradiction. So two different corners are used, and at least three sides of $\mathrm{box}(P)$ are full (Lemma 50).

- All four sides full. $T_5$ has a cell in $\mathrm{box}(T_1)$ and a cell outside; a path between them would meet a cell of the frame of $\mathrm{box}(T_1)$, which is in $T_1$. A contradiction.
- Exactly one side not full, a short one. The two sides of $T_1$ on the boundary of $R$ are full (they are at the mark), and so are both long sides. So $T_1$ contains row $q-1$ of its box. $T_5$ can leave $\mathrm{box}(T_1)$ only from column $p-1$ to column $p$. But column $p$, rows $[0,p)$, is a long side of $T_2$ and lies in $T_2$. A contradiction.
- Exactly one side not full, a long one. All short sides are full. A quarter turn of $R$ preserves the list of boxes, so we may take $T_5$ to be $p$ wide. $T_5$ meets $\mathrm{box}(T_1)$ and $\mathrm{box}(T_3)$, so $y_5 < q$ and $y_5+q-1 \geq p$; it meets $\mathrm{box}(T_4)$ and $\mathrm{box}(T_2)$, so $x_5 < q$ and $x_5+p-1 \geq p$. Column $p-1$, rows $[0,q)$, is a short side of $T_1$; so $(p-1,y_5)$ is in $T_1$ and not in $T_5$, and it is on the lower long side of $\mathrm{box}(T_5)$. Column $q$, rows $[p,p+q)$, is a short side of $T_3$; so $(q,y_5+q-1)$ is in $T_3$ and not in $T_5$, and it is on the upper long side of $\mathrm{box}(T_5)$. So neither long side of $P$ is full, a contradiction.

□

**5.5 Three occupied corners.** Let $E$ be the occupied corner between the two sides of $\mathrm{box}(P)$ with occupied ends, $A$ the other end of the long one ($p$ cells), $B$ the other end of the short one ($q$ cells); for $p = q$ name them in either way. $D$ is the corner not in $P$. By Corollary 57 two different corners are used: $E$ is, or $A$ and $B$ are. So $EA$ and $EB$ are full. Let $\alpha$ and $\beta$ be the numbers of cells of $P$ on the short side $AD$ and on the long side $BD$; $1 \leq \alpha < q$ and $1 \leq \beta < p$, as $D \notin P$. By Lemma 50, if $A$ is used, $\alpha$ is its run along $AD$, and if $B$ is used, $\beta$ is its run along $BD$. Let $\kappa_c$ $(0 \leq c < p)$ be the number of cells of $P$ on the line at distance $c$ from $EB$, and $\eta_j$ $(0 \leq j < q)$ on the line at distance $j$ from $EA$. So $\kappa_0 = q$, $\eta_0 = p$, $1 \leq \kappa_c \leq q$ and $1 \leq \eta_j \leq p$. For $p > q$, $EB$ is the only full short side of $P$ and $EA$ the only full long side.

For $p > q$, Table 1 gives the runs $(u,v)$ of a corner tile and the places of the other corners of its box, seen from its corner of $R$: "along" is on the horizontal side of $R$ at that corner, "up" on the vertical one, and "opposite" is the corner of the box opposite to the mark.

**Lemma 59** (Same orientation). *Let $p > q$. If the two corner tiles at a free side $s$ are both wide or both tall, both their runs along $s$ are whole sides of their boxes.*

*Proof.* Let the direction of $\mathrm{box}(P)$ along $s$ be that of $EA$, so both boxes have extent $p$ along $s$ and $q$ perpendicular to it; the other case is the same with $A,B$ and $p,q$ exchanged. The run along $s$ is $p$ for the marks $E$ and $A$, and $\beta < p$ for $B$. Suppose one of the marks is $B$. If both are $B$, Lemma 51 gives $\beta = p$. If the other is $A$, take $s$ to be the bottom side. Both boxes have height $q$, the opposite corners are $A$ and $B$, both occupied, and the runs add up to $a = p+\beta < 2p$. So $T_1$ contains $(0,0)$ and $(p-1,q-1)$, $T_2$ contains $(a-1,0)$ and $(a-p,q-1)$, and $p-1 \geq a-p$, against Lemma 52(b). If the other is $E$: the side $EB$ of both tiles is on the side of $R$ perpendicular to $s$, at opposite ends of the strip of the $q$ lines next to $s$; at $s$ the tile with mark $B$ has its end $B$ and the other its end $E$. The half turn of the strip maps the first tile to a copy whose side $EB$ lies where that of the second does, with $E$ at $s$, and whose box lies on the inner side; so, by the paragraph on placed copies, it maps the first tile to the second, against Lemma 55. □

**Proposition 60.** *In pattern N, $|S| = 3$ does not occur.*

*Proof. The case $p = q = k$.* Say that $T_i$ is of kind V if its mark is $A$ or $B$ and the full side at its mark is vertical (so $v_i = k$ and $u_i < k$), of kind H if its mark is $A$ or $B$ and the full side at its mark is horizontal ($u_i = k$, $v_i < k$), and of kind E if its mark is $E$ ($u_i = v_i = k$). $T_5$ has a full horizontal and a full vertical side. From $x_5 \geq 1$, $x_5 + k \leq a-1$ and $a \leq 2k$, the columns of $\mathrm{box}(T_5)$ contain $k-1$ and $a-k$; in the same way its rows contain $k-1$ and $b-k$.

- If $T_1$ is of kind H, its end $E$ is $(k-1,0)$ and the side of $T_1$ in column $k-1$ is full: $(k-1,y) \in T_1$ for $0 \leq y < k$. So the full horizontal side of $T_5$ is in a row $\geq k$. If it were the lowest row of $\mathrm{box}(T_5)$, then $y_5 \geq k$ and $y_5 + k \geq 2k \geq b$. So it is the highest row of $\mathrm{box}(T_5)$. The same holds if $T_2$ is of kind H, with column $a-k$.
- If $T_3$ or $T_4$ is of kind H, the full horizontal side of $T_5$ is the lowest row of $\mathrm{box}(T_5)$ (turn $R$ upside down).
- So kind H does not occur both at the bottom $(T_1,T_2)$ and on top $(T_3,T_4)$. In the same way kind V does not occur both on the left $(T_1,T_4)$ and on the right $(T_2,T_3)$; if it occurs on the left, the full vertical side of $T_5$ is the rightmost column of $\mathrm{box}(T_5)$, and this column is $\geq k$.
- By (1), $2k-a$ is the sum of $k-u_i$ over $T_1,T_2$ and also over $T_3,T_4$. So if $T_1$ is of kind V, some tile on top is of kind V; it is not $T_3$, so it is $T_4$, and $T_2$, $T_3$ are not of kind V, so $k-u_1 = k-u_4$. In general the tiles of kind V are none, or exactly $T_1$ and $T_4$ with $u_1 = u_4$, or exactly $T_2$ and $T_3$ with $u_2 = u_3$. In the same way the tiles of kind H are none, or exactly $T_1,T_2$, or exactly $T_3,T_4$. A pair of one kind and a pair of the other would share a tile. So only one kind occurs besides E, and not all marks are $E$ by Corollary 57.

**Table 1. Corner tiles for three occupied corners**, $p > q$.

| Mark | Wide: $(u,v)$; along, up, opposite | Tall: $(u,v)$; along, up, opposite |
|---|---|---|
| $E$ | $(p,q)$; $A, B, D$ | $(q,p)$; $B, A, D$ |
| $A$ | $(p,\alpha)$; $E, D, B$ | $(\alpha,p)$; $D, E, B$ |
| $B$ | $(\beta,q)$; $D, E, A$ | $(q,\beta)$; $E, D, A$ |

Exchange the axes and reflect, so that $T_1$ and $T_4$ are of kind V and $T_2, T_3$ of kind E. Put $\sigma = u_1 = u_4$. Then $a = k + \sigma$ and $b = 2k$.

- $T_2$ has the box $[\sigma, a) \times [0, k)$, $T_3$ the box $[\sigma, a) \times [k, 2k)$, and $(a-1, k-1)$ is the corner $A$ or $B$ of $T_2$, $(a-1, k)$ the corner $A$ or $B$ of $T_3$. The full vertical side of $T_5$ is in a column $\xi$ with $k \le \xi \le a-2$, and the rows of box$(T_5)$ contain $k-1$ and $k = b-k$, so $(\xi, k-1)$ and $(\xi, k)$ are in $T_5$. If the corner of $T_2$ at $(a-1, k-1)$ were $m_1$ (which is $A$ or $B$), the cells of $T_2$ in row $k-1$ would include the run of $m_1$, of length $\sigma$: the columns $[a-\sigma, a) = [k, a)$, with $\xi$ among them. So this corner is the other one of $A, B$, and for the same reason it is not $m_4$ ($u_4 = \sigma$). So $m_1 = m_4$, and $T_2$ is determined: mark $E$, and at $(a-1, k-1)$ the one of $A, B$ that is not $m_1$. In the same way the corner of $T_3$ at $(a-1, k)$ is the one of $A, B$ that is not $m_1$. So the reflection of $R$ in its horizontal midline exchanges $T_2$ and $T_3$ (mark $E$, same corner on the right side) and $T_1$ and $T_4$ (same mark, full side vertical). By Lemma 54, $|S|$ is even, a contradiction.

*The case $p > q$.* Lemma 59 applies to all four sides. Write w and t for wide and tall, in the order $T_1T_2T_3T_4$.

- All equal: all runs are whole sides, so all marks are $E$, against Corollary 57.
- Exactly one different, say $T_1$ (the others by symmetry): with $W, H$ the extents of $T_2, T_3, T_4$ and $H, W$ those of $T_1$, the sides $T_2T_3$ and $T_3T_4$ give $b = 2H$ and $a = 2W$. Then $2W = u_1 + u_2 \le H + W$ and $2H = v_1 + v_4 \le W + H$, so $W = H$ and $p = q$, a contradiction.
- Two equal at one side and the other two equal, of the other orientation: two opposite sides of $R$ have the lengths $2p$ and $2q$, a contradiction.
- So the orientations alternate. Reflect $R$ so that $T_1, T_3$ are wide and $T_2, T_4$ tall. A tile lacks, along the horizontal sides: $d_B = p - \beta$ if it is wide with mark $B$, $d_A = q - \alpha$ if it is tall with mark $A$, and 0 otherwise; along the vertical sides: $d_A$ if it is wide with mark $A$, $d_B$ if it is tall with mark $B$, and 0 otherwise. Both are positive. With $[\cdot]$ the value 0 or 1 of a condition, (1) gives

$$\begin{aligned} p+q-a &= d_B[m_1{=}B] + d_A[m_2{=}A] \\ &= d_B[m_3{=}B] + d_A[m_4{=}A], \\ p+q-b &= d_A[m_1{=}A] + d_B[m_4{=}B] \\ &= d_A[m_3{=}A] + d_B[m_2{=}B]. \end{aligned}$$

If $m_1 = m_3$ and $m_2 = m_4$, the half turn of $R$ permutes $T_1, \dots, T_4$, and $|S|$ is even by Lemma 54, a contradiction. So one of the differences $[m_1{=}B] - [m_3{=}B]$, $[m_2{=}A] - [m_4{=}A]$, $[m_1{=}A] - [m_3{=}A]$, $[m_2{=}B] - [m_4{=}B]$ is not 0; since $d_A, d_B > 0$, both $A$ and $B$ are used, $d_A = d_B =: d$, and

$$\begin{aligned} [m_1{=}B] - [m_3{=}B] &= [m_4{=}A] - [m_2{=}A], \\ [m_1{=}A] - [m_3{=}A] &= [m_2{=}B] - [m_4{=}B]. \end{aligned}$$

Besides the solutions of period two there are four, in which $(m_1, m_2, m_3, m_4)$ is one of $(B, E, E, A)$, $(A, B, E, E)$, $(E, A, B, E)$ and $(E, E, A, B)$. (If $m_1 = m_3$, the right sides vanish and $m_2 = m_4$. For $(m_1, m_3) = (B, E)$: $m_4 = A$, $m_2 \ne A$, and $m_2 = B$ would need $m_4 = B$; so $m_2 = E$. The pairs $(A, E)$, $(E, A)$, $(E, B)$ are the same; for $(A, B)$ and $(B, A)$ the two equations ask for $m_2 = A$ and $m_2 = B$, respectively $m_4 = A$ and $m_4 = B$.) Turn the whole tiling by a quarter turn and name the corners of the new rectangle $R(b, a)$ in the old order: the tiles that were $T_4, T_1, T_2, T_3$ are now $T_1, T_2, T_3, T_4$, and $T_1, T_3$ are again wide. This moves the four solutions around in a circle, so we may choose one of them: let the marks be $(B, E, E, A)$. Then $a = p + q - d$, $b = p + q$, $\beta = p - d$ and $\alpha = q - d$, and

- $T_1$: box $[0, p) \times [0, q)$, $E$ at $(0, q-1)$; row $q-1$ is full, $[0, p) \times \{q-1\} \subseteq T_1$; the short side $EB$ is on the left;
- $T_2$: box $[a-q, a) \times [0, p)$;
- $T_3$: box $[a-p, a) \times [p, b)$, $E$ at $(a-1, b-1)$, $B$ at $(a-1, p)$; so $[q, a) \times \{p\} \subseteq T_3$ (run $\beta = a - q$); $EB$ is on the right;
- $T_4$: box $[0, q) \times [q, b)$, $A$ at $(0, b-1)$, $E$ at $(0, q)$; the short side $EB$ is row $q$.

$T_5$ has a full vertical side, in a column $x$. If the rows of box$(T_5)$ contained $q-1$ and $p$, then $x \ge p$ by $T_1$ and $x < q$ by $T_3$. So box$(T_5)$ lies in the rows $\ge q$ or in the rows $< p$. If $T_5$ is tall, it needs $p$ rows in $[q, b-1)$ or in $[1, p)$, which have $p-1$ rows. So $T_5$ is wide. In both cases $\kappa_c = q$ for $c < \beta$: if $T_5$ is in the rows $\ge q$, the rows $[0, q)$ lie in $T_1$ and $T_2$ alone, so $[0, a-q) \times [0, q) = [0, \beta) \times [0, q)$ is in $T_1$; if $T_5$ is in the rows $< p$, the rows $[p, b)$ lie in $T_3$ and $T_4$ alone, so $[q, a) \times [p, b)$ is in $T_3$, and these are the $\beta$ columns of box$(T_3)$ next to $EB$. So $T_1$ contains $[0, \beta) \times [0, q)$, $T_4$ contains $[0, q) \times [q, q+\beta)$, and $T_3$ contains $[q, a) \times [p, b)$. The full long side of $T_5$ is horizontal, in a row $y$ and the columns $[x_5, x_5 + p)$ with $1 \le x_5 \le a - 1 - p = q - d - 1$; so $x_5 < \beta$ ($\beta = p - d$), $x_5 < q$ and $x_5 + p - 1 \ge p \ge q$. If $y < q$, $(x_5, y)$ is in $T_1$; if $q \le y < q + \beta$, it is in $T_4$; if $y \ge p$, $(x_5 + p - 1, y)$ is in $T_3$. And $q + \beta = p + q - d > p$. A contradiction.

□

**Proposition 61.** *In pattern O, $|S| = 3$ does not occur.*

*Proof.* $T_5$ has a full vertical side; let $\xi \in \{x_5, x_5 + W_5 - 1\}$ be its column. As $(\xi, 0) \in T_5$, $u_1 \le \xi \le a - 1 - u_2$. By Lemma 53 with $h = H_5$,

$$\begin{aligned} &\text{if } H_1 \le H_5 \text{ then } W_1 \le \xi; \\ &\text{if } H_2 \le H_5 \text{ then } W_2 \le a - 1 - \xi. \end{aligned} \tag{3}$$

By (1), $a = u_3 + u_4 \le W_3 + W_4$.

*The case $p = q$.* (3) gives $2p \le a - 1 \le 2p - 1$, a contradiction. So $p > q$, and Lemma 59 applies to the left, top and right side.

*$T_5$ tall.* (3) applies to both tiles at the bottom, so $W_1+W_2 \le a-1 < W_3+W_4$: there are more wide tiles on top than at the bottom. The full vertical side of $T_5$ is $EA$.

(a) *One wide on top, none at the bottom.* Reflect so that $T_4$ is wide and $T_3$ tall. $T_2, T_3$ tall: $b = 2p$ by Lemma 59. $T_1$ tall, $T_4$ wide: $b \le p+q$. A contradiction.

(b) *Two wide on top, one at the bottom.* Reflect so that $T_1$ is wide and $T_2$ tall. Lemma 59: on top $a = 2p$ and $u_3 = u_4 = p$; on the left $b = 2q$ and $v_1 = v_4 = q$. $p = H_5 \le b-1$, so $p < 2q$. By (3), $p \le \xi \le 2p-1-q$. $T_4$ has $u_4 = p$ and $v_4 = q$: its mark is $E$, and its corner $D$ is $(p-1, q)$. No other corner tile meets $\mathrm{box}(T_4) = [0,p) \times [q,2q)$ ($T_1$ is in the rows $< q$, $T_3$ in the columns $\ge p$, $T_2$ in the columns $\ge 2p-q > p$). So $(p-1, q) \in T_5$; so $x_5 \le p-1 < \xi$ and $\xi = x_5 + q - 1$. $(\xi, 0)$ is the lower right corner of $\mathrm{box}(T_5)$ and is occupied, so the bottom interval of $T_5$ ends there: $u_2 = 2p-1-\xi \ge q$, and $u_2 \le W_2 = q$. So $u_2 = q$, $\xi = 2p-1-q$ and $x_5 = 2p-2q$. The mark of $T_2$ is $E$ or $B$. The rows $[p, 2q)$ lie in $T_3$ and $T_4$ alone, so $T_4$ contains $[0,p) \times [p,2q)$; its side $EA$ is on top. So

$$\eta_j = p \quad \text{for } j < 2q-p. \tag{4}$$

In $T_5$: $[\xi+1-(2q-p), \xi+1) \times [0,p) = [3p-3q, 2p-q) \times [0,p)$ is in $T_5$. A column $c < x_5$ meets $T_1$ and $T_4$ only. $T_4$ has $\kappa_c$ cells there (mark $E$, $EB$ on the left), and $T_1$ also ($v_1 = q$: its left side is full, so it is $EB$). So $2\kappa_c = 2q$:

$$\kappa_c = q \quad \text{for } c < 2p-2q. \tag{5}$$

$T_3$ has $u_3 = p$, so its mark is $E$ or $A$.

- Mark $A$: $E$ is at $(p, 2q-1)$, $EB$ is column $p$ of $T_3$, and by (5) $T_3$ contains $[p, 3p-2q) \times [q, 2q)$. It contains $(\xi, q)$, as $p \le \xi = 2p-1-q < 3p-2q$. But $(\xi, q) \in T_5$ ($q < p$). A contradiction.
- Mark $E$: $v_3 = q$ and $v_2 = b-q = q < p$, so the mark of $T_2$ is $B$ and $\beta = q$. The corner $E$ of $T_2$ is $(2p-q, 0)$ and $EA$ is column $2p-q$; by (4) $T_2$ contains $[2p-q, p+q) \times [0,p)$. By (5) the side $BD$ of $P$ has at least $2p-2q$ cells (the full line at distance $c < 2p-2q$ from $EB$ contains the cell of $BD$ at distance $c$ from $B$), so $2p-2q \le \beta = q$ and $3p-3q \le p$. So $T_5$ and $T_2$ contain $[p, p+q) \times [q,p)$, and $T_3$ has no cell there. $T_3$ and $T_4$ are mirror images in the line between the columns $p-1$ and $p$ (both wide, mark $E$). So $T_4$ has no cell in $[p-q, p) \times [q,p)$; $T_5$ contains these cells, so $x_5 \le p-q$. But $x_5 = 2p-2q > p-q$. A contradiction.

(c) *Two wide on top, none at the bottom.* Lemma 59: $a = 2p$ and $u_3 = u_4 = p$. So the marks of $T_3, T_4$ are $E$ or $A$, and their sides $EA$ are on top. By (3), $q \le \xi \le 2p-1-q$. Reflect $R$ so that $\xi = x_5$ ($EA$ is the left side of $T_5$). $p = H_5 \le b-1$; put $\chi = b-p \ge 1$. The rows $[p, b)$ lie in $T_3$ and $T_4$ alone ($H_1 = H_2 = H_5 = p$), so

$$\eta_j = p \quad \text{for } j < \min(\chi, q). \tag{6}$$

If $\chi \ge q$, $P$ is a rectangle; so $\chi < q$ and $b-q = p-(q-\chi) < p$. In the rows $[0, b-q)$ there are only $T_1, T_2, T_5$. The columns $[0, x_5)$ there lie in $T_1$ alone ($\mathrm{box}(T_2)$ starts at $2p-q > x_5$), so $x_5 \le q$; with (3), $x_5 = q$. And $T_1$ contains $[0,q) \times [0, b-q)$. So the corner $D$ of $T_1$ is not in its bottom row, its mark is not $A$: the mark of $T_1$ is $E$ or $B$, its $EB$ is row 0, and

$$\kappa_c = q \quad \text{for } c < b-q. \tag{7}$$

- (c1) *$E$ of $T_5$ at its lower left corner* $(q, 0)$. By (7) $T_5$ contains $[q, 2q) \times [0, b-q)$. So in the rows $[0, b-q)$, $T_2$ has exactly the columns $[2q, 2p)$, and $2p-q \le 2q$.
  - Mark of $T_2$ is $A$. Then $v_2 = p$ and $v_3 = \chi < q$, so the mark of $T_3$ is $A$, the corner $E$ of $T_3$ is $(p, b-1)$, $EB$ is column $p$, and by (7) $T_3$ contains $[p, p+b-q) \times [b-q, b)$. The corner $E$ of $T_2$ is $(2p-1, p-1)$ and $EB$ is row $p-1$: $T_2$ contains $[2p-q, 2p) \times \{p-1\}$. The cell $(2p-q, p-1)$ is in both, since $p \le 2p-q < p+b-q$ (because $b > p$) and $b-q \le p-1$. A contradiction.
  - Mark of $T_2$ is $E$ or $B$. $EB$ is in row 0, and row 0 of $T_2$ has $q$ cells, so $2p-2q = q$. So $2p = 3q$, and the boxes of $T_1$, $T_5$, $T_2$ are $[0,q)$, $[q,2q)$, $[2q,3q)$ times $[0,p)$. If the mark of $T_4$ is $A$, its corner $E$ is $(p-1, b-1)$ and by (7) $T_4$ contains $[p-(b-q), p) \times [b-q, b)$; it contains $(q, p-1)$, since $p-b+q \le q < p$ and $b-q \le p-1$. But column $q$, rows $[0,p)$, is in $T_5$. So the mark of $T_4$ is $E$, and $v_4 = q$. Then $v_1 = b-q = p-(q-\chi) < p$, so the mark of $T_1$ is $B$, and its $EA$ is column $q-1$. If the mark of $T_2$ is $B$, then $v_2 = \beta = b-q$ and $v_3 = q$, so the mark of $T_3$ is $E$; then the reflection of $R$ in its vertical midline exchanges $T_1, T_2$ (tall, $B$) and $T_3, T_4$ (wide, $E$), and $|S|$ is even by Lemma 54, a contradiction. So the mark of $T_2$ is $E$, $v_2 = p$, $v_3 = \chi$, and the mark of $T_3$ is $A$. Now count columns by (2). For $c < q$, column $c$ meets $T_1$ and $T_4$, and column $2p-1-c$ meets $T_2$ and $T_3$; for $p-q \le c < q$, column $q+c$ meets $T_5$ and $T_3$. So

$$\eta_{q-1-c} + \kappa_c = b,$$
$$\eta_c + \kappa_{p-1-c} = b,$$
$$\eta_c + \kappa_{c-(p-q)} = b.$$

    From the last two with $c = p-q$ ($< q$), $\kappa_{q-1} = \kappa_0 = q$. From the first with $c = q-1$, $p + \kappa_{q-1} = b$, so $\kappa_{q-1} = \chi < q$. A contradiction.
- (c2) *$E$ of $T_5$ at its upper left corner* $(q, p-1)$. The corner $D$ of $T_5$ is its lower right corner, $EB$ is row $p-1$, and by (7) $T_5$ contains $[q, 2q) \times [q-\chi, p)$. If $2p \ge 3q$, $\mathrm{box}(T_2)$ and $\mathrm{box}(T_5)$ have no common column, and in the rows $[0, b-q)$ the columns $[q, 2q)$ lie in $T_5$ alone, $D$ included, a contradiction. So $2p < 3q$, and $2p-q$ is one of the columns $[q, 2q)$.
  - Mark of $T_2$ is $A$ or $B$: in the first case $EB$ of $T_2$ is row $p-1$ of its box, in the second $EA$ is column $2p-q$; both times $(2p-q, p-1) \in T_2$. But it is in $T_5$. A contradiction.
  - Mark of $T_2$ is $E$: $v_2 = p$, $v_3 = \chi < q$, the mark of $T_3$ is $A$, and $EB$ of $T_3$ is column $p$, rows $[b-q, b)$; so $(p, p-1) \in T_3$. But $q \le p < 2q$ (as $2p < 3q$), so $(p, p-1) \in T_5$. A contradiction.

*$T_5$ wide.* $H_5 = q$, $W_5 = p$, and the full vertical side of $T_5$ is $EB$. If $T_1$ and $T_2$ are both wide, (3) gives $2p \le a-1 \le 2p-1$, a contradiction.

(d) *One wide at the bottom.* Reflect so that $T_1$ is wide and $T_2$ tall. By (3), $p \le \xi$.

- $T_4$ wide, $T_3$ wide: Lemma 59 gives $b = 2q$ and $a = 2p$. $\mathrm{box}(T_4) = [0,p) \times [q,2q)$ meets no other tile ($T_1, T_5$ are in the rows $< q$, $T_2$ in the columns $\ge 2p-q > p$, $T_3$ in the columns $\ge p$). So $P$ is a rectangle, a contradiction.

- $T_4$ wide, $T_3$ tall: $b = 2q$ (left) and $b = 2p$ (right), a contradiction.
- $T_4$ tall, $T_3$ tall: $b = 2p$ (right) and $b = v_1 + v_4 \le p + q$, a contradiction.
- $T_4$ tall, $T_3$ wide. Then $a, b \le p + q$. If $\xi = x_5$, then $2p \le x_5 + p \le a - 1 < p + q$. So $\xi = x_5 + p - 1$; $(\xi, 0)$ is the lower right corner of box$(T_5)$ and occupied, the bottom interval of $T_5$ ends there, $a = x_5 + p + u_2$, and $u_2 = a - p - x_5 \le q - 1$. So the mark of $T_2$ is $A$: $u_2 = \alpha$, $v_2 = p$; the corner $D$ of $T_2$ is $(a - q, 0)$ and $E$ is $(a - 1, p - 1)$. Let $\Sigma = [0, x_5 + p) \times [0, q)$. We claim that $T_1$ and $T_5$ together are $\Sigma$, that $\beta = x_5$ and that $v_1 = q$.
  If $E$ of $T_5$ is $(\xi, 0)$, the bottom row of box$(T_5)$ is $EA$, so $u_1 = x_5 < p$ (as $x_5 \le a - 1 - p < q$), the mark of $T_1$ is $B$, $\beta = x_5$, $v_1 = q$, the corner $E$ of $T_1$ is $(0, q - 1)$, and row $q - 1$ contains $[0, p)$ of $T_1$. $T_5$ has $B$ at $(\xi, q - 1)$ and its run there is $\beta = x_5$: $[p, p + x_5) \times \{q - 1\}$. So row $q - 1$ of $\Sigma$ is in $T_1$ and $T_5$.
  If $E$ of $T_5$ is $(\xi, q - 1)$, then $[x_5, x_5 + p) \times \{q - 1\} \subseteq T_5$. If the mark of $T_1$ is $A$, column $p - 1$, rows $[0, q)$, is in $T_1$ $(EB)$; if it is $B$, $[0, p) \times \{q - 1\}$ is in $T_1$ $(EA)$. Both contain $(p - 1, q - 1)$, and $x_5 \le p - 1$: a contradiction. So the mark of $T_1$ is $E$, $u_1 = p$ and $v_1 = q$. The lower right corner $(\xi, 0)$ of box$(T_5)$ is $B$, and the bottom row of box$(T_5)$ is $BD$. The cells of $T_5$ on it are the columns from $u_1 = p$ to $\xi = x_5 + p - 1$ ($T_5$ has an interval between $T_1$ and $T_2$). So the cells of $P$ on $BD$ are exactly $x_5$ cells, those next to $B$. So $\beta = x_5$ here as well (if $B$ is a mark, its run is this number). The top row of box$(T_1)$ is $BD$ with $B = (0, q - 1)$: $[0, x_5) \times \{q - 1\} \subseteq T_1$. So row $q - 1$ of $\Sigma$ is in $T_1$ and $T_5$.
  In both cases the right column $\xi$ of $\Sigma$ is in $T_5$ $(EB)$, and box$(T_1)$ and box$(T_5)$ lie in $\Sigma$. $T_2, T_3, T_4$ have cells outside $\Sigma$ (their corners of $R$; $b \ge p + 1 > q$), and to enter $\Sigma$ they would have to enter row $q - 1$ or column $\xi$ of $\Sigma$. So $\Sigma = T_1 \cup T_5$, which proves the claim. So $T_2, T_3, T_4$ tile the rest: $[x_5 + p, a) \times [0, q)$ ($\alpha$ columns) and $[0, a) \times [q, b)$; and $v_4 = b - q$.
  - $v_4 < p$: $T_4$ is tall, so its mark is $B$ and $v_4 = \beta = x_5$, $b = q + \beta$. Then $v_3 = b - p = q - (p - \beta) < q$, so the mark of $T_3$ is $A$: $\alpha = q + \beta - p$, $u_3 = p$, $u_4 = a - p = x_5 + \alpha = \beta + \alpha$, and $u_4 = q$ (tall, mark $B$). So $\alpha + \beta = q$ and $p = 2\beta$. box$(T_4) = [0, q) \times [q - \beta, q + \beta)$, the corner $E$ of $T_4$ is $(q - 1, b - 1)$, and $EA$ is its column $q - 1$, so $(q - 1, q - 1) \in T_4$. But it is in $\Sigma$. A contradiction.
  - $v_4 = p$: $b = p + q$ and $v_3 = b - v_2 = q$. box$(T_4) = [0, q) \times [q, b)$, $T_3$ lies in the rows $\ge p$, $T_2$ in the rows $< p$. In the rows $[0, q)$, the $\alpha$ columns of the rest lie in $T_2$ alone, and $T_2$ has no cell of $\Sigma$; so
    $$\kappa_{p-1-c} = \alpha \quad \text{for } c < q \tag{8}$$
    (rows of $T_2$ counted from the bottom; its $EB$ is on top). The mark of $T_4$ is $E$ or $A$. If it is $E$, $EB$ of $T_4$ is on top, and row $q$ of $R$ (the lowest row of box$(T_4)$) has $\kappa_{p-1} = \alpha$ cells of $T_4$, $\kappa_{p-1-q}$ cells of $T_2$ and no cell of $T_3$ ($q < p$); so $\kappa_{p-1-q} = a - \alpha = x_5 + p > q$, a contradiction. If it is $A$, $EB$ of $T_4$ is row $q$; row $b - 1$ has $\kappa_{p-1} = \alpha$ cells of $T_4$ (also $u_4 = \alpha$) and the rest, $a - \alpha = x_5 + p > p$ cells, in $T_3$, whose box has $p$ columns. A contradiction.

(e) *None wide at the bottom.*

- $T_3, T_4$ tall: Lemma 59 gives $a = 2q$ and $b = 2p$. The rows $[p, 2p)$ lie in $T_3$ and $T_4$ alone ($H_5 = q < p$), whose boxes are disjoint. So $P$ is a rectangle, a contradiction.
- One of them wide: reflect so that $T_3$ is tall and $T_4$ wide: $b = 2p$ (right) and $b \le p + q$ (left), a contradiction.
- $T_3, T_4$ wide: $a = 2p$, $u_3 = u_4 = p$, the marks of $T_3, T_4$ are $E$ or $A$, and $EA$ is on top. $H_1 = p \le b - 1$. If $b > 2q$, row $q$ lies in $T_1$ and $T_2$ alone ($q < b - q$ and $q \ge H_5$), which have at most $2q < 2p$ cells there. So $p < b \le 2q$. A column $x$ with $q \le x < 2p - q$ does not meet $T_1, T_2$, and $T_3$ or $T_4$ has at most $q < b$ cells there; so $T_5$ meets all these columns: $x_5 \le q$ and $x_5 + p \ge 2p - q$. Reflect $R$ so that $\xi = x_5$ ($EB$ is the left side of $T_5$). $(x_5, 0)$ is the lower left corner of box$(T_5)$, occupied, so $u_1 = x_5$, and $u_1$ is $q$ or $\alpha$.
  - $u_1 = \alpha$: the mark of $T_1$ is $A$, $v_1 = p$, $v_4 = b - p \le 2q - p < q$, so the mark of $T_4$ is $A$ and $\alpha = v_4 = b - p = x_5$. The corner $E$ of $T_4$ is $(p - 1, b - 1)$, and $EB$ is its column $p - 1$. Column $p - 1$ ($q \le p - 1 < 2p - q$) has $q$ cells of $T_4$ and $\kappa_i$ cells of $T_5$ with $i = p - 1 - x_5 = 2p - 1 - b$: $\kappa_i = b - q$. Column $x_5 = b - p$ ($< q$) has $\eta_{b-p}$ cells of $T_1$ ($EA$ on the left), $\kappa_{p-1-(b-p)} = \kappa_i$ cells of $T_4$ and $\kappa_0 = q$ cells of $T_5$, and no others: $\eta_{b-p} = b - (b - q) - q = 0$. But $\eta_j \ge 1$ for $j < q$, and $b - p < q$. A contradiction.
  - $u_1 = q = x_5$: the mark of $T_1$ is $E$ or $B$, and its $EB$ is row 0. $[0, q) \times [0, b - q)$ lies in $T_1$ alone ($T_5$ is in the columns $\ge q$, $T_4$ in the rows $\ge b - q$, $T_2$ in the columns $\ge 2p - q > q$), so
    $$\kappa_c = q \quad \text{for } c < b - q. \tag{9}$$
    So $T_5$ contains $[q, b) \times [0, q)$ (as $b \le 2q \le p + q$).
    - $b < 2q$. If the mark of $T_4$ is $A$, by (9) it contains $[p - (b - q), p) \times [b - q, b)$; then $(p - 1, b - q)$ is in $T_4$ and in $T_5$ ($q \le p - 1 < b$, $b - q < q$). So the mark of $T_4$ is $E$, $v_4 = q$, $v_1 = b - q < q < p$, and the mark of $T_1$ is $B$. If the mark of $T_3$ is $A$, by (9) it contains $[p, p + b - q) \times [b - q, b)$; then $(p, b - q)$ is in $T_3$ and in $T_5$ ($p < b$). So the mark of $T_3$ is $E$, $v_3 = q$, $v_2 = b - q$, and the mark of $T_2$ is $B$. Then the reflection of $R$ in its vertical midline permutes $T_1, \dots, T_4$, and $|S|$ is even, a contradiction.
    - $b = 2q$. $T_3, T_4$ are in the rows $\ge q$ and $T_5$ in the rows $< q$. A column $x$ with $q \le x < 2p - q$ has at most $q$ cells of $T_5$ and at most $q$ of $T_3$ or $T_4$, so both are full there. So $T_4$ contains $(p - 1, q)$ and $T_3$ contains $(p, q)$: their marks are not $E$, so they are $A$, $v_4 = v_3 = \alpha$, and $v_1 = v_2 = 2q - \alpha > q$. The mark of $T_1$ is $E$ ($v_1 = p$) or $B$ ($v_1 = \beta < p$). If it is $B$, $v_2 = v_1 < p$ gives the mark $B$ for $T_2$; if it is $E$, $v_2 = p$ gives the mark $E$ or $A$ for $T_2$. If $m_1 = m_2$, the reflection in the vertical midline permutes $T_1, \dots, T_4$, a contradiction. So $m_1 = E$, $m_2 = A$, and $p = 2q - \alpha < 2q$. Put $c = p - q < q$. Column $c$ has $\eta_c$ cells of $T_1$ ($EA$ on the left), $\kappa_{p-1-c}$ of $T_4$ ($EB$ on the right), and none of $T_5$ ($c < q = x_5$). Column $2p - 1 - c$ has $\eta_c$ cells of $T_2$ (mark $A$, $EA$ on the right), $\kappa_{p-1-c}$ of $T_3$ ($EB$ on the left), and $T_5$ meets it: $2p - 1 - c = p + q - 1$ is the last column of box$(T_5)$. So the two sums differ by at least 1, but both are $b$. A contradiction. □

**5.6 Two adjacent occupied corners.** Let $A$ and $C$ be the occupied corners, $F$ the side of box$(P)$ between them, $L$ its length and $M$ the other side length of box$(P)$. One of $A, C$ is used, so $F$ is full. Let $r_A$ and $r_C$ be the runs along the sides perpendicular to $F$, for used $A$ and $C$; $1 \le r_e < M$ for each used $e$, as the other corners are not occupied. Each $r_e$ used below is the run of a used $e$: the $r_{m_i}$ are runs of marks; at the other end, $A$ and $C$ are both used in Proposition 63 (by Corollary 57) and in the types (h,v,h,v) and (v,v,h,v) of Proposition 64 (shown there), and in type (v,v,h,h) $(\star\star)$ gives the cells at both ends. *Layer* $j$ of $P$ is the line at distance $j$ from $F$, and $\lambda(j)$ is its number of cells $(0 \le j < M)$; $\lambda(0) = L$, and $f = \lambda(M-1) \ge 1$; the two ends of layer $M-1$ are the corners not in $P$, so $f \le L-2$. For $e \in \{A, C\}$, *position* $c$ from $e$ is the line perpendicular to $F$ at distance $c$ from $e$, and $X_e(c)$ is its number of cells $(0 \le c < L)$; $X_A(c) = X_C(L-1-c)$, and $X_e(0) = r_e$ for used $e$. The side $F$ of $T_i$ is horizontal (type h) or vertical (type v); for a corner tile it lies on a side of $R$. Type h: $W_i = L$, $H_i = M$, $u_i = L$, $v_i = r_{m_i}$. Type v: $W_i = M$, $H_i = L$, $u_i = r_{m_i}$, $v_i = L$. We write $m'$ for the other one of $A, C$. (If $L = M$, the types are still defined by $F$.)

**Lemma 62.** *At a free side $s$, at least one of the two corner tiles has its $F$ along $s$.*

*Proof.* Suppose both have $F$ perpendicular to the side $s$. Both boxes have extent $L$ perpendicular to $s$ and $M$ along it. With equal marks $m$, Lemma 51 gives $r_m = M$, against $r_m < M$. With different marks: $F$ of one tile is on the side of $R$ perpendicular to $s$ at one end, with its end $m$ at $s$; $F$ of the other is on the opposite side of $R$, with its end $m'$ at $s$. The half turn of the strip of the $L$ lines next to $s$ maps the first tile to a copy with $F$ on that opposite side, with its end $m'$ at $s$ and its box on the inner side; by the paragraph on placed copies this is the second tile. Against Lemma 55. □

**Proposition 63.** *In pattern N, two adjacent occupied corners do not occur.*

*Proof.* By Corollary 57 both $A$ and $C$ are used. There are four free sides and four sides $F$, each on a side of $R$, so every side of $R$ contains exactly one $F$ (Lemma 62). So going around, they all lie on the side that follows the corner or all on the side that precedes it; reflect $R$ so that $F$ is on the bottom for $T_1$, on the right for $T_2$, on top for $T_3$ and on the left for $T_4$. By (1), $a = L + r_{m_2} = L + r_{m_4}$ and $b = L + r_{m_1} = L + r_{m_3}$. If $r_A \ne r_C$, then $m_1 = m_3$ and $m_2 = m_4$, the half turn of $R$ permutes $T_1, \dots, T_4$, and $P$ has a half turn by Lemma 54, which maps $\{A, C\}$ to the other two corners, a contradiction. So $r_A = r_C =: r$ and $a = b = L + r =: s$. The frame is invariant under a quarter turn; let $F$ of $T_5$ be horizontal, $[x_5, x_5 + L) \times \{y\}$. Then $1 \le x_5 \le s - 1 - L = r - 1$, so $r \ge 2$.

$r < L$: $T_4$ contains $(x, r)$ for $x < r$ (the run at the far end $(0, r)$ of its $F$). If $r \ge L$, $T_1$, which lies in the columns $< L$, has no cell in row $r$, so $M \le r$; but $r < M$.

The following cells are in corner tiles: $(L-1, y')$ with $y' < r$ in $T_1$ (the side of box$(T_1)$ at the far end of $F$); $(r, y')$ with $y' \ge L$ in $T_3$; $(r-1, r)$ in $T_4$ (the far end of its $F$, at $(0, r)$); $(L, L-1)$ in $T_2$ (the far end of its $F$, at $(s-1, L-1)$). The columns $r-1, r, L-1, L$ are in $[x_5, x_5 + L)$, as $1 \le x_5 \le r-1$ and $r < L$. So $r + 1 \le y \le L - 2$. Row $y$ is the lowest or the highest row of box$(T_5)$, which has $M$ rows and misses the rows 0 and $s-1$; so $M \le L-2$ in both cases. By a half turn of $R$, let it be the lowest.

Consider the lines at distance $c < r$ from a side of $R$. $T_3$ is in the rows $\ge s - M \ge r + 2$ and the columns $\ge r$, $T_4$ in the rows $\ge r$ and the columns $< M \le L - 2$, $T_2$ in the columns $\ge s - M \ge r + 2$, $T_1$ in the columns $< L$, and $T_5$ in the rows $\ge r + 1$. So by (2), for $c < r$, with $e$ and $e'$ the left and right end of $F$ of $T_5$, $x' = s - L - x_5 = r - x_5$, and $X_e$ of a negative argument read as 0: for row $c$ from the bottom, column $c$ from the left and column $c$ from the right,

$$\lambda(c) + X_{m_2}(c) = s,$$
$$\lambda(c) + X_{m_1}(c) + X_e(c - x_5) = s,$$
$$\lambda(c) + X_{m_3}(c) + X_{e'}(c - x') = s.$$

With $c = x_5$ $(\le r-1)$ and $c = x'$ $(\le r-1)$: $m_1 \ne m_2$ and $m_3 \ne m_2$, as $X_e(0)$ and $X_{e'}(0)$ are both at least 1. So $m_1 = m_3$. Now $\mathrm{col}_{T_1}(c) = X_{m_1}(c) = \mathrm{col}_{T_3}(s-1-c)$ for $0 \le c < L$ and $\mathrm{col}_{T_4}(c) = \lambda(c) = \mathrm{col}_{T_2}(s-1-c)$ for $0 \le c < M$, and these tiles have no other columns. So the columns $c$ and $s-1-c$ have the same number of cells of $T_1, \dots, T_4$ in total, and by (2) of $T_5$. The columns of $T_5$ are $[x_5, x_5 + L)$; so $x_5 = x'$ and $X_e(c) = X_{e'}(c)$ for all $c$, that is, $X_A = X_C$. But the first two equations with $c = x_5$ give $X_{m_2}(x_5) - X_{m_1}(x_5) = X_e(0) \ge 1$. A contradiction. □

**Proposition 64.** *In pattern O, two adjacent occupied corners do not occur.*

*Proof.* Lemma 62 holds for the left, top and right side. If $T_4$ is of type h, the left side makes $T_1$ of type v; if $T_4$ is of type v, the top makes $T_3$ of type h and the right side makes $T_2$ of type v. Up to the reflection in the vertical midline (which exchanges $T_1, T_2$ and $T_3, T_4$), the types $(T_1, T_2, T_3, T_4)$ are (h,v,h,v), (v,v,h,h) and (v,v,h,v). Two facts, $(\star)$ and $(\star\star)$, are used below.

$(\star)$ *If* $\mathrm{col}_{T_1} + \dots + \mathrm{col}_{T_4}$ *is invariant under* $c \mapsto a - 1 - c$, *then $F$ of $T_5$ is horizontal, the left and the right margin of* box$(T_5)$ *are equal, and* $X_A = X_C$. By (2), $\mathrm{col}_{T_5}$ is invariant, and the columns of $T_5$ are those of its box, so the two margins are equal. If $F$ of $T_5$ is vertical, the columns have $\lambda(0), \dots, \lambda(M-1)$ cells, in this or the reversed order, so $L = f$, a contradiction. If it is horizontal, they have $X_e(0), \dots, X_e(L-1)$ cells, so $X_e = X_{e'}$.

*Type* (h,v,h,v). By (1), $L + r_{m_4} = a = L + \mu + r_{m_2}$ and $r_{m_1} + L = b = L + r_{m_3}$. So $m_4 \ne m_2$, both $A$ and $C$ are used, $r_A \ne r_C$, and $m_1 = m_3$. Then $\mathrm{col}_{T_1}(c) = X_{m_1}(c) = \mathrm{col}_{T_3}(a-1-c)$ and $\mathrm{col}_{T_4}(c) = \lambda(c) = \mathrm{col}_{T_2}(a-1-c)$ for all $c$, with $X_e(c) = 0$ outside $0 \le c < L$ and $\lambda(c) = 0$ outside $0 \le c < M$. By $(\star)$, $r_A = X_A(0) = X_C(0) = r_C$. A contradiction.

*Type* (v,v,h,h). By (1), $a = 2L$ (top), $b = L + r_{m_4} = L + r_{m_3}$, and $2L = r_{m_1} + \mu + r_{m_2}$. Put $r = r_{m_3} = r_{m_4}$, so $b = L + r$.

$(\star\star)$ *On each of the two sides of* box$(P)$ *perpendicular to $F$ the cells of $P$ are the $r$ cells next to $F$; so* $X_A(0) = X_C(0) = r$. *If* $m_3 = m_4$, *then* $X_A = X_C$. Suppose $m_3 \ne m_4$. Then $A$ and $C$ are used, $r_A = r_C = r$, and Lemma 50 gives the claim. Suppose $m_3 = m_4$. Then $\mathrm{col}_{T_4}(c) = X_{m_4}(c) = \mathrm{col}_{T_3}(a - 1 - c)$ and $\mathrm{col}_{T_1}(c) = \lambda(c) = \mathrm{col}_{T_2}(a-1-c)$ for all $c$, with $X_e(c) = 0$ outside $0 \le c < L$ and $\lambda(c) = 0$ outside $0 \le c < M$ (the other columns miss the boxes, as $a = 2L$), so $X_A = X_C$

by $(\star)$. If both $A$ and $C$ are used, $r_A = X_A(0) = X_C(0) = r_C$, one of them is $r$, and Lemma 50 gives the claim. If all marks are equal, Proposition 56 gives a reflection of $P$ in an axis-parallel line; it preserves $S$ and moves each corner to an adjacent one, so it exchanges $A$ and $C$, and the cells at the end that is not used are the image of the run of the used one.

So $u_1 = u_2 = r$ and $\mu = 2L - 2r$. The boxes are $\mathrm{box}(T_1) = [0,M) \times [0,L)$, $\mathrm{box}(T_2) = [2L-M,2L) \times [0,L)$, $\mathrm{box}(T_4) = [0,L) \times [b-M,b)$ and $\mathrm{box}(T_3) = [L,2L) \times [b-M,b)$; $F$ is the left column for $T_1$, the right column for $T_2$, and the top row for $T_3$ and $T_4$.

(i) *$F$ of $T_5$ vertical*, in a column $\xi$, rows $[0,L)$. Lemma 53 with $h = L = H_1 = H_2$ gives $M \le \xi$ and $M \le 2L - 1 - \xi$. $(\xi, 0)$ is an occupied corner of $\mathrm{box}(T_5)$. If $\xi = x_5$, the bottom interval of $T_5$ starts at $\xi$, and $\xi = u_1 = r < M$. If $\xi = x_5 + M - 1$, it ends at $\xi$, and $2L - 1 - \xi = u_2 = r < M$. A contradiction.

(ii) *$F$ of $T_5$ horizontal, at the bottom.* Then $\mu = L$, so $L = 2r$, $x_5 = u_1 = r$, $a = 4r$, $b = 3r$, and $\mathrm{box}(T_5) = [r,3r) \times [0,M)$. $M \le b - 1$, as $H_4 = M$.

- $M \le 2r$. A row $y$ with $2r \le y < 3r$ meets only $T_3$ and $T_4$ ($T_1, T_2$ are in the rows $< L = 2r$, $T_5$ in the rows $< M \le 2r$), in layer $3r - 1 - y$ of each: $2\lambda(c) = 4r$ for $c < r$. So the layers $0, \dots, r-1$ of $P$ are full. So $T_5$ contains $[r,3r) \times [0,r)$, $T_1$ contains $[0,r) \times [0,2r)$, $T_2$ contains $[3r,4r) \times [0,2r)$, and $T_3$ and $T_4$ contain $[0,4r) \times [2r,3r)$. All other cells of the tiles (the layers $\ge r$) are in $[r,3r) \times [r,2r)$, and each tile has such cells, since $r < M$. For $T_4$ they are in the columns $[r,2r)$ ($\mathrm{box}(T_4)$ has the columns $[0,2r)$), so at positions in $[r,2r)$ from $m_4$; for $T_3$, in the columns $[2r,3r)$, at positions in $[r,2r)$ from $m_3$; for $T_1$, in the rows $[r,2r)$, at positions in $[r,2r)$ from $m_1$; for $T_2$ from $m_2$. If $A$ and $C$ were both used, a cell of $P$ in a layer $\ge r$ would have positions $\ge r$ from $A$ and from $C$, but these add up to $L - 1 = 2r - 1$. So all marks are equal, say to $A$; by Proposition 56 $P$ has a reflection that exchanges $A$ and $C$, and the same cells are at positions $\ge r$ from $C$. A contradiction.
- $M > 2r$. Put $t = 3r - M$, so $1 \le t \le r - 1$. The row at distance $t$ from the top meets $T_3$ and $T_4$ in layer $t$, $T_5$ in its layer $M - 1$ (its highest row is $M - 1 = 3r - 1 - t$), and not $T_1, T_2$ (they are in the rows $< 2r$): $2\lambda(t) + f = 4r$. Column $t$ from the left meets $T_1$ in layer $t$ and $T_4$ in position $t$ from $m_4$, and not $T_5$ (columns $\ge r$), $T_2$ (columns $\ge 4r - M = r + t$) or $T_3$: $\lambda(t) + X_{m_4}(t) = 3r$. From the right, $\lambda(t) + X_{m_3}(t) = 3r$. So $X_{m_3}(t) = X_{m_4}(t) = r + f/2$, and by $(\star\star)$, in both cases (marks of $T_3, T_4$ different, or equal), $X_A(t) = X_C(t) = r + f/2$. Row $t$ from the bottom meets $T_1, T_2$ in positions $t$ from $m_1, m_2$, $T_5$ in layer $t$, and $T_3, T_4$ in layer $M - 1$ (their lowest row is $b - M = t$): $2(r + f/2) + (2r - f/2) + 2f = 4r$, so $f = 0$. A contradiction.

(iii) *$F$ of $T_5$ horizontal, on top* of its box: row $M-1$, columns $[x_5, x_5 + L)$, $1 \le x_5 \le L - 1$. The bottom interval of $T_5$ is its layer $M - 1$: $\mu = f \le L - 2$, so $2r = 2L - f \ge L + 2$. $T_4$ contains $(L - 1, y)$ for $L \le y < L + r = b$ (the run at the far end of its $F$, by $(\star\star)$). $(L - 1, M - 1)$ is in $T_5$ and $M - 1 \le b - 1$, so $M \le L$. A row at distance $c < r$ from the top meets only $T_3$ and $T_4$ ($T_1, T_2$ are in the rows $< L$, $T_5$ in the rows $< M \le L$): $2\lambda(c) = 2L$. So the layers $0, \dots, r-1$ are full, $T_1$ contains $[0,r) \times [0,L)$, and $T_2$ contains $[2L-r,2L) \times [0,L)$. Row $M - 1 \le L - 1$ of $T_5$ gives $x_5 \ge r$ and $x_5 + L \le 2L - r$. So $2r \le L$, a contradiction.

*Type (v,v,h,v).* By (1), $b = 2L$ (left), $r_{m_3} = L$ (right), $a = L + r_{m_4}$ (top), and $r_{m_1} + \mu + r_{m_2} = a$. Name $m_3 = C$; so $r_C = L < M$. If all marks were $C$, $a = 2L$ and $2L + \mu = a$. So $A$ is used; put $r = r_A$. $M = H_3 \le b - 1 = 2L - 1$, so $r < M \le 2L - 1$. The boxes are $\mathrm{box}(T_1) = [0,M) \times [0,L)$, $\mathrm{box}(T_2) = [a-M,a) \times [0,L)$, $\mathrm{box}(T_3) = [a-L,a) \times [2L-M,2L)$ and $\mathrm{box}(T_4) = [0,M) \times [L,2L)$; $F$ is the left column for $T_1$ and $T_4$, the right column for $T_2$, and the top row for $T_3$.

(i) *$F$ of $T_5$ vertical*: as in (i) above, $M \le \xi$ and $M \le a - 1 - \xi$, and $\xi = u_1 \le \max(r_A, r_C) < M$ or $a - 1 - \xi = u_2 < M$. A contradiction.

(ii) *$F$ of $T_5$ at the bottom of its box*: $\mu = L$, so $r_{m_4} = r_{m_1} + r_{m_2}$. If $m_4 = A$, $r$ would be the sum of two of the numbers $r, L$, so $r = 2L$; but $r < 2L - 1$. So $m_4 = C$: $L$ is the sum of two of $r, L$, so $m_1 = m_2 = A$, $L = 2r$ and $a = 2L$. $T_1$ has in row $L - 1$ (the far end $C$ of its $F$) the columns $[0,L)$, and $T_2$ the columns $[L,2L)$. But $M > L$, so $T_5$ meets row $L - 1$. A contradiction.

(iii) *$F$ of $T_5$ on top of its box*: row $M-1$, columns $[x_5, x_5 + L)$; the bottom interval of $T_5$ is its layer $M - 1$, $\mu = f$, and the two lower corners of its box are not occupied. $M > L$, so $T_5$ meets row $L - 1$. $T_1$ has in row $L - 1$ the first $r_{m'_1}$ cells, $T_2$ the last $r_{m'_2}$:

$$r_{m'_1} + r_{m'_2} \le a - 1, \tag{10}$$

and $f = a - r_{m_1} - r_{m_2}$, $a = L + r_{m_4}$.

- $m_1 = m_2 = A$: (10) gives $2L \le L + r_{m_4} - 1$, so $m_4 = A$, $r \ge L + 1$, and $f = L + r - 2r < 0$. A contradiction.
- $m_1 = m_2 = C$: $f = r_{m_4} - L \ge 1$, so $m_4 = A$, $r = L + f$, and (10) gives $2r \le L + r - 1$. A contradiction.
- $\{m_1, m_2\} = \{A, C\}$: $f = r_{m_4} - r \ge 1$, so $m_4 = C$, $a = 2L$, $f = L - r$, and $r < L$; $1 \le x_5 \le L - 1$. $T_3$ has, at the far end $A$ of its $F$, in column $L$, the rows $[2L - r, 2L)$. $(L, M - 1)$ is in $T_5$, so $M \le 2L - r$.
  - $(m_1, m_2) = (A, C)$. The bottom interval of $T_5$ is $[r, L)$, and the lower left corner of its box is not occupied: $1 \le x_5 \le r - 1$. $T_4$ has, at the lower end $A$ of its $F$, in row $L$, the columns $[0,r)$; so $(x_5, L) \in T_4$. If $C$ is at the upper left corner $(x_5, M - 1)$ of $\mathrm{box}(T_5)$, $T_5$ contains $(x_5, y)$ for $M - L \le y \le M - 1$; as $M - L \le L - r < L \le M - 1$, $(x_5, L) \in T_5$, a contradiction. So $A$ is there, and $T_5$ contains $(x_5, y)$ for $M - r \le y \le M - 1$; $L$ is not among these, so $M - r > L$: $L + r + 1 \le M \le 2L - r$. Count, with $X_e(j) = 0$ outside $0 \le j < L$, $\lambda(j) = 0$ outside $0 \le j < M$, and $\lambda'(j) = \lambda(M - 1 - j)$. The column at distance $c$ from the right has, from $T_2$, $T_3$, $T_5$, and $T_1$ with $T_4$,

$$\begin{aligned} 2L = \lambda(c) + X_C(c) + X_C(c - (L - x_5)) \\ + 2\lambda'(c - (2L - M)), \end{aligned}$$

    and the row at distance $c$ from the top has, from $T_3$, $T_4$, $T_5$, $T_1$ and $T_2$,

$$\begin{aligned} 2L = \lambda(c) + X_C(c) + \lambda(c - (2L - M)) \\ + X_C(c - L) + X_A(c - L). \end{aligned}$$

    Take $c = 2L - M$. Then $c \le L - r - 1 < L - x_5$ and $c < L$. So $2\lambda'(0) = \lambda(0)$: $2f = L$ and $r = L/2$, against $2r + 1 \le L$.

– $(m_1, m_2) = (C, A)$. The bottom interval of $T_5$ is $[L, 2L-r)$, and the lower right corner of its box is not occupied: $x_5 + L - 1 \geq 2L - r$. Put $o = L - x_5$ (its right margin), so $1 \leq o \leq r-1$. If $C$ is at the upper right corner, $T_5$ contains $(x_5 + L - 1, y)$ for $M - L \leq y \leq M - 1$, with $y = L-1$ among them; $T_2$ has in row $L-1$ (the far end $C$ of its $F$) the columns $[L, 2L)$, and $x_5 + L - 1 \geq L$, a contradiction. So $A$ is there. The column at distance $c$ from the right has, from $T_2$, $T_3$, $T_5$, and $T_1$ with $T_4$,

$$2L = \lambda(c) + X_C(c) + X_A(c-o) + 2\lambda'(c-(2L-M)),$$

and the row at distance $c$ from the top has, from $T_3$, $T_4$, $T_5$, $T_1$ and $T_2$,

$$2L = \lambda(c) + X_C(c) + \lambda(c-(2L-M)) + X_A(c-L) + X_C(c-L).$$

Take $c = o \leq r-1$. Then $2L - M \geq r > o$ and $L > o$. So $X_A(0) = 0$, a contradiction. □

**5.7 Two opposite occupied corners.** Let $A$ and $C$ be the occupied corners. By Corollary 57 both are used. For $j \in \{A, C\}$, $h_j$ and $v_j$ are the runs of $j$ along the long and along the short side (for $p = q$ fix one direction as the long one; the first step below shows $p > q$ before it uses these names); $h_j < p$ and $v_j < q$, as the other corners are not occupied, and $j'$ is the other one of $A, C$. $X_j(c)$ is the number of cells of $P$ at distance $c$ from $j$ in the long direction ($0 \leq c < p$; the line perpendicular to the long sides), and $Y_j(c)$ in the short direction ($0 \leq c < q$). $X_j(0) = v_j$, $Y_j(0) = h_j$, $X_{j'}(c) = X_j(p-1-c)$ and $Y_{j'}(c) = Y_j(q-1-c)$. The corner of box$(T_i)$ opposite to the mark $m_i$ is $m_i'$, and it is occupied.

**The boxes alternate.** At a free side the two boxes have different extents perpendicular to the side. Take the bottom and suppose $H_1 = H_2 = H$. $T_1$ contains $(0,0)$ and $(W_1-1, H-1)$, and $T_2$ contains $(a-1, 0)$ and $(a-W_2, H-1)$. As $u_1 < W_1$ and $u_2 < W_2$, $a < W_1 + W_2$ and $W_1 - 1 \geq a - W_2$, against Lemma 52(b). So $p > q$, and (with the left, top and right side) $T_1, T_3$ have one orientation and $T_2, T_4$ the other. Reflect $R$ so that $T_1, T_3$ are wide and $T_2, T_4$ tall:

$$\begin{aligned}
&\mathrm{box}(T_1) = [0,p) \times [0,q), && u_1 = h_{m_1},\ v_1 = v_{m_1};\\
&\mathrm{box}(T_2) = [a-q,a) \times [0,p), && u_2 = v_{m_2},\ v_2 = h_{m_2};\\
&\mathrm{box}(T_3) = [a-p,a) \times [b-q,b), && u_3 = h_{m_3},\ v_3 = v_{m_3};\\
&\mathrm{box}(T_4) = [0,q) \times [b-p,b), && u_4 = v_{m_4},\ v_4 = h_{m_4}.
\end{aligned}$$

Put $H_\Delta = h_C - h_A$, $V_\Delta = v_C - v_A$, $\theta = [m_1{=}C] - [m_3{=}C]$ and $\theta' = [m_2{=}C] - [m_4{=}C]$. Then (1) says

$$H_\Delta\theta + V_\Delta\theta' = \begin{cases} -\mu & \text{(O)},\\ 0 & \text{(N)}, \end{cases} \qquad V_\Delta\theta - H_\Delta\theta' = 0. \tag{11}$$

**Proposition 65.** *In pattern N, two opposite occupied corners do not occur.*

*Proof.* If $H_\Delta$ and $V_\Delta$ are not both 0, the system (11) has determinant $-(H_\Delta^2 + V_\Delta^2)$, so $\theta = \theta' = 0$: $m_1 = m_3$ and $m_2 = m_4$, the half turn of $R$ permutes $T_1, \dots, T_4$, and $P$ has a half turn by Lemma 54, which exchanges $A$ and $C$, so $H_\Delta = V_\Delta = 0$, a contradiction. So $h_A = h_C =: h$, $v_A = v_C =: v$, and $a = b = h + v =: s$. Put $N_p = s - p$ and $N_q = s - q$. Since $h < p$ and $v < q < p$, $N_p < v < q$ and $N_p < N_q$. The frame is invariant under a quarter turn of $R$, so we may let $T_5$ be wide.

*$m_1 = m_3$ and $m_2 = m_4$ is impossible.* Then $P$ has a half turn, which exchanges $A$ and $C$, so a corner tile is determined by its corner of $R$ and its orientation. So the quarter turn of $R$ permutes $T_1, \dots, T_4$, and $p = q$ by Lemma 54, a contradiction.

*Ratios.* Let $j_L, j_R, j_B, j_T$ be the occupied corners of $T_5$ on the left, right, bottom and top side of its box, and $o_L = x_5$, $o_R = s - p - x_5$, $o_B = y_5$, $o_T = s - q - y_5$ its margins, all $\geq 1$; $o_L + o_R = N_p$. For $c < N_p$, the line at distance $c$ from a side of $R$ meets only the two corner tiles at that side and $T_5$ ($T_3$ is in the columns $\geq N_p$ and the rows $\geq N_q$, $T_2$ in the columns $\geq N_q$, $T_4$ in the rows $\geq N_p$, and in the same way for the others). So, with $t_L(c) = X_{j_L}(c - o_L)$, $t_R(c) = X_{j_R}(c - o_R)$, $t_B(c) = Y_{j_B}(c - o_B)$ and $t_T(c) = Y_{j_T}(c - o_T)$, read as 0 for a negative argument, (2) gives for $c < N_p$ for the left, bottom, right and top side of $R$

$$\begin{aligned}
X_{m_1} + Y_{m_4} + t_L &= s,\\
Y_{m_1} + X_{m_2} + t_B &= s,\\
Y_{m_2} + X_{m_3} + t_R &= s,\\
Y_{m_3} + X_{m_4} + t_T &= s,
\end{aligned}$$

and $t_L(o_L) = t_R(o_R) = v$, $t_B(o_B) = t_T(o_T) = h$.

- Three marks equal, the fourth different. The two sides of $R$ that do not end at the corner of the fourth, a horizontal one and a vertical one, have the same corner terms, so $t_V(c) = t_H(c)$ for $c < N_p$, where $V$ and $H$ name these two sides. $o_V < N_p$, as $o_L + o_R = N_p$ and both are $\geq 1$. At $c = o_V$, $t_V(c) = v > 0$, so $t_H(o_V) > 0$ and $o_H \leq o_V$; at $c = o_H$, $t_H(c) = h > 0$, so $o_V \leq o_H$. So $o_V = o_H$, and $h = v$.
- Two and two, equal at neighbouring corners (the only other case, as period two is excluded). Then the corner terms of left and right together and of bottom and top together have the same sum, $X_A + X_C + Y_A + Y_C$, so $t_L + t_R = t_B + t_T$ for $c < N_p$. Let $o$ be the smaller of $o_L, o_R$ and $o'$ the smaller of $o_B, o_T$. At $c = o < N_p$ the left side of this identity is $v$ or $2v$, so $o' \leq o$; at $c = o'$ the right side is $h$ or $2h$, so $o \leq o'$. So $o = o'$, and $v$ or $2v$ equals $h$ or $2h$: $h = v$, $h = 2v$ or $v = 2h$.

*Divisibility.* Write $X_j(z)$ and $Y_j(z)$ for the polynomials with these coefficients, and $[s] = 1 + z + \cdots + z^{s-1}$. All columns, columns from the right, rows, and rows from the top give, by (2),

$$\begin{aligned}
X_{m_1} + z^{N_p}X_{m_3'} + z^{o_L}X_{j_L} + Y_{m_4} + z^{N_q}Y_{m_2'} &= s[s],\\
X_{m_3} + z^{N_p}X_{m_1'} + z^{o_R}X_{j_R} + Y_{m_2} + z^{N_q}Y_{m_4'} &= s[s],\\
X_{m_2} + z^{N_p}X_{m_4'} + Y_{m_1} + z^{N_q}Y_{m_3'} + z^{o_B}Y_{j_B} &= s[s],\\
X_{m_4} + z^{N_p}X_{m_2'} + Y_{m_3} + z^{N_q}Y_{m_1'} + z^{o_T}Y_{j_T} &= s[s].
\end{aligned}$$

Here $j_R = j_L'$ and $j_T = j_B'$ ($T_5$ has one occupied corner on each side of its box, opposite ones). This is $\mathbf{M}(z)\, w = s[s]\,(1,1,1,1)^{\mathsf{T}}$ for $w = (X_A, X_C, Y_A, Y_C)^{\mathsf{T}}$ and a $4 \times 4$ matrix $\mathbf{M}(z)$ over $\mathbb{Z}[z]$ whose entries are sums of powers of $z$. At $z = 1$, let $\epsilon_i = 1$ if $m_i = A$ and $-1$ if $m_i = C$, $\tau = 1$ if $j_L = A$, else $-1$, and $\omega = 1$ if $j_B = A$, else $-1$. In the sums

and differences of the first two and of the last two equations, and of $X_A, X_C$ and $Y_A, Y_C$, the matrix splits into $\left(\begin{smallmatrix}3&2\\2&3\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\epsilon_1-\epsilon_3+\tau & \epsilon_4-\epsilon_2\\ \epsilon_2-\epsilon_4 & \epsilon_1-\epsilon_3+\omega\end{smallmatrix}\right)$, so

$$\det \mathbf{M}(1) = 5((\epsilon_1-\epsilon_3+\tau)(\epsilon_1-\epsilon_3+\omega)+(\epsilon_2-\epsilon_4)^2). \quad (12)$$

It is odd, since $\tau$ and $\omega$ are odd and the differences of the $\epsilon_i$ are even. In the case "two and two, neighbouring" the differences are 2 or $-2$, and the bracket is $x+4$ with $x$ a product of two numbers that are both in $\{1,3\}$ or both in $\{-1,-3\}$; so it is 5, 7 or 13, and (12) is not divisible by 3.

Let $\gamma$ be the greatest common divisor of all coefficients of the four polynomials. $\gamma$ divides $s$: $o_L \geq 1$, $N_p = o_L + o_R \geq 2$ and $N_q > N_p$, so the coefficient of $z^0$ in the first equation is $X_{m_1}(0) + Y_{m_4}(0) = s$. Let $\ell$ be a prime that divides $s/\gamma$. Then $\mathbf{M}(z)\, w/\gamma = (s/\gamma)[s](1,1,1,1)^{\mathsf{T}}$ is 0 modulo $\ell$, and $w/\gamma$ is not 0 modulo $\ell$, so $\det \mathbf{M}(z)$ is 0 modulo $\ell$ ($\mathbb{F}_\ell[z]$ is a domain), and $\ell$ divides $\det \mathbf{M}(1)$.

- $h = v$: $s = 2v$ and $\gamma$ divides $v = X_A(0)$, so 2 divides $s/\gamma$ and $\det \mathbf{M}(1)$, which is odd. A contradiction.
- $h = 2v$ (two and two): $s = 3v$, $\gamma$ divides $v$, and 3 divides $s/\gamma$ and $\det \mathbf{M}(1)$, a contradiction. $v = 2h$: $s = 3h$ and $\gamma$ divides $h = Y_A(0)$; the same. □

**Proposition 66.** *In pattern O, two opposite occupied corners do not occur.*

*Proof.* As $\mu > 0$, (11) gives $(\theta, \theta') \neq (0,0)$, and $H_\Delta$, $V_\Delta$ are not both 0. There are three cases.

(a) $\theta = 0$, $\theta' \neq 0$: $H_\Delta = 0$ and $V_\Delta \theta' = -\mu$; so $m_1 = m_3$, $m_2 \neq m_4$, $h_A = h_C$ and $v_{m_4} = v_{m_2} + \mu$.

(b) $\theta' = 0$, $\theta \neq 0$: $V_\Delta = 0$ and $H_\Delta \theta = -\mu$; so $m_2 = m_4$, $m_1 \neq m_3$ and $h_{m_3} = h_{m_1} + \mu$.

(c) $\theta, \theta' \neq 0$: $V_\Delta = H_\Delta \theta'/\theta = H_\Delta \theta\theta'$ and $2H_\Delta\theta = -\mu$; with $\delta = \mu/2$, $h_{m_3} = h_{m_1} + \delta$, $v_{m_4} = v_{m_2} + \delta$, $m_1 \neq m_3$, $m_2 \neq m_4$, and $|H_\Delta| = |V_\Delta| = \delta$.

(In (a), $h_{m_1} - h_{m_3} = H_\Delta\theta$ and $v_{m_2} - v_{m_4} = V_\Delta\theta'$.) $T_1$ contains $(p-1, q-1)$ and $T_2$ contains $(a-q, p-1)$ (the corners opposite to the marks). $T_5$ contains exactly one of the two lower corners of its box, $Z$, and $Z'$ at the opposite upper corner. Lemma 52(a) applies to $T = T_1$ (rows $[0,q)$, cells $(0,0)$ and $(p-1, q-1)$) and $T' = T_5$ ($H_5 \geq q$), so

$$T_5 \text{ has a cell in row } q-1 \text{ in a column } \geq p. \quad (13)$$

*$T_5$ wide.* Suppose $Z$ is at the lower right. $T_5$ contains $(x_5 + p - 1, 0)$ and $(x_5, q-1)$ and lies in the rows $[0,q)$, so Lemma 52(b) gives $x_5 \geq p$. But $x_5 + p = u_1 + \mu = h_{m_1} + h_Z \leq 2p - 2$, a contradiction. So $Z$ is at the lower left: $x_5 = u_1 = h_{m_1}$, $\mu = h_Z$, and $Z'$ is at $(x_5 + p - 1, q-1)$. As $x_5 + p \leq a - 1 = x_5 + h_Z + v_{m_2} - 1$,

$$p + 1 \leq h_Z + v_{m_2}. \quad (14)$$

- Case (a): $h_Z = h_A = h_C$ and $\mu = h_Z$, so $v_{m_4} = v_{m_2} + h_Z \geq p + 1 > q$, a contradiction.
- $Z = m_1$ (cases (b), (c)): $T_5 = T_1 + (h_{m_1}, 0)$, with the same orientation and corner. In row $q-1$, $T_1$ has the run of $m_1'$ that ends at column $p-1$, the columns $[p - h_{m_1'}, p)$, and $T_5$ the columns $[x_5 + p - h_{m_1'}, x_5 + p)$. They are disjoint, so $h_{m_1'} \leq x_5 = h_{m_1}$. But $m_1' = m_3$ and $h_{m_3} > h_{m_1}$, a contradiction. So $Z = m_3$.
- Case (b): $h_{m_3} = h_{m_1} + \mu = h_{m_1} + h_Z > h_Z$, so $Z \neq m_3$, a contradiction.
- Case (c), $Z = m_3$: $2\delta = \mu = h_{m_3} = h_{m_1} + \delta$, so $h_{m_1} = \delta$, $h_{m_3} = 2\delta$ and $x_5 = \delta$. Name $m_1 = A$ and $m_3 = C$, so $h_A = \delta$ and $h_C = 2\delta$, and put $v = v_{m_2}$, so $v_{m_4} = v + \delta$. Then $a = h_{m_3} + v_{m_4} = v + 3\delta$ and $b = v_{m_1} + h_{m_4} = h_{m_2} + v_{m_3}$. For $(m_2, m_4) = (A, C)$: $v_A = v$, $v_C = v + \delta$, $b = v + 2\delta$; for $(C, A)$: $v_C = v$, $v_A = v + \delta$, $b = v + 2\delta$. Put $N_p = b - p \geq 1$ ($H_2 = p \leq b - 1$) and $N_q = b - q > N_p$. The right margin of $T_5$ is $a - x_5 - p = N_p$. $v + \delta = v_{m_4} < q$ gives $N_q < \delta$. For $c < \delta$, the columns $c$ from the left miss $T_5$ ($x_5 = \delta$), $T_2$ (columns $\geq a - q = N_q + \delta$) and $T_3$ (columns $\geq a - p = N_p + \delta$). So

$$X_{m_1}(c) + Y_{m_4}(c) = b \quad \text{for } c < \delta. \quad (15)$$

The row $c$ from the bottom meets $T_1$ in $Y_{m_1}(c)$, $T_2$ in $X_{m_2}(c)$, $T_5$ in $Y_Z(c)$, $T_3$ in $Y_{m_3'}(c - N_q)$, and $T_4$ in $X_{m_4'}(c - N_p)$ (read as 0 for a negative argument and beyond the box).

  - $(m_2, m_4) = (A, C)$. Row $N_q$: $Y_A(N_q) + X_A(N_q) + Y_C(N_q) + Y_A(0) + X_A(N_q - N_p) = a$, and (15) gives $X_A(N_q) + Y_C(N_q) = b$. So $Y_A(N_q) + X_A(N_q - N_p) = a - b - h_A = \delta - \delta = 0$. But $X_A(N_q - N_p) \geq 1$, as $0 \leq N_q - N_p = p - q < p$. A contradiction.
  - $(m_2, m_4) = (C, A)$. By (15), $X_A + Y_A = b$ for $c < \delta$. The columns $c < \delta$ from the right meet $T_2$ in $Y_C(c)$, $T_3$ in $X_C(c)$, $T_5$ in $X_{Z'}(c - N_p) = X_A(c - N_p)$, and not $T_1$ (columns $< p = a - N_p - \delta$) or $T_4$ (columns $< q = a - N_q - \delta$):

$$Y_C(c) + X_C(c) + X_A(c - N_p) = b.$$

The rows $c < \delta$ from the top meet $T_3$ in $Y_C(c)$, $T_4$ in $X_A(c)$, $T_1$ in $Y_{m_1'}(c - N_q) = Y_C(c - N_q)$, $T_2$ in $X_{m_2'}(c - N_p) = X_A(c - N_p)$, and $T_5$ in $Y_{Z'}(c - N_q) = Y_A(c - N_q)$:

$$a = Y_C(c) + X_A(c) + Y_C(c - N_q) + X_A(c - N_p) + Y_A(c - N_q).$$

For $c < N_q$ the last two give $X_A(c) - X_C(c) = a - b = \delta$. At $c = N_q$ ($< \delta$), row $N_q$ from the bottom is $Y_A(N_q) + X_C(N_q) + Y_C(N_q) + Y_A(0) + X_C(N_q - N_p) = a$; subtracting the right column $N_q$ gives $Y_A(N_q) + Y_A(0) + X_C(N_q - N_p) - X_A(N_q - N_p) = \delta$. As $N_q - N_p < N_q$, $Y_A(N_q) = 2\delta - h_A = \delta$, and by (15) $X_A(N_q) = b - \delta$. Row $N_q$ from the top then gives $Y_C(N_q) = a - (b - \delta) - Y_C(0) - X_A(N_q - N_p) - Y_A(0) = 2\delta - 2\delta - \delta - X_A(N_q - N_p) < 0$. A contradiction.

*$T_5$ tall.* Suppose $Z$ is at the lower left: $T_5$ contains $(x_5, 0)$ and $(x_5 + q - 1, p - 1)$, $T_2$ contains $(a-1, 0)$ and $(a-q, p-1)$, both in the rows $[0,p)$. Lemma 52(b), mirrored, gives $x_5 + q - 1 < a - q$. But $x_5 = u_1$ and $a = x_5 + v_Z + v_{m_2} \leq x_5 + 2q - 2$, a contradiction. So $Z$ is at the lower right: $\mu = v_Z$, $u_1 + \mu = x_5 + q$, and by (13), $x_5 + q - 1 \geq p$:

$$h_{m_1} + v_Z \geq p + 1. \quad (16)$$

From (1), $h_{m_3} + v_{m_4} = h_{m_1} + v_Z + v_{m_2} \geq p + 1 + v_{m_2}$, and $h_{m_3} \leq p - 1$, so $v_{m_4} \geq v_{m_2} + 2$. So case (b) does not hold.

- $Z = m_2$: $T_5 = T_2 - (v_{m_2}, 0)$ (the right edge of its box is at $a - v_{m_2}$: its last column is $a - v_{m_2} - 1$). In row $p-1$, $T_2$

has the run of $m'_2$, the columns $[a-q, a-q+v_{m'_2})$, and $T_5$ the same moved by $-v_{m_2}$. They are disjoint, so $v_{m'_2} \le v_{m_2}$. But $m'_2 = m_4$ and $v_{m_4} > v_{m_2}$, a contradiction. So $Z = m_4$.

- Case (a): $v_{m_4} = v_{m_2} + \mu = v_{m_2} + v_Z > v_Z$, so $Z \neq m_4$, a contradiction.
- Case (c), $Z = m_4$: $2\delta = \mu = v_{m_4} = v_{m_2} + \delta$, so $v_{m_2} = \delta$, $v_{m_4} = 2\delta \le q-1$, $a = h_{m_1} + 3\delta$, and (16) reads $h_{m_1} + 2\delta \ge p+1$. Consider the cell $(a-q, q-1)$. $T_1$ has in row $q-1$ the run of $m'_1 = m_3$, the columns $[p-h_{m_3}, p)$; and $p - h_{m_3} = p - h_{m_1} - \delta \le \delta - 1 \le a - q$ (as $a - q = h_{m_1} + 3\delta - q \ge p - q + \delta + 1$), while $a - q = h_{m_3} + v_{m_4} - q < p$. So the cell is in $T_1$. $T_2$ has in column $a-q$ the run of $m'_2 = m_4$, long, downwards from $(a-q, p-1)$: the rows $[p - h_{m_4}, p)$; $h_{m_4} \ge h_{m_1} \ge p + 1 - 2\delta \ge p - q + 2$ ($h_{m_4}$ is $h_{m_1}$ or $h_{m_1} + \delta$), so $q-1$ is among them, and the cell is in $T_2$. A contradiction. □

### 5.8 Proof of Theorem 2 and of Theorem 1.

*Proof of Theorem 2.* In a tiling as in this section, $|S|$ is 1, 2, 3 or 4. $|S| = 1$ is excluded by Corollary 57, $|S| = 4$ by Proposition 58, $|S| = 3$ by Propositions 60 and 61, and $|S| = 2$ by Propositions 63, 64, 65 and 66. □

*Proof of Theorem 1.* Let $T_1, \dots, T_5$ be a 5-copy tiling of a rectangle by a non-rectangular polyomino $P$; after a translation, a grid isometry, the rectangle is $R(a,b)$. A copy of $P$ has at least two rows and two columns, as $P$ is not a rectangle, so $a, b \ge 2$ and the four corner cells of $R(a,b)$ are four different cells. By Proposition 3 no tile contains two of them, so they lie in four distinct tiles, against Theorem 2. □

**Corollary 67** (Order five). *No polyomino has order* 5.

*Proof.* A rectangle tiles itself, so a rectangular polyomino has order 1. A polyomino of order 5 is therefore not a rectangle, and five congruent copies of it tile a rectangle, against Theorem 1. □

## 6. THE SEARCH: AN INDEPENDENT COMPUTATION

This section proves Theorem 7 and reports a computation that checks Theorem 1 for tiles with at most 460 cells, in the sense of Section 1. No other proof in this paper uses this section.

**Tuples, solutions and families.** For a box $B = R(w,h)$, the eight symmetries of the square, each followed by the translation that puts the image of $B$ with its lower-left cell at $(0,0)$, are named $s_0, \dots, s_7$: $s_0(x,y) = (x,y)$, $s_1(x,y) = (w-1-x, y)$, $s_2(x,y) = (x, h-1-y)$ and $s_3(x,y) = (w-1-x, h-1-y)$ keep the two axis directions, and $s_4(x,y) = (y,x)$, $s_5(x,y) = (h-1-y, x)$, $s_6(x,y) = (y, w-1-x)$ and $s_7(x,y) = (h-1-y, w-1-x)$ exchange them. A *placement* is a triple $(t,x,y)$ with $s_t(B) + (x,y)$ inside $R$; it stands for $u \mapsto s_t(u) + (x,y)$. It is *vertical* if $t \ge 4$. A *tuple* is a list of $k$ placements $g_1, \dots, g_k$, and a *solution* is a set $X \subseteq B$ such that $g_1(X), \dots, g_k(X)$ partition $R$. Each family comes with a set of *known cells* $K$ of $B$ containing $(0,0)$, and a solution is *proper* if it is connected and contains $K$.

- $I(k)$, here with $k = 5$: rectangles with $2 \le a \le b$; boxes with $w \le a$, $h \le b$, $wh > n$, $w + h - 1 \le n$; $g_1 = (0,0,0)$, the images of $B$ under $g_2, g_3, g_4$ contain the corners $(a-1, 0)$, $(a-1, b-1)$, $(0, b-1)$, and $g_5$ is any placement. $K$ is $(0,0)$ and the preimages of these corners under $g_2, g_3, g_4$ respectively.
- $H_1(k)$, $k \ge 3$: $2 \le a \le b-1$; $w = a$, $2 \le h \le a$, $ah > n$, $a + h - 1 \le n$; $g_1 = (0,0,0)$, every other placement is vertical with $0 \le x \le a - h$ and $1 \le y \le b - a$, $g_2 = (t, 0, b-a)$, and if $h < a$ also $g_3 = (t', a-h, b-a)$; the remaining placements are in increasing order. $K$ is the bottom row of $B$, the preimage of $(0, b-1)$ under $g_2$ and, if $h < a$, that of $(a-1, b-1)$ under $g_3$.
- $H_2(k)$, $k \ge 3$: $a \ge 2$; $w = a$, $h \ge 2$, $ah > n$, $a + h - 1 \le n$; $g_1 = (0,0,0)$, $g_2 = (t, 0, b-h)$ with $t \in \{2,3\}$; $a \le b - 2$, $h \le a$, and the other placements are vertical with $0 \le x \le a - h$ and $1 \le y \le b - a - 1$, in increasing order. $K$ is the bottom row of $B$.

Here $n \ge 1$, $R = R(a,b)$ satisfies $ab = kn$, and increasing order is that of one fixed numbering of all placements (any one will do; different placements get different numbers). In $I(5)$ the image of $B$ under $g_2$ contains a corner of $R$, so $B$ is non-empty and $w, h \ge 1$; in $H_1$ and $H_2$, $w = a \ge 2$ and $h \ge 2$ are required. If $w = 1$ or $h = 1$, then $wh = w + h - 1$, which the conditions $w + h - 1 \le n < wh$ exclude. So $w \ge 2$ and $h \ge 2$, and $n \ge 3$: for $n = 1$ and $n = 2$ the families are empty, as they should be, since there every polyomino is rectangular.

*Proof of Theorem 7.* Let $P$ be a non-rectangular polyomino with $n$ cells and $T_1, \dots, T_5$ a 5-copy tiling of $R(a,b)$ by $P$; in case (I) take $a \le b$ after a transposition. We may replace $P$ by any congruent copy. In case (I) let $T_1, \dots, T_4$ be the corner tiles in counterclockwise order from $(0,0)$, let $P = T_1$ and $B = \mathrm{box}(P)$, which has its lower-left cell at $(0,0)$; in cases (II.2) and (II.3) let $P = T$ be the tile containing row 0 and $B = \mathrm{box}(T) = [0,a) \times [0,h)$ with $h = d$. Each tile is $T_i = f_i(P)$ for an isometry $f_i$, and $f_i(B) = \mathrm{box}(T_i)$ lies in $R$; so $f_i$ agrees on $B$ with the placement $g_i = (t_i, x_i, y_i)$, where $s_{t_i}$ is the symmetry with its linear part and $(x_i, y_i)$ is the lower-left cell of $\mathrm{box}(T_i)$. Since $P$ is connected and not its box, $wh > n$ and $w + h - 1 \le n$. In case (I) the box of $T_i$ contains the corner cell of $R$ in $T_i$ as its own corner cell, and its preimage is in $P$, so $P$ is a proper solution of a tuple in $I(5)$. In case (II.2), Lemmas 14 and 20 give the bounds of $H_1(5)$: the vertical tiles have $1 \le y_i \le b-a$ and $0 \le x_i \le a-h$, the tile on $(0, b-1)$ has lower-left cell $(0, b-a)$, and if $h < a$ the tile on $(a-1, b-1)$ is a different one, with lower-left cell $(a-h, b-a)$; and $P \supseteq K$. In case (II.3), $T'$ is placed at $(0, b-h)$ by a symmetry that keeps the axis directions and turns row 0 into the top row of the box, so $t \in \{2,3\}$, the vertical tiles avoid rows 0 and $b-1$, and $P$ is a proper solution of a tuple in $H_2(5)$. In case (II.1), and for any 3-copy tiling, three of the tiles tile a rectangle; as in the proof of Theorem 9, the tiling of that rectangle has one tile on row 0 after an isometry, at most one other horizontal tile, and at least one vertical tile, so the same argument with $k = 3$ gives a proper solution of a tuple in $H_1(3)$ or $H_2(3)$. The tiles whose placement the family does not fix are numbered in the increasing order of their placements, which are distinct because the tiles are. In every case the solution is

congruent to $P$, so it is not a rectangle. By Theorem 16 these are all cases, against the hypothesis. □

The Lean proof of Theorem 7 is another one: it goes through Theorem 9 and Proposition 3 and never uses the families $H_1$ and $H_2$. The arguments for these two families above are written proofs and are not formalized.

**The search.** For each $n$ the program takes $X \subseteq B$ as the unknown; each cell of $B$ is undecided, in $X$ or out of $X$. Tile $i$ covers a cell $c$ of $R$ exactly when the cell of $B$ that $g_i$ sends to $c$ is in $X$, and every cell of $R$ must be covered exactly once. The possible owners of $c$ are the pairs of a tile and a cell of $B$ that the tile sends to $c$; an owner is selected when its cell is in $X$ and a candidate while its cell is undecided. The program starts from $X \supseteq K$; it rejects the tuple if some cell of $R$ has no possible owner, excludes each cell of $B$ that two tiles send to the same cell of $R$, and puts into $X$ the sole possible owner of a cell of $R$ that has only one. Whenever a cell of $B$ gets a value, the exact-cover rules are applied at its images: a cell of $R$ with two selected owners, or with no selected owner and no candidate, is a contradiction; when a cell of $R$ has one selected owner, its other candidates are excluded; when it has no selected owner and one candidate, that candidate is put into $X$. When these rules give nothing more, the configuration is rejected if a cell of $X$ is not connected to $(0,0)$ through cells not excluded; otherwise every undecided cell not so connected is excluded and the exact-cover rules are applied again, until nothing changes. The program then makes one pass of failed-literal probing over the cells of $B$: an undecided cell is probed only if some tile sends it to a cell of $R$ that has no selected owner and exactly two candidates. Each value of the cell, in $X$ first, is set, followed by the exact-cover rules alone (not the connectivity step), and undone; if a value leads to a contradiction, the cell gets the other value, and if that leads to a contradiction too, the configuration is rejected. If a pass sets a value, the connectivity step and another pass follow, until a pass sets nothing; only then is the first undecided cell tried both in and out of $X$. Every value these steps set holds in every solution below the current configuration, so no step removes a leaf that meets all the conditions; probing removes branches that end in a contradiction, so it changes the number of nodes and not the leaves that are solutions. The leaves reached without contradiction are exactly the proper solutions of the tuple, so a search that returns no non-rectangular leaf for all tuples at a given $n$ settles the hypothesis of Theorem 7 at that $n$. The program prints every leaf, including rectangular ones, and a separate reader checks the printed cells and the coverage of the tuple lists.

**What was computed.** The registered reader's verdict on the run of record is `VERDICT CLEAN` with exit code 0. It reports that the run covered all tuples of the five families for $n = 1, \dots, 460$, in 2,300 log files, one per family and $n$, and 2,599,555,992,856 placement tuples. That number is also the count of tuples the definitions of the families give, which `recompute.py` computes from them; for $n = 1$ and $n = 2$ there are no tuples. Which tuples the program visited is what the verdict reports and is not checked here. Two AI checkers of two model families read the result before it was credited; their reruns, the planted-variant calibration and the equality control of the two versions of the program are described in the credit record, a copy of which is in `data/records/`. The per-search records they cite are kept with the project and are not in this package, so those counts are reported there and not checked here.

**Table 2. What the verdict and counters of the run of record report.** For $n = 1, \dots, 460$; a leaf line is a rectangular solution, and "other tilings" are proper solutions that are not rectangles.

| Quantity | Value | Source |
|---|---|---|
| Families per $n$ | 5 | Section 6 |
| Log files | 2,300 | one per family and $n$ |
| Tuples covered, as reported | 2,599,555,992,856 | verdict in `data/` |
| Leaf lines, as reported | 23,136 | counters in `data/records/` |
| Other tilings found | 0 | counters in `data/records/` |

**Limits.** All logs come from one search source; the two versions of the program differ only in probing. For five and three copies the positive controls are rectangular solutions only, so a lost non-rectangular solution would leave no trace in the logs; the proofs carry that case. The probing counters are a report that nothing in the logs ties to. The checkers re-ran a sample of the searches, not all of them, and this package checks the verdict and the digest list, not the logs. Separately, a direct enumeration of all 14,996,820,360 free polyominoes with at most 21 cells, the rectangular ones included, found no 5-copy tiling of any rectangle by a non-rectangular one (computed, not proved; its credit record is in `data/records/`). The computation says nothing for $n > 460$; Theorem 1 is proved for all $n$.

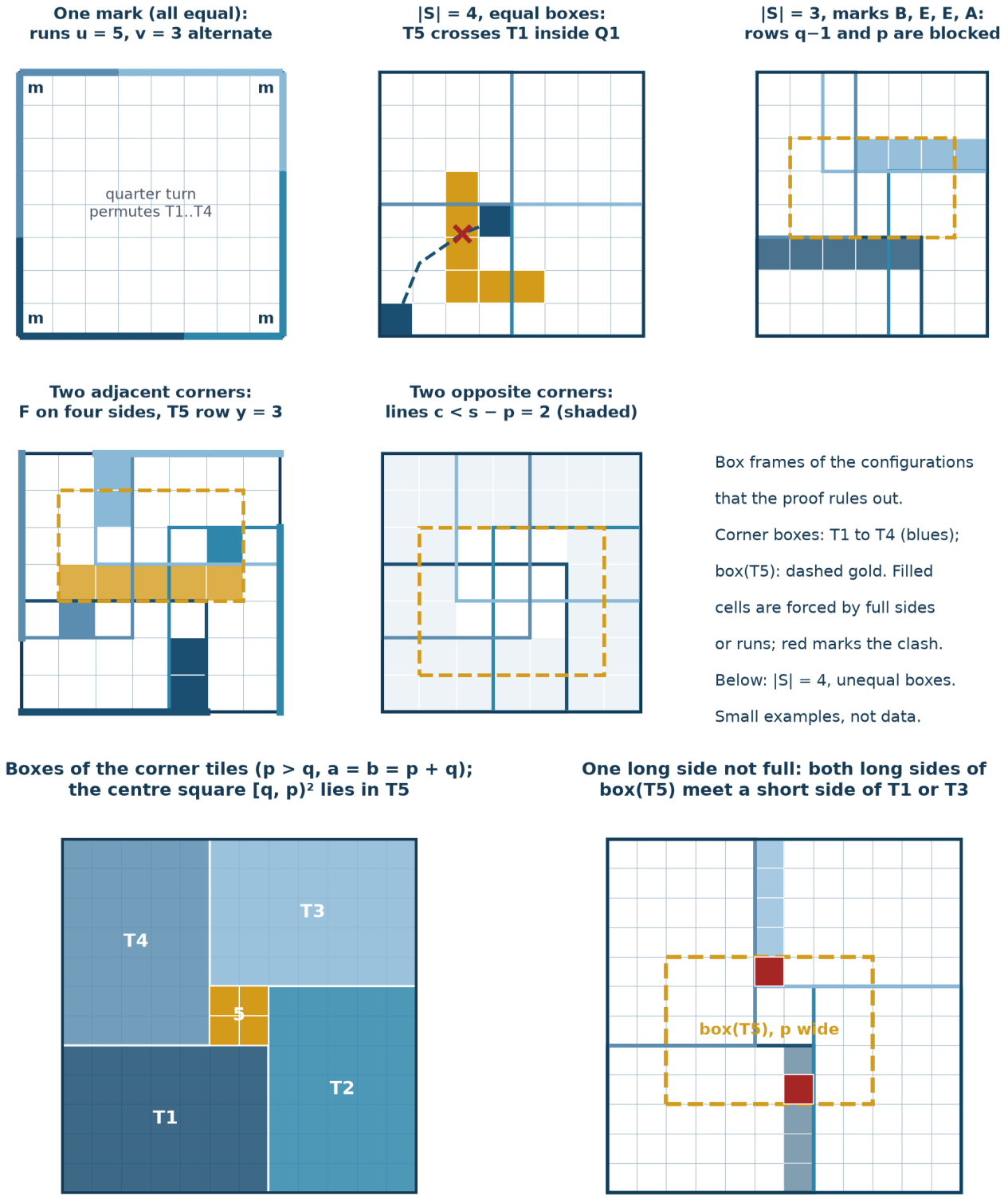


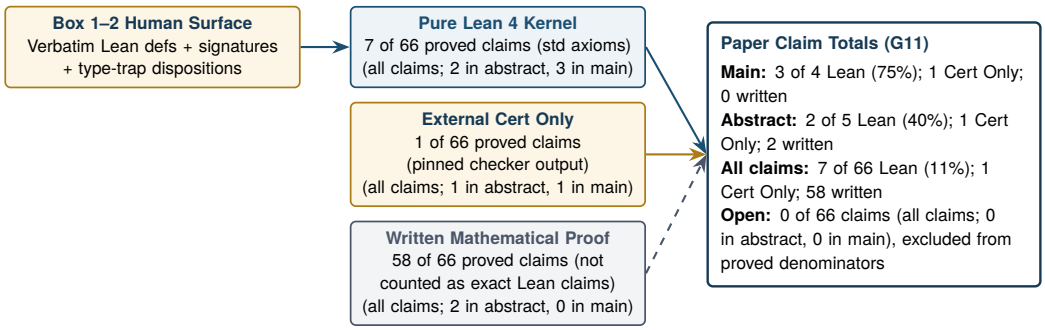

**Figure 4. One frame for each class of occupied corners; the pinwheel.** Boxes of the corner tiles $T_1$ to $T_4$ (blues) and of $T_5$ (dashed gold); filled cells are forced by full sides or runs, and red marks a clash. One mark (Proposition 56); $|S| = 4$ with equal boxes (Proposition 58); $|S| = 3$ with marks $B, E, E, A$ (Proposition 60); two adjacent occupied corners (Proposition 63); two opposite occupied corners, with the lines at distance $c < N_p$ shaded (Proposition 65). Bottom: four occupied corners with unequal boxes (Proposition 58). For $p > q$ the four corner boxes form a pinwheel in the square $R(p+q, p+q)$ and leave out a central square, which $T_5$ contains. When exactly one long side of $P$ is not full, the cells marked on the two long sides of box$(T_5)$ lie in $T_1$ and in $T_3$, so neither long side of $P$ is full. Small examples of configurations that the proof rules out, not data; `figures/case_one_frames.out` records the boxes drawn and the checks they pass.

## 7. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

### What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit c9d2bec53c2d (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 4 below.

**Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)**




**Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)**

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 13/991 public theorems (source-level approx) | WARN: Fidelity: 7/7 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

**Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)**

**PolyominoOddOrder.Cell (PolyominoOddOrder/Defs.lean:24-24) — Section 1 (cells)**

```
abbrev Cell : Type := ℤ × ℤ
```

**PolyominoOddOrder.Adj (PolyominoOddOrder/Defs.lean:30-32)**

```
def Adj (c d : Cell) : Prop :=
  (c.1 = d.1 ∧ (c.2 + 1 = d.2 ∨ d.2 + 1 = c.2)) ∨
    (c.2 = d.2 ∧ (c.1 + 1 = d.1 ∨ d.1 + 1 = c.1))
```

**PolyominoOddOrder.EdgeConnected (PolyominoOddOrder/Defs.lean:36-38)**

```
def EdgeConnected (S : Finset Cell) : Prop :=
  ∀ c ∈ S, ∀ d ∈ S,
    Relation.ReflTransGen (fun x y : Cell => x ∈ S ∧ y ∈ S ∧ Adj x y) c d
```

**PolyominoOddOrder.IsPolyomino (PolyominoOddOrder/Defs.lean:42-43) — Section 1 (polyomino)**

```
def IsPolyomino (P : Finset Cell) : Prop :=
  P.Nonempty ∧ EdgeConnected P
```

**PolyominoOddOrder.GridIso (PolyominoOddOrder/Defs.lean:51-59) — Section 1 (grid isometry)**

```
structure GridIso where
  /-- Exchange the two coordinates first. -/
  swap : Bool
  /-- Then negate the first coordinate. -/
  negX : Bool
  /-- Then negate the second coordinate. -/
  negY : Bool
  /-- Finally translate by this vector. -/
  shift : Cell
```

**PolyominoOddOrder.GridIso.apply (PolyominoOddOrder/Defs.lean:63-67)**

```
def GridIso.apply (g : GridIso) (c : Cell) : Cell :=
  ((if g.negX then -(if g.swap then c.2 else c.1) else (if g.swap then c.2 else c.1))
      + g.shift.1,
    (if g.negY then -(if g.swap then c.1 else c.2) else (if g.swap then c.1 else c.2))
      + g.shift.2)
```

**PolyominoOddOrder.Congruent (PolyominoOddOrder/Defs.lean:70-71)**

```
def Congruent (S T : Finset Cell) : Prop :=
  ∃ g : GridIso, S.image g.apply = T
```

**PolyominoOddOrder.R (PolyominoOddOrder/Defs.lean:76-77) — Section 1 (the rectangle)**

```
def R (a b : ℤ) : Finset Cell :=
  Finset.Ico 0 a ×ˢ Finset.Ico 0 b
```

**PolyominoOddOrder.xSpan (PolyominoOddOrder/Defs.lean:82-83)**

```
def xSpan (S : Finset Cell) : Finset ℤ :=
  S.biUnion fun c => S.biUnion fun d => Finset.Icc c.1 d.1
```

**PolyominoOddOrder.ySpan (PolyominoOddOrder/Defs.lean:86-87)**

```
def ySpan (S : Finset Cell) : Finset ℤ :=
  S.biUnion fun c => S.biUnion fun d => Finset.Icc c.2 d.2
```

**PolyominoOddOrder.boundingBox (PolyominoOddOrder/Defs.lean:91-92) — Section 1 (box)**

```
def boundingBox (S : Finset Cell) : Finset Cell :=
  xSpan S ×ˢ ySpan S
```

**PolyominoOddOrder.IsRectangular (PolyominoOddOrder/Defs.lean:95-96) — Section 1 (rectangular)**

```
def IsRectangular (S : Finset Cell) : Prop :=
  S = boundingBox S
```

**PolyominoOddOrder.bottomRow (PolyominoOddOrder/Defs.lean:99-100)**

```
def bottomRow (a b : ℤ) : Finset Cell :=
  (R a b).filter fun c => c.2 = 0
```

**PolyominoOddOrder.rightCol (PolyominoOddOrder/Defs.lean:103-104)**

```
def rightCol (a b : ℤ) : Finset Cell :=
  (R a b).filter fun c => c.1 = a - 1
```

**PolyominoOddOrder.topRow (PolyominoOddOrder/Defs.lean:107-108)**

```
def topRow (a b : ℤ) : Finset Cell :=
  (R a b).filter fun c => c.2 = b - 1
```

**PolyominoOddOrder.leftCol (PolyominoOddOrder/Defs.lean:111-112)**

```
def leftCol (a b : ℤ) : Finset Cell :=
  (R a b).filter fun c => c.1 = 0
```

**PolyominoOddOrder.corner (PolyominoOddOrder/Defs.lean:121-122) — Section 2 (corner cells)**

```
def corner (a b : ℤ) : Fin 4 → Cell :=
  ![(0, 0), (a - 1, 0), (a - 1, b - 1), (0, b - 1)]
```

**PolyominoOddOrder.Meets (PolyominoOddOrder/Defs.lean:125-126)**

```
def Meets (S T : Finset Cell) : Prop :=
  ∃ c, c ∈ S ∧ c ∈ T
```

**PolyominoOddOrder.IsTiling (PolyominoOddOrder/Defs.lean:141-147) — Section 1 (k-copy tiling)**

```
structure IsTiling (P : Finset Cell) (a b : ℤ) {k : ℕ} (t : Fin k → Finset Cell) :
  ↪ Prop where
  /-- Each tile is the image of `P` under a grid isometry. -/
  congruent : ∀ i, Congruent P (t i)
  /-- Distinct tiles share no cell. -/
  disjoint : ∀ i j, i ≠ j → Disjoint (t i) (t j)
  /-- The tiles together are exactly `R a b`. -/
  cover : Finset.univ.biUnion t = R a b
```

**PolyominoOddOrder.CornersInDistinctTiles (PolyominoOddOrder/Defs.lean:152-153) — Theorem 2 (four corner cells in four distinct tiles)**

```
def CornersInDistinctTiles (a b : ℤ) {k : ℕ} (t : Fin k → Finset Cell) : Prop :=
  ∀ (i : Fin k) (j j' : Fin 4), corner a b j ∈ t i → corner a b j' ∈ t i → j = j'
```

**PolyominoOddOrder.boxSymm (PolyominoOddOrder/Defs.lean:180-182)**

```
def boxSymm (w h : ℤ) (t : Fin 8) (c : Cell) : Cell :=
  ![(c.1, c.2), (w - 1 - c.1, c.2), (c.1, h - 1 - c.2), (w - 1 - c.1, h - 1 - c.2),
    (c.2, c.1), (h - 1 - c.2, c.1), (c.2, w - 1 - c.1), (h - 1 - c.2, w - 1 - c.1)] t
```

**PolyominoOddOrder.Placement (PolyominoOddOrder/Defs.lean:194-201) — Section 6 (placement)**

```
structure Placement where
  /-- The symmetry `s_t`, `t = 0, ..., 7`. -/
  t : Fin 8
  /-- The first coordinate of the translation. -/
  x : ℤ
  /-- The second coordinate of the translation. -/
  y : ℤ
deriving DecidableEq
```

**PolyominoOddOrder.Placement.apply (PolyominoOddOrder/Defs.lean:205-206)**

```
def Placement.apply (p : Placement) (w h : ℤ) (u : Cell) : Cell :=
  ((boxSymm w h p.t u).1 + p.x, (boxSymm w h p.t u).2 + p.y)
```

**PolyominoOddOrder.IsPlacement (PolyominoOddOrder/Defs.lean:210-211)**

```
def IsPlacement (a b w h : ℤ) (p : Placement) : Prop :=
  (R w h).image (p.apply w h) ⊆ R a b
```

**PolyominoOddOrder.Placement.IsVertical (PolyominoOddOrder/Defs.lean:214-215)**

```
def Placement.IsVertical (p : Placement) : Prop :=
  4 ≤ (p.t : ℕ)
```

**PolyominoOddOrder.IsSolution (PolyominoOddOrder/Defs.lean:226-233)**

```
structure IsSolution (a b w h : ℤ) {k : ℕ} (g : Fin k → Placement) (X : Finset Cell) :
    Prop where
  /-- `X` is a set of cells of `B = R w h`. -/
  sub : X ⊆ R w h
  /-- The images of `X` under two different entries of the tuple share no cell. -/
  disjoint : ∀ i j, i ≠ j → Disjoint (X.image ((g i).apply w h)) (X.image ((g
    ↪ j).apply w h))
  /-- The images of `X` together are exactly `R a b`. -/
  cover : Finset.univ.biUnion (fun i => X.image ((g i).apply w h)) = R a b
```

**PolyominoOddOrder.Placement.preimage (PolyominoOddOrder/Defs.lean:238-239)**

```
def Placement.preimage (p : Placement) (w h : ℤ) (c : Cell) : Finset Cell :=
  (R w h).filter fun u => p.apply w h u = c
```

**PolyominoOddOrder.preimageAt (PolyominoOddOrder/Defs.lean:243-245)**

```
def preimageAt (w h : ℤ) {k : ℕ} (g : Fin k → Placement) (j : ℕ) (c : Cell) :
    Finset Cell :=
  Finset.univ.biUnion fun i : Fin k => if (i : ℕ) = j then (g i).preimage w h c else ∅
```

**PolyominoOddOrder.IsProperSolution (PolyominoOddOrder/Defs.lean:254-256) — Section 6 (proper solution)**

```
def IsProperSolution (a b w h : ℤ) {k : ℕ} (g : Fin k → Placement) (K X : Finset
  ↪ Cell) :
    Prop :=
  IsSolution a b w h g X ∧ EdgeConnected X ∧ K ⊆ X
```

**PolyominoOddOrder.FamilyI (PolyominoOddOrder/Defs.lean:278-288) — Section 6, I(k)**

```
def FamilyI (k : ℕ) (num : Placement → ℕ) (n : ℕ) (a b w h : ℤ)
    (g : Fin k → Placement) : Prop :=
  4 ≤ k ∧ 1 ≤ n ∧ a * b = (k : ℤ) * n ∧
  2 ≤ a ∧ a ≤ b ∧
  w ≤ a ∧ h ≤ b ∧ (n : ℤ) < w * h ∧ w + h - 1 ≤ n ∧
  (∀ i, IsPlacement a b w h (g i)) ∧
  (∀ i : Fin k, (i : ℕ) = 0 → g i = ⟨0, 0, 0⟩) ∧
  (∀ i : Fin k, (i : ℕ) = 1 → corner a b 1 ∈ (R w h).image ((g i).apply w h)) ∧
  (∀ i : Fin k, (i : ℕ) = 2 → corner a b 2 ∈ (R w h).image ((g i).apply w h)) ∧
  (∀ i : Fin k, (i : ℕ) = 3 → corner a b 3 ∈ (R w h).image ((g i).apply w h)) ∧
  (∀ i j : Fin k, 4 ≤ (i : ℕ) → i < j → num (g i) < num (g j))
```

**PolyominoOddOrder.knownI (PolyominoOddOrder/Defs.lean:294-296)**

```
def knownI (a b w h : ℤ) {k : ℕ} (g : Fin k → Placement) : Finset Cell :=
  {(0, 0)} ∪ preimageAt w h g 1 (corner a b 1) ∪ preimageAt w h g 2 (corner a b 2) ∪
    preimageAt w h g 3 (corner a b 3)
```

**PolyominoOddOrder.FamilyH1 (PolyominoOddOrder/Defs.lean:314-325) — Section 6, H1(k)**

```
def FamilyH1 (k : ℕ) (num : Placement → ℕ) (n : ℕ) (a b w h : ℤ)
    (g : Fin k → Placement) : Prop :=
  2 ≤ k ∧ 1 ≤ n ∧ a * b = (k : ℤ) * n ∧
  2 ≤ a ∧ a ≤ b - 1 ∧
  w = a ∧ 2 ≤ h ∧ h ≤ a ∧ (n : ℤ) < a * h ∧ a + h - 1 ≤ n ∧
  (∀ i, IsPlacement a b w h (g i)) ∧
  (∀ i : Fin k, (i : ℕ) = 0 → g i = ⟨0, 0, 0⟩) ∧
  (∀ i : Fin k, (i : ℕ) ≠ 0 → (g i).IsVertical ∧
    0 ≤ (g i).x ∧ (g i).x ≤ a - h ∧ 1 ≤ (g i).y ∧ (g i).y ≤ b - a) ∧
  (∀ i : Fin k, (i : ℕ) = 1 → (g i).x = 0 ∧ (g i).y = b - a) ∧
  (h < a → 3 ≤ k ∧ ∀ i : Fin k, (i : ℕ) = 2 → (g i).x = a - h ∧ (g i).y = b - a) ∧
  (∀ i j : Fin k, (if h < a then 3 else 2) ≤ (i : ℕ) → i < j → num (g i) < num (g j))
```

**PolyominoOddOrder.knownH1 (PolyominoOddOrder/Defs.lean:335-337)**

```
def knownH1 (a b w h : ℤ) {k : ℕ} (g : Fin k → Placement) : Finset Cell :=
  bottomRow w h ∪ preimageAt w h g 1 (corner a b 3) ∪
    (if h < a then preimageAt w h g 2 (corner a b 2) else ∅)
```

**PolyominoOddOrder.FamilyH2 (PolyominoOddOrder/Defs.lean:354-366) — Section 6, H2(k)**

```
def FamilyH2 (k : ℕ) (num : Placement → ℕ) (n : ℕ) (a b w h : ℤ)
    (g : Fin k → Placement) : Prop :=
  2 ≤ k ∧ 1 ≤ n ∧ a * b = (k : ℤ) * n ∧
  2 ≤ a ∧
  w = a ∧ 2 ≤ h ∧ (n : ℤ) < a * h ∧ a + h - 1 ≤ n ∧
  (∀ i, IsPlacement a b w h (g i)) ∧
  (∀ i : Fin k, (i : ℕ) = 0 → g i = ⟨0, 0, 0⟩) ∧
  (∀ i : Fin k, (i : ℕ) = 1 → ((g i).t = 2 ∨ (g i).t = 3) ∧ (g i).x = 0 ∧ (g i).y = b
    ↪ - h) ∧
  (k = 2 → h ≤ b) ∧
  (3 ≤ k → a ≤ b - 2 ∧ h ≤ a ∧
    (∀ i : Fin k, 2 ≤ (i : ℕ) → (g i).IsVertical ∧
      0 ≤ (g i).x ∧ (g i).x ≤ a - h ∧ 1 ≤ (g i).y ∧ (g i).y ≤ b - a - 1) ∧
    (∀ i j : Fin k, 2 ≤ (i : ℕ) → i < j → num (g i) < num (g j)))
```

**PolyominoOddOrder.knownH2 (PolyominoOddOrder/Defs.lean:374-375)**

```
def knownH2 (w h : ℤ) : Finset Cell :=
  bottomRow w h
```

**PolyominoOddOrder.IsSearchSolution (PolyominoOddOrder/Defs.lean:384-394) — Theorem 7 (the five families)**

```
def IsSearchSolution (num : Placement → ℕ) (n : ℕ) (a b w h : ℤ) (X : Finset Cell) :
  ↪ Prop :=
  (∃ g : Fin 5 → Placement,
    FamilyI 5 num n a b w h g ∧ IsProperSolution a b w h g (knownI a b w h g) X) ∨
  (∃ g : Fin 5 → Placement,
    FamilyH1 5 num n a b w h g ∧ IsProperSolution a b w h g (knownH1 a b w h g) X) ∨
  (∃ g : Fin 5 → Placement,
    FamilyH2 5 num n a b w h g ∧ IsProperSolution a b w h g (knownH2 w h) X) ∨
  (∃ g : Fin 3 → Placement,
    FamilyH1 3 num n a b w h g ∧ IsProperSolution a b w h g (knownH1 a b w h g) X) ∨
  (∃ g : Fin 3 → Placement,
    FamilyH2 3 num n a b w h g ∧ IsProperSolution a b w h g (knownH2 w h) X)
```

**Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs):** PolyominoOddOrder.Witness.R_card ((R 2 6).card = 12); PolyominoOddOrder.Witness.corner_sample (corner 2 6 2 = (1, 5)).

## Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

**PolyominoOddOrder.corollaryA19_no_five_copy_tiling (PolyominoOddOrder/Main.lean:168-171)** src:52a6bdf66321f3

```
theorem corollaryA19_no_five_copy_tiling (P : Finset Cell) (hP : IsPolyomino P)
    (hnr : ¬ IsRectangular P) (a b : ℤ) (t : Fin 5 → Finset Cell) :
    ¬ IsTiling P a b t
```

**Assumptions:** P is a non-rectangular polyomino; a and b are any integers; t is a family of five sets of cells. The conclusion is that t is not a tiling of R(a, b) by P. Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**PolyominoOddOrder.theoremA18_no_case_one_tiling (PolyominoOddOrder/Main.lean:152-155)** src:836029b030354e

```
theorem theoremA18_no_case_one_tiling (P : Finset Cell) (hP : IsPolyomino P)
    (hnr : ¬ IsRectangular P) (a b : ℤ) (t : Fin 5 → Finset Cell) :
    ¬ (IsTiling P a b t ∧ CornersInDistinctTiles a b t)
```

**Assumptions:** P is a non-rectangular polyomino; a and b are any integers; t is a family of five sets of cells. The conclusion is that t is not a tiling of R(a, b) by P in which no tile holds two of the four corner cells. Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**PolyominoOddOrder.theorem21_search_bridge (PolyominoOddOrder/Main.lean:130-137)** src:d70b9405d7778e

```
theorem theorem21_search_bridge (num : Placement → ℕ) (hnum : Function.Injective num)
    (n : ℕ)
    (hsearch : ¬ ∃ (a b w h : ℤ) (X : Finset Cell),
      IsSearchSolution num n a b w h X ∧ ¬ IsRectangular X)
    (P : Finset Cell) (hP : IsPolyomino P) (hnr : ¬ IsRectangular P) (hn : P.card =
      ↪ n) :
    (∀ (a b : ℤ) (t : Fin 3 → Finset Cell), ¬ IsTiling P a b t) ∧
      (∀ (a b : ℤ) (t : Fin 5 → Finset Cell), ¬ IsTiling P a b t)
```

**Assumptions:** A fixed injective numbering of placements; the absence of a proper solution that is not a rectangle in the five families at this n is a hypothesis; P is a non-rectangular polyomino with n cells. Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**PolyominoOddOrder.theorem62_no_three_copy_tiling (PolyominoOddOrder/Main.lean:102-105)** src:7ff50a99b645f2

```
theorem theorem62_no_three_copy_tiling (P : Finset Cell) (hP : IsPolyomino P)
    (hnr : ¬ IsRectangular P) (a b : ℤ) (t : Fin 3 → Finset Cell) :
    ¬ IsTiling P a b t
```

**Assumptions:** P is a non-rectangular polyomino; a and b are any integers. In print: Maltby 1994. Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**PolyominoOddOrder.corollary1a_crossing (PolyominoOddOrder/Main.lean:37-43)** src:e5b341e692209c

```
theorem corollary1a_crossing (a b : ℤ) (ha : 2 ≤ a) (hb : 2 ≤ b)
    (S S' : Finset Cell) (hSR : S ⊆ R a b) (hS : EdgeConnected S)
    (hSl : Meets S (leftCol a b)) (hSr : Meets S (rightCol a b))
    (hS'R : S' ⊆ R a b) (hS' : EdgeConnected S')
    (hS'b : Meets S' (bottomRow a b)) (hS't : Meets S' (topRow a b)) :
    Meets S S'
```

**Assumptions:** a and b are at least 2; S and S' are edge-connected subsets of R(a, b); S meets the left and right columns, S' the bottom and top rows. Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**PolyominoOddOrder.lemma2_no_opposite_corners (PolyominoOddOrder/Main.lean:57-61)** src:935775390503f5

```
theorem lemma2_no_opposite_corners (P : Finset Cell) (hP : IsPolyomino P)
    (k : ℕ) (hk : 2 ≤ k) (a b : ℤ) (t : Fin k → Finset Cell) (ht : IsTiling P a b t)
    (i : Fin k) :
    ¬ ((0, 0) ∈ t i ∧ (a - 1, b - 1) ∈ t i) ∧ ¬ ((a - 1, 0) ∈ t i ∧ (0, b - 1) ∈ t i)
```

**Assumptions:** P is a polyomino, k is at least 2, t is a k-copy tiling of R(a, b) by P. Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**PolyominoOddOrder.lemma3_adjacent_corners_full_row (PolyominoOddOrder/Main.lean:74-79)** src:9f41e273375be7

```
theorem lemma3_adjacent_corners_full_row (P : Finset Cell) (hP : IsPolyomino P)
    (k : ℕ) (hk : 2 ≤ k) (a b : ℤ) (t : Fin k → Finset Cell) (ht : IsTiling P a b t)
    (i : Fin k) (h0 : (0, 0) ∈ t i) (h1 : (a - 1, 0) ∈ t i)
    (x : ℤ) (hx0 : 0 ≤ x) (hxa : x < a) :
    (x, 0) ∈ t i
```

**Assumptions:** P is a polyomino, k is at least 2, t is a k-copy tiling of R(a, b) by P; tile i holds (0, 0) and (a - 1, 0). Ordinary witness in Sanity.lean, bound by check-lean -f. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

**Table 3.** Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

| Declaration | Hazard | Operand | Category | Audited Disposition & Guard |
|---|---|---|---|---|
| FamilyH1 | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyH1 | nat_sub | - h | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyH1 | nat_sub | - a | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyH2 | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyH2 | nat_sub | - h | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyH2 | nat_sub | - 2 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyH2 | nat_sub | - a | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| FamilyI | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| boxSymm | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| boxSymm | nat_sub | - c.1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |

| Declaration | Hazard | Operand | Category | Audited Disposition & Guard |
|---|---|---|---|---|
| boxSymm | nat_sub | - c.2 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| corner | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| lemma2_no_opposite_corners | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| lemma3_adjacent_corners_full_row | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| rightCol | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |
| topRow | nat_sub | - 1 | intended_convention | Subtraction of integers (a, b, w, h and all coordinates are of type Int), which does not truncate. |

[a] Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

**Table 4.** Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Thm. 1: For every polyomino P that is not a rectangle and all integers a and b, there is no 5-copy tiling of R(a,b) by P (grid isometries, reflections included); no rectangle is tiled by exactly five congruent copies of P | Lean Kernel | PolyominoOddOrder.corollaryA19_no_five_copy_tiling | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. |
| Cor. 67: No polyomino has order 5: a rectangle has order 1, and a non-rectangular polyomino does not tile a rectangle with exactly five copies by Theorem 1 | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Thm. 2: For every polyomino P that is not a rectangle and all integers a and b, there is no 5-copy tiling of R(a,b) by P in which the four corner cells of R(a,b) lie in four distinct tiles | Lean Kernel | PolyominoOddOrder.theoremA18_no_case_one_tiling | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. |
| Prop. 3: In a 5-copy tiling of a rectangle by a non-rectangular polyomino no tile contains two corner cells; the four corner cells lie in four distinct tiles | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: corners_in_distinct_tiles (a frozen statement of the earlier nine-statement receipt, audited exact then). |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Prop. 4: In such a tiling, with the four corner cells in four distinct tiles, every tile meets every side of the rectangle in an interval of cells, possibly empty | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: sides_are_intervals (stronger: every k-copy tiling by a polyomino; audited exact in the earlier receipt in the five-copy form). |
| Prop. 5: In such a tiling, with the four corner cells in four distinct tiles, a tile containing a corner cell meets only the two sides at that corner | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: corner_tile_sides (a frozen statement of the earlier nine-statement receipt, audited exact then). |
| Prop. 6: In such a tiling, with the four corner cells in four distinct tiles, the tile containing no corner cell does not meet two different sides of the rectangle; it meets no side or exactly one | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: free_tile_one_side (a frozen statement of the earlier nine-statement receipt, audited exact then). |
| Thm. 7: If, over all rectangles and boxes allowed, no tuple of the families I(5), H1(5), H2(5), H1(3), H2(3) at cell count n has a proper solution that is not a rectangle, then no rectangle is tiled by exactly 3 or exactly 5 congruent copies of a non-rectangular polyomino with n cells | Lean Kernel | `PolyominoOddOrder.theorem21_search_bridge` | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. |
| Thm. 8: Computed, not proved: the registered verdict of the run of record of the search reports that the run covered every tuple of the families I(5), H1(5), H2(5), H1(3) and H2(3) for every n with 1 <= n <= 460, and that no tuple has a proper solution that is not a rectangle | Cert Only | recompute.py (b4b2bddbb840) [regenerable; replay pending] | External cert only; fidelity: n/a | A computation, not a proof: no proof in the paper uses it. Read through Theorem 7, and if the search is correct and the verdict accurate, it says that no rectangle is tiled by exactly 5, or exactly 3, congruent copies of a non-rectangular polyomino with at most 460 cells. The search program, its logs and the registered reader are kept with the project, so which tuples the program visited is what the verdict reports and is not checked here; recompute.py checks the recorded verdict text and its digest, the full SHA-256 of the digest list of the 2,300 logs and its completeness (5 families, n = 1..460), the number of tuples the definitions of the families give, and the rectangular-leaf count of the counters record. The reruns by two AI checkers are described in the credit record in data/records/. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Thm. 9: No rectangle is tiled by exactly three congruent non-rectangular polyominoes (grid isometries, reflections included) | Lean Kernel | `PolyominoOddOrder.theorem62_no_three_copy_tiling` | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. In print: Maltby 1994, read in full; not an increment over print. |
| Lem. 10: Two connected cell sets that interlace on the boundary share a cell | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: interlace. |
| Cor. 11: For a and b at least 2, a connected subset of R(a,b) meeting the left and the right column and a connected subset of R(a,b) meeting the bottom and the top row share a cell | Lean Kernel | `PolyominoOddOrder.corollary1a_crossing` | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. |
| Lem. 12: In a k-copy tiling of R(a,b) by a polyomino, k at least 2, no tile contains two opposite corner cells | Lean Kernel | `PolyominoOddOrder.lemma2_no_opposite_corners` | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. |
| Lem. 13: In a k-copy tiling of R(a,b) by a polyomino, k at least 2, a tile containing (0,0) and (a-1,0) contains every cell of row 0 | Lean Kernel | `PolyominoOddOrder.lemma3_adjacent_corners_full_row` | Axioms: Classical.choice, Quot.sound, propext; fidelity: exact | Lean 4 kernel with the axioms propext, Classical.choice and Quot.sound. Witnessed: its hypotheses are satisfiable (check-lean -f receipt of the library). A written proof is printed here. |
| Lem. 14: With T containing row 0, horizontal tiles have distinct full rows as spines and vertical boxes are d wide and a high | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: Frame.two_le_d, horizontal_spine, horizontal_spine_injective, horizontal_far_row, vertical_spine, vertical_bounds, separated_by_spine_row. Not stated there: the full side of P itself (the theorems take T as the prototype). |
| Lem. 15: The lines $y = r$ at up-type spines and $y = r + 1$ at down-type spines, with 0 and b, cut the rectangle into t blocks of at least two tiles each, so the number of tiles is at least 2t | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: Frame.clean_line, block, horizontal_spine_injective, two_mul_blocks_le. Not stated there: that a block is a union of tiles, and the clause on a block with an odd number of tiles. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Thm. 16: A 5-copy tiling by a non-rectangular polyomino has shape (I) or at least one of (II.1), (II.2), (II.3) | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: theorem6_five, TypeH1.bounds, TypeH2.bounds. Not stated there: case (I) and the reduction to a tile that contains row 0; in case (II.1), everything except that some rectangle with a columns has a 3-copy tiling by P (no cut of the given tiling, no bounds on d and a); the full side of P itself; that for every choice of the isometries some tile is vertical and one of (II.1) to (II.3) holds (one choice is supplied). |
| Cor. 17: In case (I), with the tile on (0,0) as prototype, the tiling is determined by the rectangle, the prototype and three symmetries | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 18: A vertical tile has, in the row matching column x of the row-0 tile T, as many cells as T has in column x | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: Frame.vertical_rowCount, vertical_mem. |
| Prop. 19: One tile on row 0, all others vertical, square box: the number of tiles is even | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: TypeH1.theorem22. |
| Lem. 20: Top tiles have distinct spine columns and split the columns into blocks, each with one or two top tiles and at least d columns | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: TypeH1.lemma25a to lemma25d, Frame.column_blocks, column_blocks_unique. Not stated there: that the spine columns are distinct (it follows from the separation in lemma25a). |
| Cor. 21: For five copies with T on row 0, four vertical tiles and $d < a$, the number p of top tiles is 2, 3 or 4 and a is at most p d; if $b - a > d$ then $p = 2$ | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: TypeH1.corollary23, corollary26. |
| Prop. 22: Not every vertical tile is a top tile when $d < a$; no 3-copy tiling by a non-rectangular polyomino with one tile containing row 0 and two vertical tiles | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: TypeH1.proposition28, proposition27. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Cor. 23: For five copies with T on row 0 and four vertical tiles, d < a and there are two or three top tiles | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: TypeH1. corollary23, corollary29. |
| Lem. 24: A bottom tile exists, and every vertical tile contains its s columns nearest the spine, s the number of full bottom rows of T | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: Frame. exists_fullRows, thick_spine, vertical_rowCount, TypeH1. lemma30a, lemma30cd. Not stated there: the bounds on s for a given s (an s with these bounds is supplied, and s is unique). |
| Prop. 25: In case (II.2) with two top tiles and b - a > d, the four vertical boxes sit at the two side columns | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: proposition 31, Layout31.prop31_iii, prop31_vi. Not stated there: the two sums of row counts, in (iv) and at the end of (v); the boxes in (ii) are by Frame.vertical_spine, the wall formulas of (v) by Layout31.lemma32d_A to lemma32d_D and Layout31. mirror, each top count by Frame. vertical_rowCount. |
| Lem. 26: In that layout T meets each side column in an interval from row 0 and each side column is in one of four cases A, B, C, D | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: Layout31. lemma32a, lemma32b, col0, col A, full_end, lemma32d_A to lemma32d_D, lemma36c; the other wall by Layout31.mirror. Not stated there: in (c), the clause on an inner tile, except that a = 2d and some end column of T is full; (e). |
| Prop. 27: In that layout the case pairs AA, BB, every pair with D, and CC with a - d at least d - 1 do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: Layout31. excl_AA, excl_BB, excl_D, excl_CC (without the condition on a - d); mirror images by Layout31. mirror. |
| Lem. 28: An inner top tile forces a = 2d in that layout | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: Layout31. lemma32a_inner. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Prop. 29: Case (II.2) with b - a > d does not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: Layout31. absurd, corollary35. |
| Lem. 30: In case (II.2) with b - a at most d each side column is in one of three states, with stated counts | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: lemma36a to lemma36f, CaseII2.thick_V, wall_left, wall_right, gamma_corner; the formula for t at column a-1 by TypeH1.mirror. Not stated there: in (a), the common rows of the spines, that they are at least two, and that thick spines are disjoint; in (d), that only one state holds; in (e), the second and third sentences in general (lemma25c gives the full top row of a one-tile block, gamma_corner an end cell of T in Top); in (f), the second sentence. |
| Prop. 31: In case (II.2) with b - a at most d there are not three top tiles | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: proposition 37. |
| Prop. 32: In case (II.2) with b - a at most d there are not two top tiles | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: proposition 38. |
| Lem. 34: In case (II.3) with d = a all five tiles meet both side columns in intervals, in the same order | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: lemma40a, lemma40b, lemma40c, TypeH2.top_mem. Not stated there: that T comes first and T′ last. |
| Prop. 35: Case (II.3) with a square box does not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: theorem41. |
| Cor. 36: In case (II.3) with d < a, a is at most 3 d and at most b - 2, and every vertical tile starts at a row from s to b - a - s | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: corollary42, lemma43c (a stronger last clause). |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Lem. 37: In case (II.3) with d < a a bottom tile and a top tile exist and none is both | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: lemma43a, lemma43b, lemma43c, TypeH2. top_mem, top_colCount. Not stated there: that no tile is both a bottom and a top tile (it follows from lemma43c); the rows of T′ in (a) are top_mem applied to lemma43a. |
| Prop. 38: In case (II.3) with d < a the row ranges of any two vertical tiles overlap | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: proposition 44, lemma43c. Not stated there: the last clause, on the common rows of the three spines. |
| Lem. 39: In case (II.3) with d < a each side column is in one of four states; under a half turn both are in the same state | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: lemma45a, lemma45a_right, lemma45b; f by lemma43c and proposition44. Not stated there: in (a), the bound b at most 2d in o, the wall interval and the interval Top in beta, that the two tiles in gamma are the only ones, and that only one state holds (at column a-1 the one-tile clauses by TypeH2. mirror); in (b), that the tiles in alpha and beta are the only ones, and the bound and intervals of (a). |
| Lem. 40: In case (II.3) with d < a, for different vertical tiles X and Y, Y not starting below X, the top row of X avoids the spine column of Y and the bottom row of Y avoids the spine column of X; the smaller end column of T has at most a - d cells | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: proposition 44, lemma48a, lemma48b, lemma48c (the spine rows in (a) by the overlap of proposition44). |
| Lem. 41: In case (II.3) with d < a the cells of row s outside T are the bottom rows of the bottom tiles, and the same at the top | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: counts_ bottom_top, lemma43c, lemma 52a, lemma52_row_image, square_reflect, lemma54_half_ turn, lemma54_image, lemma 52b, lemma52c. Not stated there: that no tile is both a bottom and a top tile (it follows from lemma 43c); the equalities with the cells outside T and T′ in (a) (which tiles meet the rows, and how their cells correspond under G, is stated); for the image tiling, lemma54_image exchanges the two classes in a tiling it supplies, not named as the image under G; in (b), the lengths and the two halves. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Prop. 42: In case (II.3) with d < a there is not exactly one bottom tile and one top tile | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: proposition 53. |
| Prop. 43: In case (II.3) with d < a the numbers of bottom and top tiles are equal | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: proposition 54. |
| Prop. 45: No 3-copy tiling by a non-rectangular polyomino in which T contains row 0, a second tile is horizontal with spine row b - 1, and the third is vertical | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: three_copy (stronger: no 3-copy tiling by a non-rectangular polyomino); proposition61 for this case with the second tile down-type. |
| Prop. 46: Case (II.1) does not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 33: Case (II.2) does not occur for k = 5 | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: corollary39. |
| Prop. 44: Case (II.3) does not occur for k = 5 | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, the same statement or a stronger one, not separately audited: theorem55. |
| Lem. 47: Under two directional hypotheses, a tile that meets a region cut off by a path in another tile lies in that region and has both box sides shorter | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a different form, not separately audited: enclosure, enclosure_not_ congruent; they use the component of R minus T that holds a cell of T′, in place of the path and Omega, with cells of T to the right and above, and conclude the two smaller sizes and that the tiles are not congruent. |
| Prop. 48: In case (I) the left-over tile meets no side, exactly one, or exactly two opposite sides | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. Internal Lean, a part, not separately audited: free_tile_ not_adjacent, free_tile_one_side, proposition57d. Not stated there: the values of p and q (from congruence, Congruent. dims); the bottom and top case is the transpose of proposition 57d (IsTiling.transpose). |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Lem. 50: In case (I), the cells of P on a side of its box at a used corner form an interval from that corner; a side with both ends occupied, one of them used, lies in P | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 51: Two corner tiles at a free side with the same mark and the same direction are mirror images, and both runs along the side are half the side | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 52: Let T, T′ be disjoint connected sets of cells with no cell below row 0, T in the rows [0,H) containing (x_1,0) and (x_2,H-1), and T′ containing (x′_1,0) with x_1 < x′_1 and a cell in some row at least H - 1. (a) T′ has a cell (x′,H-1) with x′ > x_2. (b) If T′ also lies in the rows [0,H), every cell (x′_2,H-1) of T′ has x′_2 > x_2 | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 53: In pattern O, a column of T5 of height at least the height of a bottom corner tile bounds that tile | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 54: If a symmetry of R permutes the four corner tiles, the conjugate symmetry maps P, box(P) and the set S of occupied corners to themselves and is of the same kind; for a reflection in an axis-parallel line or a half turn \|S\| is even; for a quarter turn p = q and \|S\| is 0 or 4 | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 55: The two corner tiles at a free side are not half turns of each other in the strip of their common height | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 56: If all four marks are equal, all four corners of the box are occupied in pattern N, and P has an axis-parallel reflection in pattern O | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Cor. 57: One occupied corner does not occur; with three, or two opposite, or two in pattern N, two different corners are used | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 58: Four occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Lem. 59: With three occupied corners and p > q, two corner tiles at a free side with the same orientation have whole runs along it | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 60: In pattern N three occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 61: In pattern O three occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |

| Paper Claim | Proof Tier | Lean Decl / Artifact | Actual Axioms & Fidelity | Assumptions & What You Must Trust |
|---|---|---|---|---|
| Lem. 62: With two adjacent occupied corners, at a free side at least one corner tile has its full side along it | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 63: In pattern N two adjacent occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 64: In pattern O two adjacent occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 65: In pattern N two opposite occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |
| Prop. 66: In pattern O two opposite occupied corners do not occur | Written | Written proof | Lean: none; fidelity: n/a | Written proof; source notes read by AI checkers. No human referee; not counted in Lean. |

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 2d67af1fe218 | Lean mode: committed_receipt | Closure SHA-256: 4acbadc5fa09 | Git anchor: c9d2bec53c2d | Axioms SHA-256: 5a9371226bbf | Fidelity SHA-256: 772ecd39941a.

## 8. PRIOR WORK AND OPEN QUESTIONS

Stewart and Wormstein [1] proved that if three congruent copies of a connected polyomino tile a rectangle, then the polyomino is rectangular; in particular no polyomino has order 3. Their paper states that it is open whether a polyomino can have an odd order greater than one. Maltby [2] proved that if a rectangle is dissected into three congruent pieces then the pieces are rectangles, for the class of pieces listed in the appendix of that paper, which contains every polyomino tile; Theorem 9 is a special case. Winslow's survey [3] asks whether some polyomino has order 5 as Open Problem 11 in the author's version (the numbering of the published version is not checked [to check]). The count of corner cells in the proof of Theorem 9, and the interval and bounding-box arguments of Sections 2 and 4, are forms of tools in the proof of Stewart and Wormstein (its step 1 and remark (b)). They treat three copies only, and they write that their proof probably cannot be extended to odd orders of five or more. Maldonado and Roldán-Pensado [5] prove that no rectangle is tiled by 5 or by 7 copies of a convex polygon unless the polygon is a rectangle; a polyomino that is not a rectangle is not convex, so their theorem does not cover Theorem 1.

A search for a proof or a counterexample of Theorem 1, and for any result on odd orders of polyominoes, was logged on 2026-11-25 and 2026-11-26, by problem number, by short keywords and by phrases; a copy of the log is in the package. No proof and no counterexample was found in the searches logged; every source read there that states a status calls order 5 open. The OpenAlex searches include the works that cite Stewart and Wormstein, Winslow, and Reid; none of them says that it settles five copies. The searches do not include the full text of CORE (titles and abstracts only), the free-text search of Semantic Scholar (throttled), Golomb's book on polyominoes, the published version of Winslow's survey, and several other sources listed with the log. An earlier search, also in the package, looked for a published bound of the form "no polyomino with at most $N$ cells tiles a rectangle with exactly five copies" and found none. Two unpublished computations give bounded results of this kind. A public repository dated 2026-11-22 [7] reports an exhaustive search in which no non-rectangular polyomino with at most 24 cells tiles a rectangle with exactly 5, 7 or 9 copies; it states that the result has not been checked by a third party, and its French summary says that the size 24 is still running. A comment of 2016 on MathOverflow [8] reports the same for at most 13 cells, with a few 13-cell polyominoes left. Theorem 1 covers every number of cells for five copies; these searches also treat seven and nine copies. A comment of 2022 in the same thread, in reply to a question about rectangles tiled by an odd number of congruent convex polygons, says that the case of five pieces "was resolved negatively in a paper" and links to the article of Yuan, Zamfirescu and Zamfirescu [10] [to check], which we have not read. Rao, Ren and Wang [9] and Maldonado and Roldán-Pensado [5] report that this article proves, for five tiles, that a square tiled by congruent copies of a convex polygon is tiled by rectangles, and Maldonado and Roldán-Pensado report that a conjecture attributed there to Danzer, for five congruent polygons that need not be convex, remains open. A polyomino that is not a rectangle is not convex, so these results on convex tiles do not cover Theorem 1. Talon's thesis [6] searches for the orders of small polyominoes and leaves some undecided [to check]. Reid's pages on prime rectangles and his paper on rectangles and half strips [4] list odd-count rectangle tilings for particular polyominoes; none of the listed prime rectangles uses 3 or 5 copies, and the listed odd counts begin at 11.

One question is open: does some polyomino have an odd order greater than 5? The proof here uses five copies at every step, in the count of tiles in case (II) and in the four corner tiles with one tile left over in case (I).

## 9. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

The verdict text, the list of log digests, the script `recompute.py`, the figure scripts, the search log and the trust-boundary ledgers are in the paper's package; the logs themselves and the program that produced them are kept with the project and will be archived with the final version. To rebuild, with `ROOT` the repository root: `ourotrix paper trust -p . -r ROOT -g`, then `ourotrix paper check -p . -r ROOT -b -x -B`. The Lean library is public at `https://github.com/mathdiscoveries/polyomino-odd-order`, commit `c10ddcb1e6b0`; it contains, byte for byte, the files that the Lean receipt of Section 7 lists.

**Use of AI.** AI research agents built and directed by Tyler B. Hughes found the reduction, wrote the proofs, the search programs, the figures and this text, and checked the proofs and the computation with other AI agents from several model families. No human mathematician has yet refereed the proof. The seven statements listed in Section 1 are also proved in Lean 4; the proofs printed here are the written ones.

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