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Working Draft

The Last Exception for Two-Colour Rado Numbers of $x + y + c = kz$ for Every $k \geq 3$

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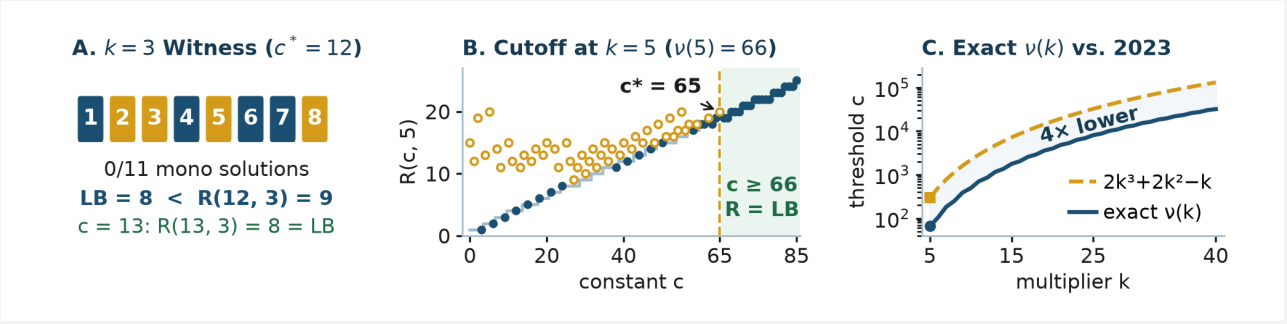
Supporting Information: `recompute.py`, *core Lean 4 library* (`lean/RadoNu/`), `PALOMAR-2026-11-23-000035`

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For $k \geq 3$ and valid $c \geq 0$ (k odd or c even), let $R(c,k)$ be the two-colour Rado number of $x+y+c = kz$, bounded below by $\mathrm{LB}(c,k) = \lceil (2\lceil (c+2)/k \rceil + c)/k \rceil$ (Martinelli and Schaal 2007, Lemma 2). Kim, Song, and Hwang (2023) proved $R(c,k) = \mathrm{LB}(c,k)$ for $k \geq 5$ and valid $c \geq 2k^3+2k^2-k$, asking for the exact cutoff $\nu(k)$ and conjecturing $\nu(5) = 66$. For every $k \geq 3$, we determine the exact last exception $c^*(k) = k(k^2+k-4)/2$ (odd k) and $k(k^2-2)/2$ (even k): $R(c,k) = \mathrm{LB}(c,k)$ for every valid $c > c^*(k)$, while an explicit 2-colouring of $[1, \mathrm{LB}(c^*(k),k)]$ avoids monochromatic solutions. Thus the least valid threshold is $\nu(k) = c^*(k)+1$ (odd k) and $c^*(k)+2$ (even k) $(13, 30, 66, 104, 183, 250, 388, 492, \ldots)$, improving the 2023 bound by a factor tending to 4 and settling $\nu(5) = 66$. Reflecting $x \mapsto N+1-x$ sends the problem to $x+y = kz+b$ ($b \geq 1$), where we prove forcing on $[1, n_1(k,b)]$ ($b \geq k+1$) and $[1, N_2(k,b)]$ ($1 \leq b \leq k$), and bridge $N = \mathrm{LB}(c,k)$ to those bounds. All six results are proved unconditionally in core Lean 4 (`lean/RadoNu/`, 68 declarations, zero Mathlib imports).

Verification Scope (G11): WARN: Human referee: pending
Main results: 6 of 6 in Lean Kernel (100%)
Abstract claims: 6 of 6 in Lean Kernel (100%)

Full Trust Boundary: Section 5.

MSC 2020: 05D10, 11B75 **Keywords:** Rado numbers, generalized Schur numbers, Ramsey theory on the integers, reflection argument, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Colour each integer in $\{1, \ldots, n\}$ either red or blue. A *solution* of the linear equation $x + y + c = kz$ is an ordered triple $(x,y,z) \in \{1, \ldots, n\}^3$ (with repeats permitted) satisfying $x + y + c = kz$. The *two-colour Rado number* $R(c,k)$ is the least positive integer n such that every 2-colouring of $\{1, \ldots, n\}$ admits a monochromatic solution. When k is even and c is odd, colouring integers by parity avoids monochromatic solutions on every finite stretch; Jones and Schaal [2] (`[to check]`) established that $R(c,k)$ is finite precisely when c is *valid*, meaning that k is odd or c is even.

Martinelli and Schaal [6] (Lemma 2, p. 36) proved via a two-block colouring that every valid $c \geq 0$ satisfies the general lower bound

$$R(c,k) \geq \mathrm{LB}(c,k) := \left\lceil \frac{2\lceil (c+2)/k \rceil + c}{k} \right\rceil. \tag{1}$$

Call a valid constant $c \geq 0$ an *exception* if $R(c,k) > \mathrm{LB}(c,k)$. Earlier work settled small multipliers: Kézdy, Snevily, and White [3] determined the case $k = 3$, showing that the last exception is $c = 12$ and hence equality holds for all $c \geq 13$, while Guo [1] (`[to check]`) proved $R(c,4) = \mathrm{LB}(c,4)$ for even $c \geq 34$. Kim, Song, and Hwang [5] (Theorem 1) proved that for every $k \geq 5$ and every valid $c \geq 2k^3 + 2k^2 - k$, one has $R(c,k) = \mathrm{LB}(c,k)$. In the Remark following their Theorem 1, they defined $\nu(k)$ as the smallest positive threshold from which $R(c,k) = \mathrm{LB}(c,k)$ holds for all valid $c \geq \nu(k)$, noted $\nu(3) = 13$ and $\nu(4) = 30$, exhibited a boundary colouring at $k = 5, c = 65$ showing $\nu(5) \geq 66$, posed the problem of determining $\nu(k)$ for all k, and conjectured that $\nu(5) = 66$.

We determine the exact last exception $c^*(k)$ for every integer $k \geq 3$ by an elementary reflection proof that uses no computer search and does not invoke the threshold theorem of [5]. Define

$$c^*(k) := \begin{cases} \dfrac{k(k^2+k-4)}{2}, & \text{if } k \text{ is odd,} \\ \dfrac{k(k^2-2)}{2}, & \text{if } k \text{ is even.} \end{cases} \tag{2}$$

Theorem 1 (Exact Last Exception for Every $k \geq 3$). *Let $k \geq 3$ be an integer, and define $c^*(k)$ by (2) and $\mathrm{LB}(c,k)$ by (1). Then:*

1. *For every valid integer $c > c^*(k)$ (that is, k odd or c even), the two-colour Rado number of $x + y + c = kz$ satisfies $R(c,k) = \mathrm{LB}(c,k)$.*
2. *At $c = c^*(k)$ (which is valid for every $k \geq 3$), there exists a 2-colouring of $\{1,\dots,\mathrm{LB}(c^*(k),k)\}$ with no monochromatic solution of $x + y + c^*(k) = kz$. In particular, $\{1,\dots,\mathrm{LB}(c^*(k),k)\}$ is not forced, so any Rado number R for $(c^*(k),k)$ satisfies $R > \mathrm{LB}(c^*(k),k)$.*

As an immediate consequence, we obtain the exact threshold characterization and resolve the conjecture $\nu(5) = 66$ of Kim, Song, and Hwang [5].

Corollary 2 (Threshold Characterization $\nu(k)$). *For every integer $k \geq 3$ and every integer $v \geq 0$, the property "$R(c,k) = \mathrm{LB}(c,k)$ for every valid $c \geq v$" holds if and only if $v > c^*(k)$. Consequently, the least integer threshold is $c^*(k)+1$, and the least valid threshold $\nu(k)$ (matching the convention $\nu(4) = 30$ of [5]) is*

$$\nu(k) = \begin{cases} c^*(k) + 1 = \dfrac{k(k^2+k-4)}{2} + 1, & \text{if } k \text{ is odd,} \\ c^*(k) + 2 = \dfrac{k(k^2-2)}{2} + 2, & \text{if } k \text{ is even.} \end{cases}$$

For $k = 3,\dots,10$, Corollary 2 gives the least valid cutoffs $\nu(k) = 13, 30, 66, 104, 183, 250, 388, 492$, reproducing $\nu(3) = 13$ [3] and $\nu(4) = 30$ [1, 5] and reducing the threshold $2k^3 + 2k^2 - k$ of [5] by a factor that tends to 4 as $k \to \infty$.

Corollary 3 (Resolution of the Conjecture $\nu(5) = 66$). *For $k = 5$, $R(c,5) = \mathrm{LB}(c,5)$ holds for every valid $c \geq 66$ (in particular $R(66,5) = 19$), whereas it fails for $c = 65$ (since $[1,19]$ admits a good 2-colouring for $x + y + 65 = 5z$). Thus $\nu(5) = 66$.*

The proof of Part (1) of Theorem 1 passes through the interval reflection $x \mapsto N + 1 - x$ on $[1,N]$ with $N = \mathrm{LB}(c,k)$, which sends $x + y + c = kz$ to the reflected equation $x + y = kz + b$ with $b = c - (k-2)(N+1)$. We establish two unconditional forcing theorems for the reflected equation.

Theorem 4 (Reflected-Equation Forcing for $b \geq k+1$). *Let $k \geq 3$ and $b \geq k+1$, with b even if k is even. Set $h_0(k,b) = \lceil (k+b)/2 \rceil$ and $n_1(k,b) = \lceil (kh_0(k,b) + b)/2 \rceil$. Then every 2-colouring of $\{1,\dots,n_1(k,b)\}$ contains a monochromatic solution of $x + y = kz + b$.*

Theorem 5 (Reflected-Equation Forcing for $1 \leq b \leq k$). *Let $k \geq 3$ and $1 \leq b \leq k$, with b even if k is even. Let $j_0(k,b) \in \{k+1, k+2\}$ be the least integer $j \geq k+1$ such that $kj + b$ is even, and set $N_2(k,b) = (kj_0(k,b) + b)/2$. Then every 2-colouring of $\{1,\dots,N_2(k,b)\}$ contains a monochromatic solution of $x + y = kz + b$.*

Combined with Martinelli and Schaal's negative-constant lower bound [6] (Lemma 3, p. 37), Theorem 4 also determines $R(-b,k) = n_1(k,b)$ for every $k \geq 3$ and every valid $b \geq k+1$. Figure 1 maps the full proof architecture from the reflection transformation through Lean 4 kernel verification.

Concept in Action: Exact Threshold and LB Check

```
def check_k(k):
    cs = k*(k*k+k-4)//2 if k%2 else k*(k*k-2)//2
    nv = cs + (1 if k % 2 else 2)
    lv = -(-((2 * -(-(cs + 2) // k)) + cs) // k)
    return cs, nv, lv
assert check_k(3) == (12, 13, 8)
assert check_k(5) == (65, 66, 19)
assert check_k(6) == (102, 104, 23)
```

Roadmap. Section 2 fixes notation matching `lean/RadoNu/RadoNu.lean` verbatim and works through the concrete cases $k = 3, c \in \{12, 13\}$ by hand. Section 3 gives the complete self-contained proofs of Theorems 4 and 5, the arithmetic reflection bridge (Lemma 6), the boundary colourings at $c^*(k)$, Theorem 1, and Corollaries 2–3. Section 4 reports independent computational confirmations and the empirical exception landscape for $k \leq 9$. Section 5 presents the machine-generated Lean 4 Reader's Trust Boundary, Section 6 reviews prior work and open questions, and Section 7 documents reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND WORKED EXAMPLES

Throughout, $\mathbb{N} = \{0,1,2,\dots\}$ and $[a,b] = \{m \in \mathbb{Z} : a \leq m \leq b\}$. Fix integers $k \geq 3$ and $c \geq 0$. A 2-colouring of $[1,n]$ is represented by a function $\chi : \mathbb{N} \to \{\text{false}, \text{true}\}$ (which we refer to as *blue* and *red*, or 0 and 1). We adopt the exact predicates formalized in `lean/RadoNu/RadoNu.lean`:

- $\mathrm{Valid}(c,k)$ holds if $k \equiv 1 \pmod 2$ or $c \equiv 0 \pmod 2$.
- $\mathrm{HasMono}(c,k,n,\chi)$ holds if there exist $x,y,z \in [1,n]$ with $x + y + c = kz$ and $\chi(x) = \chi(y) = \chi(z)$.
- $\mathrm{Forced}(c,k,n)$ holds if $\mathrm{HasMono}(c,k,n,\chi)$ holds for every $\chi : \mathbb{N} \to \{\text{false}, \text{true}\}$.
- $\mathrm{IsRado}(c,k,R)$ holds when $\mathrm{Forced}(c,k,R)$ holds while $\mathrm{Forced}(c,k,n)$ fails for all $n < R$. Since $\mathrm{Forced}(c,k,0)$ is false, any such R is unique and satisfies $R \geq 1$; we write $R(c,k) = R$.

For the reflected equation $x + y = kz + b$ ($b \geq 1$) on $[1,N]$, a colouring χ is *good* (predicate $\mathrm{Good}(k,b,N,\chi)$) if there is no triple $(x,y,z) \in [1,N]^3$ with $x + y = kz + b$ and $\chi(x) = \chi(y) = \chi(z)$.

Convention on $\nu(k)$ for even k. Because $c^*(k) = k(k^2-2)/2$ is even whenever k is even, the integer $c^*(k) + 1$ is odd and therefore not valid. In the Remark of [5], the authors quote both $\nu(2) = 1$ (literal least positive integer) and $\nu(4) = 30 = c^*(4) + 2$ (least *valid* integer). Theorem 1 avoids this ambiguity by characterizing the exact last exception $c^*(k)$, while Corollary 2 states both readings explicitly: the equivalence holds for every integer $v > c^*(k)$, giving least integer threshold $c^*(k) + 1$ and least valid threshold $\nu(k) = c^*(k) + 2$ for even k.

Worked example: $k = 3, c = 13$ versus $c = 12$. At $k = 3$ and $c = 13$, (1) gives $\mathrm{LB}(13,3) = \lceil (2\cdot 5 + 13)/3 \rceil = 8$. On $[1,7]$, the Martinelli–Schaal two-block colouring with $\{1,2,3,4\}$ blue and $\{5,6,7\}$ red avoids monochromatic solutions because every solution has $z \geq 5$ (red) while two red inputs force $3z \geq 23$, hence $z \geq 8$. On $[1,8]$, there are 10 solutions ($x \leq y$):

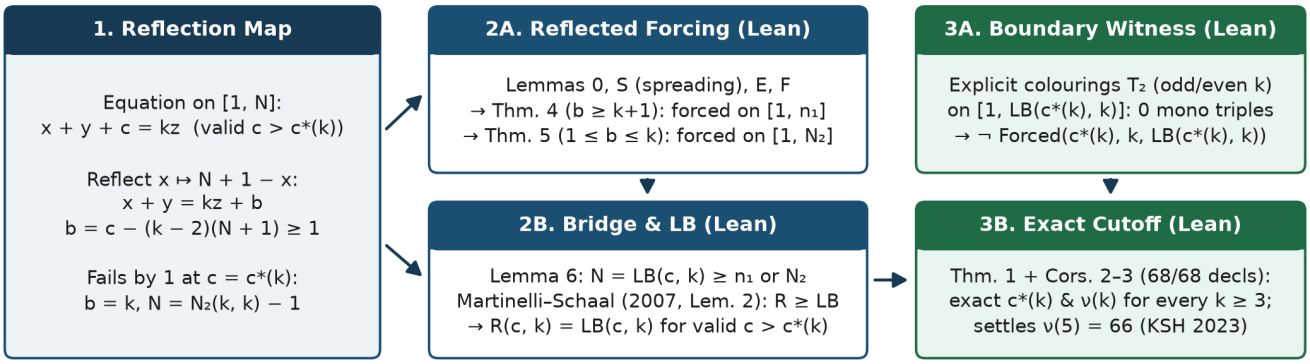


Figure 1. Discovery and proof architecture for Theorem 1 and Corollaries 2–3. Every proved theorem, lemma, and corollary in this manuscript is formalized unconditionally in pure core Lean 4 (`lean/RadoNu/`, `Init` prelude only, zero Mathlib imports, 68 verified declarations, standard axioms only). Independent exact Python backtracking and SAT checks confirm small k. Unsettled: individual exception values $R(c,k)$ for $0 < c \le c^*(k)$; finite existence of $R(c^*(k),k)$ cites Jones and Schaal [2] (`[to check]`).

$(1,1,5)$, $(1,4,6)$, $(2,3,6)$, $(1,7,7)$, $(2,6,7)$, $(3,5,7)$, $(4,4,7)$, $(3,8,8)$, $(4,7,8)$, and $(5,6,8)$. Assuming without loss of generality that 1 is blue, every colour is forced without branching: $(1,1,5)$ forces 5 red; $(1,7,7)$ forces 7 red; $(3,5,7)$ forces 3 blue; $(4,4,7)$ forces 4 blue; $(1,4,6)$ forces 6 red; $(2,6,7)$ forces 2 blue; $(5,6,8)$ forces 8 blue; finally $(3,8,8)$ is all blue $(3+8+13=24)$. Thus $R(13,3)=8=\mathrm{LB}(13,3)$. Under the reflection $x \mapsto 9-x$, the 10 solutions map bijectively onto the solutions of $x+y=3z+4$ in $[1,8]^3$ ($b=13-1\cdot 9=4\ge k+1$, $h_0=4$, $n_1(3,4)=8$).

At $c=12=c^*(3)$, $\mathrm{LB}(12,3)$ is again 8, yet the colouring $(1,0,0,1,0,1,1,0)$ of $[1,8]$ (the $k=3$ case of our boundary colouring in Section 3) has 0 monochromatic triples among its 11 solutions in $[1,8]^3$, so $R(12,3)=9>8$.

3. PROOF OF THE MAIN RESULTS

Forcing Lemmas for the Reflected Equation. Fix integers $k \ge 3$, $b \ge 1$ with b even whenever k is even, and $N \ge 1$. Suppose χ is a good 2-colouring of $[1,N]$ for $x+y=kz+b$. By symmetry of the two colours, assume $\chi(1)$ is blue (false). Recall the *forcing rule*: if $(x,y,z) \in [1,N]^3$ satisfies $x+y=kz+b$ and two of the positions share a colour, the remaining entry must have the opposite colour; in particular, if (x,x,z) is a solution then x and z have opposite colours.

Define the following auxiliary quantities:

- $H := \lfloor (2N-b)/k \rfloor$, the largest z occurring in a solution inside $[1,N]^3$;
- for any integer j with $kj+b$ even, $x_j := (kj+b)/2$; whenever $i \equiv j \pmod 2$ and $x_i, x_j, (i+j)/2 \in [1,N]$, the triple $(x_i, x_j, (i+j)/2)$ is a solution because $x_i + x_j = k(i+j)/2 + b$, and $x_j \le N$ holds precisely when $j \le H$;
- H^* is the largest integer $j \le H$ such that $kj+b$ is even ($H^*=H$ if k is even; $H^* \in \{H-1, H\}$ if k is odd);
- $h_0 := \lceil (k+b)/2 \rceil$;
- write $b = km_1 + s_1$ with $2 \le s_1 \le k+1$, so $m_1 = \lfloor (b-2)/k \rfloor \ge -1$ (with $m_1 = -1$ only when $b=1$, giving $s_1 = k+1$).

Let t denote the least red integer in $[1,N]$ when one exists.

Step 1 (Upper bound on t, Lemma 0). *If $h_0 \le N$, then a red integer exists and $2 \le t \le h_0$.*

Indeed, $k+b-h_0 = \lfloor (k+b)/2 \rfloor \in \{h_0-1, h_0\} \subseteq [1,h_0]$, so $(h_0, k+b-h_0, 1)$ is a solution in $[1,h_0]^3$. Since 1 is blue, $[1,h_0]$ cannot be entirely blue; hence the least red integer t satisfies $2 \le t \le h_0$. (Formalized as `RadoNu.lemma0`.)

In Steps 2–4 below, we assume $2 \le t \le N$, $[1,t-1]$ is blue, and t is red.

Step 2 (Spreading, Lemma S). *Every integer z with $t \le z \le H^*$ is red.*

Let $Q = \{j \in \mathbb{Z} : j \equiv H^* \pmod 2\}$. Because k is even whenever b is even and kH^*+b is even, $kj+b$ is even for every $j \in Q$. Let q be the least element of Q with $q \ge t$, so $q \in \{t, t+1\}$, and put $a = q-2 \ge 0$. If $H^* < q$, then $H^* \le q-2 \le t-1$ (since $H^*, q \in Q$) and the claim is vacuous. Assume $q \le H^*$. Notice $x_a \ge 1$: if $a \ge 1$ then $ka+b \ge 4$, while if $a=0$ then $0 \in Q$ forces b even, so $b \ge 2$ and $x_0 = b/2 \ge 1$.

Base: If $q=t$, then (x_t, x_t, t) forces x_t blue (since t is red), and then $(x_{t-2}, x_t, t-1)$ forces $x_a = x_{t-2}$ red (since $t-1$ and x_t are blue). If $q = t+1$, then $(x_{t-1}, x_{t-1}, t-1)$ forces $x_a = x_{t-1}$ red (since $t-1$ is blue), and then (x_{t-1}, x_{t+1}, t) forces $x_q = x_{t+1}$ blue (since t and x_{t-1} are red). In both cases, x_a is red and x_q is blue.

Induction: Suppose $i \in Q$ satisfies $q \le i \le H^*$ and x_j is blue for all $j \in Q \cap [q,i]$. Then every $z \in [t,i]$ is red: $z=t$ is red by hypothesis; any $z \in Q \cap [q,i]$ is forced red by (x_z, x_z, z); and any $z \in (t,i) \setminus Q$ has $z-1, z+1 \in Q \cap [q,i]$ and is forced red by (x_{z-1}, x_{z+1}, z). If furthermore $i+2 \le H^*$, then $z' := (i+2+a)/2$ is an integer satisfying $q = a+2 \le z' \le i$ (as $i \ge q = a+2$), so z' is red. Since x_a is red, (x_a, x_{i+2}, z') forces x_{i+2} blue. By induction, x_j is blue for all $j \in Q \cap [q, H^*]$,

and hence every $z \in [t, H^*]$ is red. Every index used lies in $[0, H^*]$, so every number lies in $[1, N]$. (Formalized as `RadoNu.lemmaS`.)

Step 3 (Two red images, Lemma E). *Suppose $t+1 \le H$ and $s_1 \le 2t-2$. Suppose further that either (i) $m_1 \le t-2$, or (ii) $m_1 \ge t-1$, $k+b-1 \le N$, and $m_1+2 \le H^*$. Then χ is not good.*

Set $w := \max(2, t-m_1)$ and $z_0 := w + m_1$. We have $2 \le w \le 2t-2$: if $m_1 \ge 0$ then $w \le t \le 2t-2$, whereas if $m_1 = -1$ then $b = 1$ and $s_1 = k+1 \le 2t-2$ implies $t \ge 3$ and $w = t+1 \le 2t-2$. Choose $x_a = \lfloor s_1/2 \rfloor$, $x_b = \lceil s_1/2 \rceil$, $z_a = \lfloor w/2 \rfloor$, and $z_b = \lceil w/2 \rceil$. Since $2 \le s_1 \le 2t-2$ and $2 \le w \le 2t-2$, all four integers lie in $[1, t-1]$ and are blue. Define

$$e_a := kz_a + b - x_a, \qquad e_b := kz_b + b - x_b.$$

Clearly $e_a, e_b \ge k+b-(k+1) \ge 1$ (indeed ≥ 2 when $b \ge 2$; for $b = 1$, $x_a \le \lfloor (k+1)/2 \rfloor < k+1$ so $e_a, e_b \ge 1$), and

$$e_a + e_b = kw + 2b - s_1 = k(w+m_1) + b = kz_0 + b.$$

In case (i), $w = t - m_1$ and $z_0 = t$ (which is red). Also $z_b \le (w+1)/2$ and $x_a \ge (s_1-1)/2$, while $e_a, e_b \le (k(w+1) + 2b - s_1)/2 = (k(t+1)+b)/2 \le N$ because $t+1 \le H$. Thus (x_a, e_a, z_a) and (x_b, e_b, z_b) force e_a and e_b red, making (e_a, e_b, t) a red solution.

In case (ii), $w = 2$, $z_a = z_b = 1$, $e_a, e_b \le k+b-1 \le N$, and $z_0 = m_1 + 2 \in (t, H^*]$ is red by Step 2. Again e_a and e_b are forced red, making (e_a, e_b, z_0) a red solution. (Formalized as `RadoNu.lemmaE1` and `RadoNu.lemmaE2`.)

Step 4 (Clash at $k+1$, Lemma F). *Suppose $t \le k$, $H^* \ge k+1$, and $e := k(t-1) + b - x \le N$, where $x := \min(t-1, H^*-k)$. Then χ is not good.*

Indeed, $1 \le x \le t-1$, so x is blue, while $y := x + k$ satisfies $t < k+1 \le y \le H^*$, so y is red by Step 2. Also $e \ge k+b-(t-1) \ge 2$. Since $(x, e, t-1)$ has x and $t-1$ blue, e is forced red. But $y + e = (x+k) + k(t-1) + b - x = kt + b$, so (y, e, t) is a red solution. (Formalized as `RadoNu.lemmaF`.)

Proofs of Theorems 4 and 5.

Proof of Theorem 4. Here $b \ge k+1$ and $N = n_1(k,b) = \lceil (kh_0 + b)/2 \rceil$. Then $2N - b \in \{kh_0, kh_0+1\}$, so $H = h_0 \ge k+1$ and $m_1 \ge 0$. From $h_0 \ge (k+b)/2$ we have

$$4N \ge 2(kh_0 + b) \ge k^2 + (k+2)b, \tag{3}$$

and hence $4N - 4(k+b-1) \ge (k-2)^2 + b(k-2) \ge 0$, giving $k+b-1 \le N$. Suppose for contradiction that χ is a good 2-colouring of $[1, N]$ with 1 blue. By Step 1, $2 \le t \le h_0$. We split into three cases.

Case 1: $t = h_0$. Then $[1, h_0 - 1]$ is blue and $h_0 - 1 \ge k$. For each $v \in \{N-1, N\}$, put $z = \lceil (v-b+1)/k \rceil$ and $x = kz + b - v$, so $1 \le x \le k \le h_0 - 1$. By (3), $4N \ge k^2 + (k+2)b \ge 4b + 4$, so $v \ge N - 1 \ge b$ and thus $z \ge 1$. Moreover $v \le N \le (kh_0 + b + 1)/2 \le k(h_0-1) + b - 1$ (since $k(h_0-2) + b - 3 \ge 0$), so $z \le h_0 - 1$. Hence (x, v, z) has x and z blue, forcing v red. Thus both $N-1$ and N are red, as is $h_0 = t$. If $kh_0 + b$ is even then (N, N, h_0) is a red solution; if $kh_0 + b$ is odd then $(N-1, N, h_0)$ is a red solution, contradiction.

Case 2: $t \le h_0 - 1$ and $s_1 \le 2t-2$. Then $t+1 \le H$. If $m_1 \le t-2$, Step 3(i) yields a contradiction. Otherwise $m_1 \ge t-1 \ge 1$; we already have $k+b-1 \le N$, and

$$H^* \ge h_0 - 1 \ge \frac{k + km_1 + 2}{2} - 1 = \frac{k(m_1+1)}{2} \ge m_1 + 2$$

since $(k-2)(m_1+1) \ge 2$. Step 3(ii) yields a contradiction.

Case 3: $t \le h_0 - 1$ and $s_1 \ge 2t-1$. Then $s_1 \ge 3$ and $t \le (s_1+1)/2 \le (k+2)/2 \le k$. We first verify $H^* \ge k+1$: otherwise, since $H = h_0 \ge k+1$ and $H^* \ge H-1$, we must have $h_0 = k+1$ and $H^* = k$, forcing k odd and $k(k+1) + b$ odd, hence b odd; combined with $h_0 = k+1$ this forces $b \in \{k+1, k+2\}$ and therefore $b = k+2$, giving $s_1 = 2$, contradicting $s_1 \ge 3$. It remains to check $e \le N$ in Step 4 with $x = \min(t-1, H^*-k)$:

- If $H^* \ge k+t-1$, then $x = t-1$ and $e = (k-1)(t-1) + b$. By (3), it suffices that $4(k-1)(t-1) \le k^2 + (k-2)b$; using $t-1 \le k/2$ and $b \ge k+1$, the left side is at most $2k^2 - 2k$ while the right side is at least $2k^2 - k - 2$, which holds for $k \ge 3$.
- If $k+1 \le H^* \le k+t-2$, then $(k + km_1 + s_1)/2 \le h_0 \le H^* + 1 \le k+t-1 \le k + (s_1-1)/2$, forcing $km_1 \le k-1$, hence $m_1 = 0$ and $b = s_1 \le k+1$. Thus $b = k+1$, $h_0 = k+1 = H^*$ (k odd), and $t \le (k+1)/2$. Here $x = 1$ and $e = k(t-1) + k = kt \le k(k+1)/2 \le (k+1)^2/2 = N$.

In both subcases Step 4 yields a contradiction. □

Proof of Theorem 5. Here $1 \le b \le k$ and $N = N_2(k,b) = x_{j_0}$, so $H = H^* = j_0 \ge k+1$, $h_0 \le k < H$, and $N \ge (k(k+1) + b)/2 \ge h_0$. Suppose χ is a good 2-colouring of $[1, N]$ with 1 blue. By Step 1, $2 \le t \le h_0 \le k$, so $t+1 \le H$.

If $b \ge 2$, then $m_1 = 0$ and $s_1 = b$. If $b = 1$, then $m_1 = -1$, $s_1 = k+1$, and $s_1 \le 2t-2$ is impossible because $t \le h_0 = (k+1)/2$ (k is odd when $b = 1$).

- If $s_1 \le 2t-2$, then $b \ge 2$ and $m_1 = 0 \le t-2$, so Step 3(i) gives a contradiction.
- If $s_1 \ge 2t-1$, then $x = \min(t-1, H^*-k) \ge 1$ and $e \le k(t-1) + b - 1$. When $b \ge 2$, we have $t-1 \le (b-1)/2$ and

  $$e \le \frac{(k+2)(b-1)}{2} \le \frac{(k+2)(k-1)}{2} < \frac{k(k+1)+b}{2} \le N.$$

  When $b = 1$, $t-1 \le (k-1)/2$ and $e \le k(k-1)/2 < N$. Thus Step 4 gives a contradiction. □

Arithmetic Reflection Bridge and the Upper Half. For any integers k, c and $N \ge 1$, the involution $x \mapsto N+1-x$ on $[1, N]$ maps solutions of $x+y+c = kz$ bijectively onto solutions of $x+y+c' = kz$ with $c' = (k-2)(N+1) - c$. Thus $[1, N]$ admits a good 2-colouring for $x+y+c = kz$ if and only if it admits one for $x+y = kz+b$ where $b = c - (k-2)(N+1)$.

Lemma 6 (Arithmetic Reflection Bridge). *Let $k \ge 3$, let $c > c^*(k)$ be valid, and put $N = \mathrm{LB}(c,k)$. Then there exists an integer $b \ge 1$ such that:*

1. *$c = (k-2)(N+1) + b$, and b is even whenever k is even;*
2. *if $b \ge k+1$, then $N \ge n_1(k,b)$;*
3. *if $b \le k$, then $N \ge N_2(k,b)$.*

Proof. Set $b = c - (k-2)(N+1)$ and $H = \lfloor (2N-b)/k \rfloor$. Since $c+2 = k(N+1) - (2N-b)$, we have $\lceil (c+2)/k \rceil = N+1-H$,

and substituting into (1) gives

$$N = \mathrm{LB}(c,k) = \left\lceil \frac{k(N+1) - 2H + b}{k} \right\rceil = N + 1 - \left\lfloor \frac{2H - b}{k} \right\rfloor.$$

Hence $\lfloor (2H-b)/k \rfloor = 1$, that is,

$$k \le 2H - b \le 2k - 1. \tag{4}$$

Proof of $b \ge 1$ and parity. If $b \le 0$, then (4) yields $2H \le 2k - 1 + b \le 2k - 1$, so $H \le k - 1$, $2N - b < k^2$, and $N < k^2/2$. Therefore

$$c = (k-2)(N+1) + b < \frac{(k-2)(k^2+2)}{2} \le c^*(k),$$

contradicting $c > c^*(k)$. When k is even, both c and $k-2$ are even, so $b = c - (k-2)(N+1)$ is even.

Case $b \ge k+1$. By (4), $2H \ge k + b$, so $H \ge \lceil (k+b)/2 \rceil = h_0(k,b)$. Thus $2N - b \ge kH \ge k h_0(k,b)$, giving $N \ge \lceil (k h_0(k,b) + b)/2 \rceil = n_1(k,b)$.

Case $1 \le b \le k$. Suppose for contradiction that $N \le N_2(k,b) - 1$. Then $c = (k-2)(N+1) + b \le (k-2)N_2(k,b) + b$.

- If b is even, then $j_0(k,b) = k+1$ and $N_2(k,b) = (k^2 + k + b)/2$, so $c \le (k-2)(k^2+k+b)/2 + b$, which increases with b. For even k, setting $b = k$ gives $c \le k(k^2-2)/2 = c^*(k)$; for odd k, $b \le k-1$ gives $c \le (k^3 - 3k)/2 < c^*(k)$.
- If b is odd, then k is odd, $j_0(k,b) = k+2$, and $N_2(k,b) = (k^2 + 2k + b)/2$, so $c \le (k-2)(k^2+2k+b)/2 + b$; setting $b = k$ gives $c \le k(k^2+k-4)/2 = c^*(k)$.

Every subcase contradicts $c > c^*(k)$, so $N \ge N_2(k,b)$. □

Notice that at $c = c^*(k)$ itself, Lemma 6 misses by exactly one: in both parities one has $b = k$ and $\mathrm{LB}(c^*(k),k) = N_2(k,k) - 1$.

Lower Half and Conclusion of Theorem 1.

Proof of Theorem 1. Part (1) (Upper half). Fix $k \ge 3$ and a valid constant $c > c^*(k)$, and set $N = \mathrm{LB}(c,k)$. By Lemma 6, the integer $b = c - (k-2)(N+1) \ge 1$ is even when k is even, with $N \ge n_1(k,b)$ if $b \ge k+1$ and $N \ge N_2(k,b)$ if $b \le k$. By Theorem 4 (if $b \ge k+1$) or Theorem 5 (if $b \le k$, restricting any good colouring of $[1,N]$ to $[1, N_2(k,b)]$), no 2-colouring of $[1,N]$ is good for $x+y = kz+b$. Reflecting via $x \mapsto N+1-x$, every 2-colouring of $[1,N]$ has a monochromatic solution of $x + y + c = kz$, so Forced(c,k,N) holds.

For the reverse inequality (Lemma 2 of Martinelli and Schaal [6], formalized as `RadoNu.lower`), put $A = \lceil (c+2)/k \rceil$ and colour $x \in [1, N-1]$ red if $x < A$ and blue if $x \ge A$. Any red triple (x,y,z) satisfies $kz = x + y + c \ge c + 2$, contradicting $z \le A - 1$ since $k(A-1) < c + 2$. Any blue triple satisfies $kz = x + y + c \ge 2A + c$, contradicting $z \le N - 1$ since $k(N-1) < 2A + c$. Hence Forced(c,k,n) fails for every $n < N$, proving $R(c,k) = \mathrm{LB}(c,k)$.

Part (2) (Lower half at $c = c^*(k)$). Write $c^*(k) = km$, where $m = (k^2 + k - 4)/2$ for odd k and $m = (k^2-2)/2$ for even k. Any solution (x,y,z) of $x + y + c^*(k) = kz$ has $z = m + u$ with $x + y = ku$ and $u \ge 1$.

Odd $k = 2j+1$ ($j \ge 1$). Here $m = kj + 2j - 1 = k(j+1) - 2$, and direct evaluation of (1) gives $N = \mathrm{LB}(c^*(k),k) = k(j+2) - 1 = m + k + 1$, so $1 \le u \le k+1$. Colour $x \in [1,N]$ with colour 1 if $1 \le (x \bmod k) \le j$, or if $k \mid x$ and $x \ge k(\lfloor j/2 \rfloor + 2)$; colour 0 otherwise. Suppose $(x,y,z) \in [1,N]^3$ solves $x + y = ku$. If x and y are not multiples of k, their nonzero residues r and $k - r$ modulo k lie one in $[1,j]$ (colour 1) and one in $[j+1, 2j]$ (colour 0), so the triple is not monochromatic. Otherwise $x = kv$ and $y = kv'$ with $v, v' \in [1, j+1]$ and $u = v + v' \ge 2$. Then $z = m + u = k(j+1) + (u-2)$ has quotient $j + 1 \ge \lfloor j/2 \rfloor + 2$ and residue $u - 2 \in [0, 2j]$. Consequently, z has colour 1 whenever $2 \le u \le j+2$ (residue in $[0,j]$) and colour 0 whenever $j+3 \le u \le k+1$ (residue in $[j+1, 2j]$). If x and y both have colour 0, then $v, v' \le \lfloor j/2 \rfloor + 1$, giving $u = v + v' \le j + 2$, so z has colour 1. If x and y both have colour 1, then $v, v' \ge \lfloor j/2 \rfloor + 2$, giving $u = v + v' \ge 2\lfloor j/2 \rfloor + 4 \ge j + 3$, so z has colour 0. No monochromatic solution exists.

Even $k \ge 4$. Set $q = k(k-2)/2$. Direct evaluation of (1) gives $N = \mathrm{LB}(c^*(k),k) = q + 2k - 1 = m + k$, so $1 \le u \le k$. Colour $x \in [1,N]$ with colour 0 on $\{q+1\} \cup [q+k, q+2k-2]$, and colour 1 elsewhere. Since $m = q + k - 1$, the values $z = m + u$ have colour 0 for $1 \le u \le k-1$ and colour 1 for $u = k$ ($z = N$). For a monochromatic colour-1 solution we must have $u = k$, hence $x + y = k^2$. Without N, every colour-1 element is at most $q + k - 1 = k^2/2 - 1$, so $x + y \le k^2 - 2 < k^2$; with $x = N = q + 2k - 1$, we get $y = k^2 - N = q + 1$, which has colour 0 (and $2N = k^2 + 2k - 2 > k^2$). For a monochromatic colour-0 solution we must have $u \le k - 1$, hence $x + y \le k(k-1)$. However, $2(q+1) = k^2 - 2k + 2 \equiv 2 \not\equiv 0 \pmod{k}$, while any other pair of colour-0 elements sums to at least $(q+1) + (q+k) = k^2 - k + 1 > k(k-1)$. Again no monochromatic solution exists.

In both parities, the interval $[1, \mathrm{LB}(c^*(k),k)]$ is not forced for $c^*(k)$; by monotonicity of Forced, every R satisfying IsRado$(c^*(k),k,R)$ must exceed $\mathrm{LB}(c^*(k),k)$. □

Proof of Corollaries 2 and 3. If $v > c^*(k)$, Part (1) of Theorem 1 gives $R(c,k) = \mathrm{LB}(c,k)$ for all valid $c \ge v$. Conversely, if $v \le c^*(k)$, then $c = c^*(k)$ is valid (clearly for odd k, and $c^*(k) = k(k^2-2)/2$ is even for even k) and fails $R(c,k) = \mathrm{LB}(c,k)$ by Part (2). This proves Corollary 2. Specializing to $k = 5$ gives $c^*(5) = 65$, $\nu(5) = 66$, and $R(66,5) = \mathrm{LB}(66,5) = 19$, proving Corollary 3. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Although Theorem 1 is proved unconditionally for every $k \ge 3$ with zero computer search, independent exact arithmetic and backtracking checks confirm every small case and map the irregular exception regime below $c^*(k)$.

Table 1 lists $c^*(k)$, $\nu(k)$, $\mathrm{LB}(c^*(k),k)$, Kim, Song, and Hwang's threshold $2k^3 + 2k^2 - k$ [5], and their ratio for $k = 3, \dots, 12$ alongside $k = 30, 100, 1000$ (reproduced by `recompute.py`). Figure 3 displays the exact threshold against the 2023 bound and the convergence of the ratio $(2k^3 + 2k^2 - k)/\nu(k)$ to 4 (4.470 at $k = 5$, 4.140 at $k = 30$, and 4.004 at $k = 1000$).

The standalone script `recompute.py` runs in under 3 seconds and verifies:

1. All 13 rows of Table 1 and the closed form $\mathrm{LB}(c^*(k),k) = N(k)$ for $k = 3, \dots, 200$ (0 failures).
2. Direct brute-force checks of the explicit T_2 boundary colourings over all 32,956 solutions $(x \le y)$ for $k = 3, \dots, 30$ (0 monochromatic solutions).

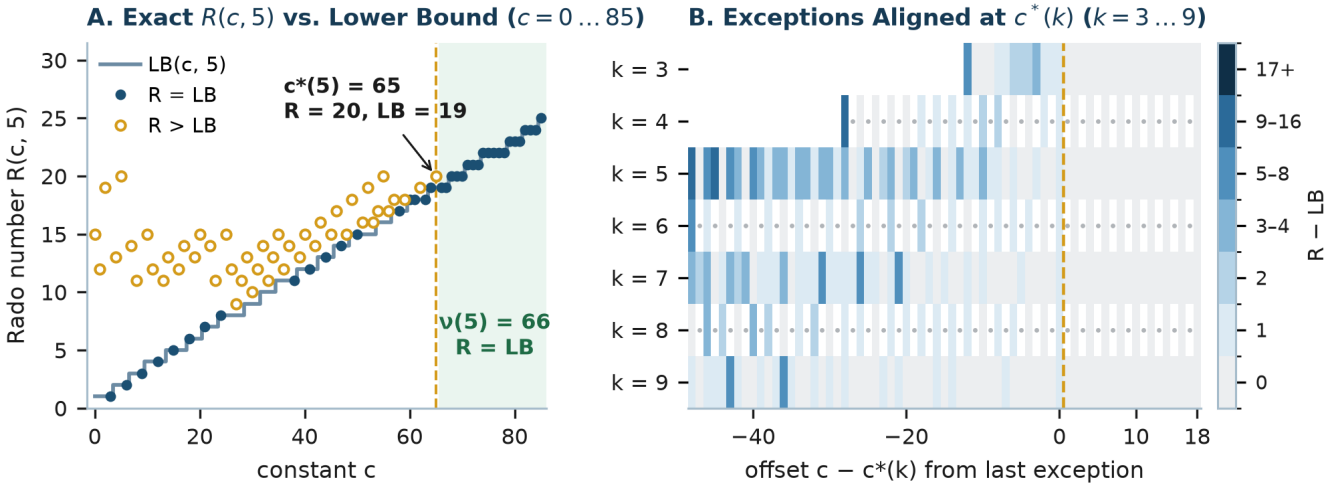


Figure 2. Empirical exception landscape below $c^*(k)$. Panel A: exact $R(c,5)$ versus $\mathrm{LB}(c,5)$ for $c = 0,\dots,85$ (86 values; 48 exceptions in $0\dots65$, ending at $c^*(5) = 65$ with $R(65,5) = 20 > 19 = \mathrm{LB}(65,5)$, and $R = \mathrm{LB}$ for all $c \geq 66$). Notice $R(64,5) = \mathrm{LB}(64,5)$ just before the final exception at $c = 65$. Panel B: excess $R(c,k) - \mathrm{LB}(c,k)$ aligned at $c - c^*(k)$ across $k = 3,\dots,9$ (952 exact values from `values.tsv`; dotted cells are invalid odd c at even k). Every row terminates at column 0 with zero exceptions to the right.

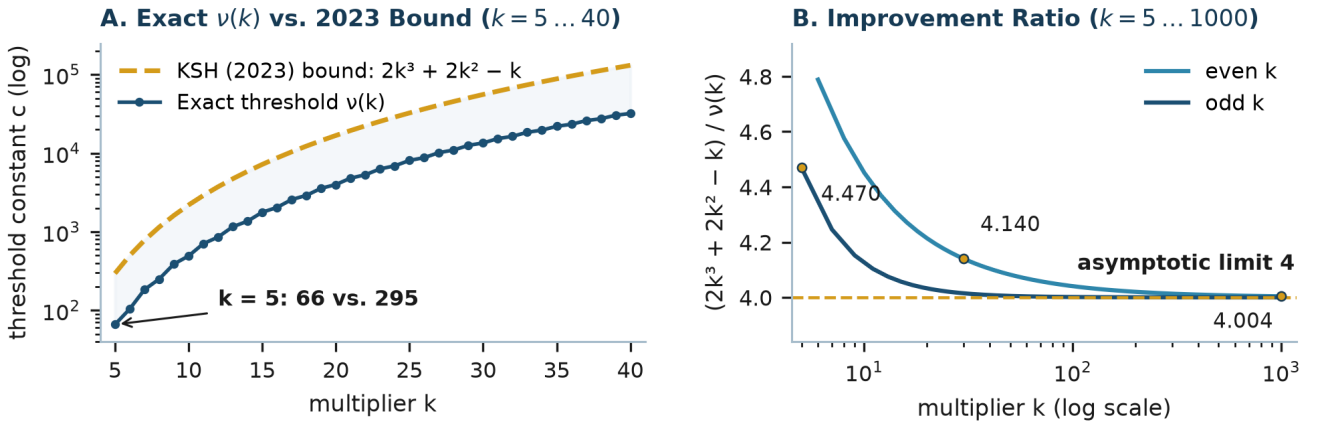


Figure 3. Comparison of the exact threshold $\nu(k)$ against the printed upper bound $2k^3 + 2k^2 - k$ of Kim, Song, and Hwang [5]. Panel A: log-scale thresholds for $k = 5,\dots,40$, dropping from 295 to 66 at $k = 5$. Panel B: improvement ratio $(2k^3 + 2k^2 - k)/\nu(k)$ for odd and even $k \in [5, 1000]$, converging monotonically to 4.

3. Lemma 6 across all 49,191 valid constants c with $c^*(k) < c \leq c^*(k) + 3k^2$ for $k = 3,\dots,40$ (0 violations), plus the exact boundary miss $b = k$, $N = N_2(k,k) - 1$ at $c = c^*(k)$ for all 38 values $k = 3,\dots,40$.
4. Exact backtracking for $k = 3,4,5,6$ over all valid $c \leq c^*(k) + 20$, confirming $R(c^*(k),k) = 9,12,20,24$ against $\mathrm{LB} = 8,11,19,23$, with $10,9,48,33$ exceptions in $0\dots c^*(k)$ and 0 exceptions in $(c^*(k), c^*(k)+20]$.

The extended archive table `values.tsv` (used for Figure 2) records 952 exact values $R(c,k)$ across $k = 3,\dots,9$ through $c^*(k) + 20$, giving exception counts $10,9,48,33,147,94,317$ in $0\dots c^*(k)$ and 0 exceptions above $c^*(k)$.

Table 1. Exact last exception $c^*(k)$, least valid cutoff $\nu(k)$, $\mathrm{LB}(c^*(k),k)$, Kim–Song–Hwang [5] bound $2k^3+2k^2-k$ ($k \geq 5$), and improvement ratio.

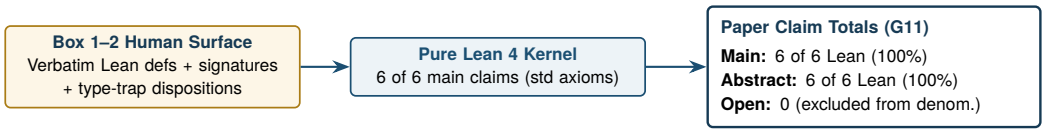
k	$c^*(k)$	$\nu(k)$	$\mathrm{LB}(c^*)$	2023 bound	Ratio
3	12	13	8	—	—
4	28	30	11	—	—
5	65	66	19	295	4.470
6	102	104	23	498	4.788
7	182	183	34	777	4.246
8	248	250	39	1,144	4.576
9	387	388	53	1,611	4.152
10	490	492	59	2,190	4.451
11	704	705	76	2,893	4.104
12	852	854	83	3,732	4.370
30	13,470	13,472	479	55,770	4.140
100	499,900	499,902	5,099	2,019,900	4.041
1000	499,999,000	499,999,002	500,999	2,001,999,000	4.004

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 2f09f46076a2 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Artifacts & Remaining Claims:** No external certificate artifacts are required; any claims marked Written or Unsettled rely on standard written proofs or remain explicitly unproved.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 68/68 public theorems (source-level approx) | WARN: Fidelity: 6/6 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

RadoNu.T2.LB (RadoNu/RadoNu.lean:46-46) — Section 1, eq. (1) (Martinelli-Schaal lower bound LB(c, k))

```
def LB (c k : Nat) : Nat := (2 * ((c + 2 + k - 1) / k) + c + k - 1) / k
```

RadoNu.T2.cstar (RadoNu/RadoNu.lean:49-50) — Section 1, eq. (2) (exact last exception c*(k))

```
def cstar (k : Nat) : Nat :=
  if k % 2 = 1 then k * (k * k + k - 4) / 2 else k * (k * k - 2) / 2
```

RadoNu.Valid (RadoNu/RadoNu.lean:977-977) — Section 2 (validity predicate: k odd or c even)

```
def Valid (c k : Nat) : Prop := k % 2 = 1 ∨ c % 2 = 0
```

RadoNu.HasMono (RadoNu/RadoNu.lean:964-966) — Section 2 (existence of a monochromatic solution of x + y + c = kz in [1, n]^3)

```
def HasMono (c k n : Nat) (col : Nat → Bool) : Prop :=
  ∃ x y z : Nat, (1 ≤ x ∧ x ≤ n) ∧ (1 ≤ y ∧ y ≤ n) ∧ (1 ≤ z ∧ z ≤ n) ∧
    x + y + c = k * z ∧ col x = col y ∧ col y = col z
```

RadoNu.Forced (RadoNu/RadoNu.lean:970-970) — Section 2 (every 2-colouring of [1, n] has a monochromatic solution)

```
def Forced (c k n : Nat) : Prop := ∀ col : Nat → Bool, HasMono c k n col
```

RadoNu.IsRado (RadoNu/RadoNu.lean:974-974) — Section 2 (exact two-colour Rado number R(c, k) = R)

```
def IsRado (c k R : Nat) : Prop := Forced c k R ∧ ∀ n, n < R → ¬ Forced c k n
```

RadoNu.Good (RadoNu/RadoNu.lean:364-366) — Section 2 (2-colouring avoiding monochromatic solutions of x + y = kz + b on [1, N])

```
def Good (k b N : Nat) (col : Nat → Bool) : Prop :=
  ∀ x y z : Nat, 1 ≤ x → x ≤ N → 1 ≤ y → y ≤ N → 1 ≤ z → z ≤ N →
    x + y = k * z + b → ¬ (col x = col y ∧ col y = col z)
```

RadoNu.h0 (RadoNu/RadoNu.lean:745-745) — Section 1 & 3 (upper bound ceil((k + b)/2) on the least red integer)

```
def h0 (k b : Nat) : Nat := (k + b + 1) / 2
```

RadoNu.n1 (RadoNu/RadoNu.lean:748-748) — Section 1 & 3 (forcing interval bound ceil((k*h0 + b)/2) for b >= k + 1)

```
def n1 (k b : Nat) : Nat := (k * h0 k b + b + 1) / 2
```

RadoNu.j0 (RadoNu/RadoNu.lean:880-880) — Section 1 & 3 (least integer j >= k + 1 with k*j + b even)

```
def j0 (k b : Nat) : Nat := if (k * (k + 1) + b) % 2 = 0 then k + 1 else k + 2
```

RadoNu.N2 (RadoNu/RadoNu.lean:883-883) — Section 1 & 3 (forcing interval bound (k*j0 + b)/2 for 1 <= b <= k)

```
def N2 (k b : Nat) : Nat := (k * j0 k b + b) / 2
```

Supporting Semantic Closure Index (non-Box-1 helpers): RadoNu.T2.col (RadoNu/RadoNu.lean), RadoNu.T2.colOdd (RadoNu/RadoNu.lean), RadoNu.T2.colEven (RadoNu/RadoNu.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

RadoNu.R17_3_1_rado (RadoNu/RadoNu.lean:1252-1260) src:697712dbb33a03

```
theorem R17_3_1_rado (k : Nat) (hk : 3 ≤ k) :
    (∀ c, cstar k < c → Valid c k → IsRado c k (LB c k)) ∧
      ¬ Forced (cstar k) k (LB (cstar k) k) ∧
      ∀ R, IsRado (cstar k) k R → LB (cstar k) k < R
```

Assumptions: Any integer k >= 3 (core Lean 4 pure kernel, Init prelude only, zero Mathlib). Part (2) proves non-forcing at LB(c*(k), k) and LB(c*(k), k) < R for any Rado number R; finite existence of R(c*(k), k) cites Jones and Schaal (2004). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RadoNu.nu_iff (RadoNu/RadoNu.lean:1265-1273) src:4f3af5829b87b8

```
theorem nu_iff (k : Nat) (hk : 3 ≤ k) (v : Nat) :
    (∀ c, v ≤ c → Valid c k → IsRado c k (LB c k)) ↔ cstar k < v
```

Assumptions: Any integer k >= 3 and any integer cutoff v >= 0 (core Lean 4 pure kernel, zero Mathlib). Least integer cutoff is c*(k) + 1; least valid cutoff nu(k) is c*(k) + 1 (odd k) and c*(k) + 2 (even k). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RadoNu.nu_5_eq_66 (RadoNu/RadoNu.lean:1290-1294) src:f7b64d36a442ed

```
theorem nu_5_eq_66 :
    (∀ c, 66 ≤ c → Valid c 5 → IsRado c 5 (LB c 5)) ∧
    ¬ (∀ c, 65 ≤ c → Valid c 5 → IsRado c 5 (LB c 5))
```

Assumptions: Closed statement settling Kim, Song, and Hwang's (2023) conjecture nu(5) = 66 in core Lean 4. Applicability: Unconditional (closed statement; no outer premise witness needed).

RadoNu.thm1 (RadoNu/RadoNu.lean:869-875) src:50fda43dbcace8

```
theorem thm1 {k b : Nat} (hk : 3 ≤ k) (hb : k + 1 ≤ b)
    (hpar : k % 2 = 0 → b % 2 = 0) (col : Nat → Bool) :
    ¬ Good k b (n1 k b) col
```

Assumptions: Any integers k >= 3 and b >= k + 1 with b even if k is even, and any 2-colouring col : Nat -> Bool (core Lean 4 pure kernel, zero Mathlib). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RadoNu.thm2 (RadoNu/RadoNu.lean:950-956) src:f53fc36f1d7728

```
theorem thm2 {k b : Nat} (hk : 3 ≤ k) (hb1 : 1 ≤ b) (hbk : b ≤ k)
    (hpar : k % 2 = 0 → b % 2 = 0) (col : Nat → Bool) :
    ¬ Good k b (N2 k b) col
```

Assumptions: Any integers k >= 3 and 1 <= b <= k with b even if k is even, and any 2-colouring col : Nat -> Bool (core Lean 4 pure kernel, zero Mathlib). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RadoNu.lemmaA (RadoNu/RadoNu.lean:1069-1128) src:1ce5980fef2433

```
theorem lemmaA {k c : Nat} (hk : 3 ≤ k) (hc : cstar k < c) (hval : Valid c k) :
    ∃ b, 1 ≤ b ∧ c = (k - 2) * (LB c k + 1) + b ∧ (k % 2 = 0 → b % 2 = 0) ∧
      (k + 1 ≤ b → n1 k b ≤ LB c k) ∧ (b ≤ k → N2 k b ≤ LB c k)
```

Assumptions: Any integer k >= 3 and any valid c > c*(k) (core Lean 4 pure kernel, zero Mathlib). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
N2	nat_div	/ 2	intended_convention	Exact integer division by 2 proved by RadoNu.j0_facts whenever k is odd or b is even.
LB	nat_sub	- 1	intended_convention	Ceiling encoding (a + k - 1) / k has a + k >= 5 > 1 for all k >= 3 (verified by ceil_spec).
LB	nat_div	/ k	intended_convention	Divisor k >= 3 > 0; ceiling identity (a + k - 1) / k = ceil(a / k) proved in RadoNu.ceil_spec.
col	nat_mod	% 2	intended_convention	Standard parity test modulo 2 in boundary witness definition.
colEven	nat_sub	- 2	intended_convention	Evaluated only for even k >= 4 (k - 2 >= 2 and 2*k - 2 >= 6; zero Nat underflow).
colEven	nat_div	/ 2	intended_convention	Exact integer division by 2 for even k >= 4 (proved in T2.q_even).
colOdd	nat_mod	% k	intended_convention	Residue modulo odd k >= 3 > 0 (only used as existential witness).
colOdd	nat_sub	- 1	intended_convention	For odd k >= 3, k - 1 >= 2 is even with zero Nat underflow.
colOdd	nat_div	/ 2	intended_convention	Intended floor division for j = (k - 1)/2 and floor(j/2) in existential witness T2.colOdd.
cstar	nat_mod	% 2	intended_convention	Standard parity branch modulo 2.

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
cstar	nat_sub	- 4	intended_convention	For k >= 3, k^2 + k >= 12 > 4; exact identity 2*cstar k + 4*k = k^3 + k^2 proved in two_cstar.
cstar	nat_div	/ 2	intended_convention	Exact integer division by 2 in both parities (proved in RadoNu.two_cstar).
cstar	nat_sub	- 2	intended_convention	For even k >= 4, k^2 >= 16 > 2; exact identity 2*cstar k + 2*k = k^3 proved in two_cstar.
Valid	nat_mod	% 2	intended_convention	Standard parity predicate modulo 2.
h0	nat_div	/ 2	intended_convention	Intended ceiling division by 2.
j0	nat_mod	% 2	intended_convention	Standard parity test modulo 2.
lemmaA	nat_sub	- 2	intended_convention	Hypothesis 3 <= k guarantees k - 2 >= 1 with zero Nat underflow.
lemmaA	nat_mod	% 2	intended_convention	Standard parity implication modulo 2.
n1	nat_div	/ 2	intended_convention	Intended ceiling division by 2.
thm1	nat_mod	% 2	intended_convention	Standard parity hypothesis modulo 2.
thm2	nat_mod	% 2	intended_convention	Standard parity hypothesis modulo 2.

[a]Covers the specified Lean 4 hazard set (`Nat` subtraction/division/mod, totalized real/rational division, `Real.log`/`rpow`, `Finset.range`, `Classical.choose`, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Exact last exception c*(k) for every k >= 3 (R(c, k) = LB(c, k) for valid c > c*(k), and non-forcing at LB(c*(k), k))	Lean Kernel	RadoNu.R17_3_1_rado	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Core Lean 4 pure kernel (Init prelude only, zero Mathlib) + Box 1 defs for all k >= 3; finiteness of R(c*(k), k) cites Jones and Schaal (2004).
Cor. 2: Threshold equivalence (R(c, k) = LB(c, k) for valid c >= v if and only if v > c*(k)), giving exact nu(k) for every k >= 3	Lean Kernel	RadoNu.nu_iff	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Core Lean 4 pure kernel (zero Mathlib) + Box 1 defs for all k >= 3 and v >= 0.
Cor. 3: Exact resolution of Kim, Song, and Hwang's (2023) conjecture nu(5) = 66	Lean Kernel	RadoNu.nu_5_eq_66	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Unconditional core Lean 4 pure kernel (zero Mathlib) + Box 1 defs.
Thm. 4: Reflected-equation forcing on [1, n1(k, b)] for every k >= 3 and parity-compatible b >= k + 1	Lean Kernel	RadoNu.thm1	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Core Lean 4 pure kernel (zero Mathlib) + Box 1 defs for all k >= 3 and valid b >= k + 1.
Thm. 5: Reflected-equation forcing on [1, N2(k, b)] for every k >= 3 and parity-compatible 1 <= b <= k	Lean Kernel	RadoNu.thm2	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Core Lean 4 pure kernel (zero Mathlib) + Box 1 defs for all k >= 3 and valid 1 <= b <= k.
Lem. 6: Arithmetic reflection bridge showing N = LB(c, k) meets n1(k, b) or N2(k, b) for every valid c > c*(k)	Lean Kernel	RadoNu.lemmaA	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Core Lean 4 pure kernel (zero Mathlib) + Box 1 defs for all k >= 3 and valid c > c*(k).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: b08a198fcf60 | Lean mode: committed_receipt | Closure SHA-256: d0e36352d4dc | Git anchor: 2f09f46076a2 | Axioms SHA-256: d2b7692dc252 | Fidelity SHA-256: 2c56e4451b19.

6. PRIOR WORK AND OPEN QUESTIONS

Jones and Schaal [2] (`[to check]`) established the validity criterion for the existence of $R(c,k)$ and conjectured that $R(c,k)$ agrees with the lower bound for large c. Martinelli and Schaal [6] proved the general lower bound (1) (Lemma 2, p. 36) and the negative-constant lower bound (Lemma 3, p. 37). Exact cutoffs for small multipliers were obtained by Kézdy, Snevily, and White [3] for $k = 3$ ($\nu(3) = 13$, together with the weaker odd-k lower bound $\nu(k) \geq (k^3 - k)/2 + 1$) and by Guo [1] (`[to check]`) for $k = 4$ ($R(c,4) = \mathrm{LB}(c,4)$ for even $c \geq 34$, tightened to $\nu(4) = 30$ in [5]). For small positive odd constants, Kim,

Hwang, and Song [4] (`[to check]`) studied $x+y+c=kz$ when c and k are odd with $k \geq c+6$. Kim, Song, and Hwang [5] established $R(c,k) = \mathrm{LB}(c,k)$ for $k \geq 5$ and valid $c \geq 2k^3+2k^2-k$, posed determining $\nu(k)$ for all k, and conjectured $\nu(5) = 66$ via an explicit good colouring of $[1,19]$ at $c = 65$.

Three independent logged literature searches passed `litsearch.py check` (exit 0: two rounds logging 33 and 26 queries with 22 and 20 candidates across arXiv, OEIS, OpenAlex, Semantic Scholar, zbMATH Open, Math Stack Exchange, MathOverflow, DataCite, Zenodo, HAL, OSF, and GitHub, plus a keyed CORE run returning 71 records across 7 answered requests and 12 candidates, supplemented by a 5-query Google Scholar sweep). No prior determination of $c^*(k)$ or $\nu(k)$ for general $k \geq 3$ was found in the searches logged. *Unread or unsearched items* (`[to check]` in `references-checked.md`): full texts of Jones and Schaal [2], Guo [1], and Kim, Hwang, and Song [4]; theses by Myers (2015), Bergjord (2023), and Rahman (2025); paywalled Kim, Song, and Hwang (2026); direct Google Scholar cited-by traversal of [5]; MathSciNet full text; and Landman and Robertson's monograph.

Open questions. (1) Characterize the irregular exception values $R(c,k)$ for $0 < c \leq c^*(k)$ (Figure 2 shows non-monotone structure with $10, 9, 48, 33, 147, 94, 317$ exceptions for $k = 3, \ldots, 9$). (2) Formalize finite existence of $R(c^*(k),k)$ directly in Lean 4 (currently cited from [2] `[to check]`; our Lean proof establishes $\neg\,\mathrm{Forced}(c^*(k),k,\mathrm{LB}(c^*(k),k))$ and $\forall R,\ \mathrm{IsRado}(c^*(k),k,R) \to \mathrm{LB}(c^*(k),k) < R$ unconditionally).

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

Manuscript sources, figure scripts, and `recompute.py` live in `papers/rado-numbers-last-exception/draft1/`, and the pure core Lean 4 formalization lives in `lean/RadoNu/` (68 verified declarations, zero Mathlib imports, zero `sorry`, zero `native_decide`, standard axioms only). Running `python3 papers/rado-numbers-last-exception/draft1/recompute.py` replays all tables, worked examples, T_2 checks ($k \leq 30$), Lemma 6 checks ($k \leq 40$), and exact backtracking values ($k \leq 6$). The standalone Lean 4 package is archived under Zenodo DOI `10.5281/zenodo.24490762` (concept DOI `10.5281/zenodo.24490761`) and registered as Palomar entry `PALOMAR-2026-11-23-000035` (mechanical Lean checking plus automated editorial review; not human peer review). A Lean Pool port (PR #997) received maintainer review on 2026-11-24, which corrected the Martinelli–Schaal Lemma 2 attribution incorporated here.

AI research agents built and directed by Tyler B. Hughes discovered the reflection proof, produced the Lean 4 formalization, executed computational confirmations, and drafted this manuscript. No human mathematician has yet refereed the proof.

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