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Working Draft

Factor-Language Rigidity of the Rote–Fibonacci Word Under Avoidance of xxx^R

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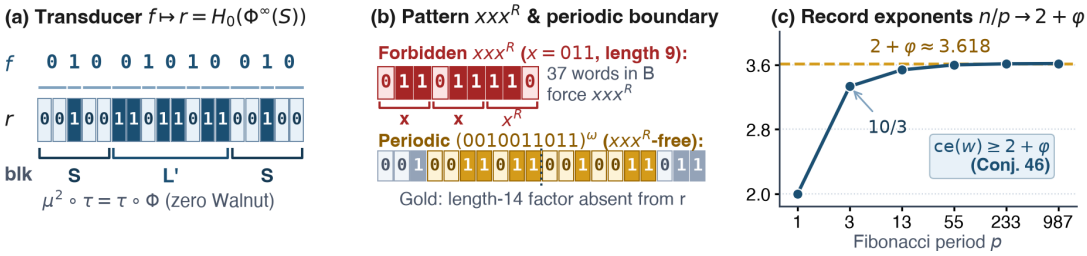
Supporting Information: `recompute.py`, audited Lean 4 library (`lean/RoteFibonacciReversal/`), and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: A binary word avoids xxx^R if no factor has the form xxx^R with x nonempty. Du, Mousavi, Schaeffer, and Shallit (2014; with Rowland, 2017) proved that the Rote–Fibonacci word r avoids xxx^R with critical exponent $2 + \varphi$ ($\varphi = (1+\sqrt{5})/2$) and conjectured (Conjecture 46 / Conjecture 10) that every infinite binary avoider has critical exponent $\geq 2+\varphi$. We prove factor-language rigidity: every one-sided recurrent binary word avoiding xxx^R is strictly periodic or has the exact same finite factors as r, and the same dichotomy holds for every two-sided recurrent binary avoider. Via minimal subshifts and Fibonacci-period powers in r, this settles Conjecture 46: every infinite binary word avoiding xxx^R has critical exponent $\geq 2 + \varphi$. We also prove $\mathrm{Fact}(g_{\mathrm{rf}}(b_5)) = \mathrm{Fact}(r)$ for Badkobeh and Ochem's projection $g_{\mathrm{rf}}(b_5)$ ($g_{\mathrm{rf}}(b_5)[n+1] = \overline{r[n]}$), giving the same rigidity for $g_{\mathrm{rf}}(b_5)$. All five main results are proved unconditionally in Lean 4 (`v4.35.0`) + Mathlib (`lean/RoteFibonacciReversal/`, 306 audited declarations, zero `sorry`, zero `native_decide`, and zero Walnut dependence).

Verification Scope (G11): WARN: Human referee: pending
Main results: 5 of 5 in Lean Kernel (100%) · 1 open (not counted)
Abstract claims: 5 of 5 in Lean Kernel (100%)

Full Trust Boundary: Section 5.

MSC 2020: 68R15, 05A05, 11B85 **Keywords:** combinatorics on words, patterns with reversal, Rote–Fibonacci word, factor-language rigidity, critical exponent, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

In combinatorics on words, a central question is how strongly forbidding a single pattern constrains the global structure of an infinite sequence. Write x^R for the reversal of a finite binary word $x \in \{0,1\}^+$; a binary word avoids the *pattern with reversal* xxx^R if no factor of it has the form xxx^R with x nonempty. For instance, 000 ($x = 0$), 010110 ($x = 01$), and 011011110 ($x = 011$) are instances of xxx^R, whereas 0110 avoids the pattern.

Let $f = 01001010010010100101\ldots$ be the Fibonacci word (the fixed point of $\mu : 0 \mapsto 01, 1 \mapsto 0$) and $\varphi = (1+\sqrt{5})/2$ be the golden ratio. Applying the alternating run-length transducer of Du, Mousavi, Schaeffer, and Shallit [4]—where each 0 in f emits two copies of the current bit, each 1 emits one copy, and the bit flips after every letter of f—produces the *Rote–Fibonacci word*

$$r = 00100110110110010011011011 0010\ldots$$

whose factor complexity is $p_r(n) = 2n$ for every $n \geq 1$ (a Rote sequence [6] `[to check]`; Theorem 43 of [4]). Using the Walnut decision procedure, Du et al. [4, 3] proved that r avoids xxx^R (Theorem 37 of [4]) and has critical exponent $2 + \varphi \approx 3.618034$ (Corollary 39 of [4]), and posed Conjecture 46 [4] (Conjecture 10 of [3]): every infinite binary word avoiding xxx^R has critical exponent at least $2 + \varphi$. Independently, Currie and Rampersad [2] analyzed the polynomial growth rate of finite binary words avoiding xxx^R via a multi-scale block decomposition, and Badkobeh and Ochem [1] remarked in their concluding section that a binary projection $g_{\mathrm{rf}}(b_5)$ appears to share the factor language of r while noting that their finite-forbidden-set method cannot prove such characterizations.

We show that the single pattern xxx^R completely pins down the factor language of r among all recurrent aperiodic binary words, settling Conjecture 46 and Badkobeh–Ochem's factor-equality remark unconditionally.

Theorem 1 (One-Sided Rote–Fibonacci Rigidity for xxx^R). *Let $w \in \{0,1\}^{\mathbb{N}}$ be a recurrent one-sided infinite binary word that contains no factor of the form xxx^R with x nonempty. Then either w is strictly periodic, or $\mathrm{Fact}(w) = \mathrm{Fact}(r)$.*

The proof operates on two-sided words via an infinite multi-scale descent through Currie and Rampersad's suitable-pair hierarchy, giving the matching two-sided statement directly.

Theorem 2 (Two-Sided Rote–Fibonacci Rigidity for xxx^R). *Let $w \in \{0,1\}^{\mathbb{Z}}$ be a recurrent two-sided infinite binary word avoiding nonempty factors of the form xxx^R. Then either w is periodic, or $\mathrm{Fact}(w) = \mathrm{Fact}(r)$.*

The aperiodicity alternative in Theorems 1 and 2 is necessary: by Theorem 48 of [4], the period-10 word z^ω with $z = r[0\ldots9] = 0010011011$ avoids xxx^R and matches r on its first 11 letters, yet contains the length-14 factor 00110110010011, which is absent from the 28 length-14 factors of r.

Dropping recurrence altogether yields Du, Mousavi, Schaeffer, and Shallit's Conjecture 46 [4] (Conjecture 10 of [3]) as an unconditional consequence via minimal subshifts.

Corollary 3 (Critical-Exponent Lower Bound: Conjecture 46). *Every infinite binary word $w \in \{0,1\}^{\mathbb{N}}$ avoiding nonempty factors of the form xxx^R satisfies $\mathrm{ce}(w) \geq 2+\varphi = (5+\sqrt{5})/2$ in the extended reals.*

In addition, we prove Badkobeh and Ochem's factor-equality remark [1] for $g_{\mathrm{rf}}(b_5)$ from first principles, transferring rigidity directly to their sequence.

Proposition 4 (Factor Equality for Badkobeh–Ochem's $g_{\mathrm{rf}}(b_5)$). *Let $b_5 = f_5^\omega(0)$ be the fixed point of $f_5 : 0 \mapsto 01,\ 1 \mapsto 23,\ 2 \mapsto 4,\ 3 \mapsto 21,\ 4 \mapsto 0$ and $g_{\mathrm{rf}} : 0 \mapsto 01,\ 1 \mapsto 10,\ 2 \mapsto \varepsilon,\ 3 \mapsto 11,\ 4 \mapsto 00$. Then $\mathrm{Fact}(g_{\mathrm{rf}}(b_5)) = \mathrm{Fact}(r)$.*

Theorem 5 (Rigidity in Badkobeh–Ochem's $g_{\mathrm{rf}}(b_5)$ Formulation). *Every recurrent one-sided infinite binary word $w \in \{0,1\}^{\mathbb{N}}$ avoiding nonempty factors of the form xxx^R is either strictly periodic or satisfies $\mathrm{Fact}(w) = \mathrm{Fact}(g_{\mathrm{rf}}(b_5))$.*

Concept in Action: Transducer, Avoidance, and the Period-10 Boundary

```
f = "0"
for _ in range(9):
    f = "".join("01" if c == "0" else "0" for c in f)
r = "".join(str(i % 2) * (2 - int(c)) for i, c in enumerate(f))
p, bad = r[:10] * 4, "00110110010011"
def has_xxxr(w):
    return any(w[i:i+k] == w[i+k:i+2*k] == w[i+2*k:i+3*k][::-1]
               for k in range(1, len(w)//3 + 1)
               for i in range(len(w) - 3*k + 1))
assert not has_xxxr(r[:200]) and not has_xxxr(p)
assert bad in p and bad not in {r[i:i+14] for i in range(len(r)-14)}
```

Eliminating Walnut and Currie–Rampersad Lemma 3. Two structural features make the proof completely self-contained and machine-checked from first principles (Figure 1):

1. Currie and Rampersad's published proof of their Lemma 4 [2] relies on their Lemma 3, which has posted errata. We bypass Lemma 3 entirely by constructing 37 explicit symbolic certificates in the free monoid with reversal and complement involutions, verified unconditionally for every suitable pair inside Lean's kernel via `decide +kernel` (zero axioms; Lemma 6).
2. Rather than importing Du et al.'s Walnut proof that r avoids xxx^R (Theorem 37 of [4]), we connect the Fibonacci morphism μ to Currie and Rampersad's block morphism $\Phi : S \mapsto SL,\ L \mapsto SLL$ via $\tau(S) = 010$, $\tau(L) = 01010$ and the exact identity $\mu^2 \circ \tau = \tau \circ \Phi$. This proves algebraically that every level-i suitable-pair block image of every factor of $\Phi^\omega(S)$ occurs inside r.

Roadmap. Section 2 fixes notation matching our Lean 4 library verbatim. Section 3 proves Proposition 4, Theorems 1–5, Lemma 6, and Corollary 3. Section 4 summarizes independent computational cross-checks and states the remaining open periodic-classification problem (Conjecture 7). Section 5 presents the machine-generated Lean 4 Reader's Trust Boundary, and Sections 6–7 review prior work, search caveats, and reproducibility.

2. DEFINITIONS, NOTATION, AND SETUP

We write $\{0,1\}$ for the binary alphabet (represented in Lean as `Bool` with `false` $= 0$ and `true` $= 1$), $\overline{b} = 1 - b$ for bitwise complement, $\overline{w}$ for letterwise complement of a finite word $w \in \{0,1\}^*$, and w^R for its reversal.

Factors, recurrence, and periodicity. For a one-sided infinite word $x : \mathbb{N} \to \Sigma$, the slice $x[i\ldots i+n-1]$ has length n with k-th letter $x(i+k)$ (`sliceOne`); $u \in \Sigma^*$ is a *factor* of x ($u \in \mathrm{Fact}(x)$, Lean `IsFactorOne`) if $u = x[i\ldots i+|u|-1]$ for some $i \in \mathbb{N}$. Two one-sided words have the same factors (`HasSameFactorsOne`) when $\mathrm{Fact}(x) = \mathrm{Fact}(y)$. The word x is *recurrent* (`RecurrentOne`) if every factor $u \in \mathrm{Fact}(x)$ occurs at two indices $i < j$ (equivalently, infinitely often); it is *uniformly recurrent* (`UniformlyRecurrentOne`) if for every $u \in \mathrm{Fact}(x)$ there exists $R \in \mathbb{N}$ such that every length-$(R+|u|)$ factor of x contains u. It is *strictly periodic* (`PeriodicOne`) if there exists $p \geq 1$ with $x(n+p) = x(n)$ for all $n \in \mathbb{N}$. For two-sided words $x : \mathbb{Z} \to \Sigma$, the corresponding notions `sliceTwo`, `IsFactorTwo`, `RecurrentTwo` (every factor occurs at some $i < j$ in $\mathbb{Z}$), and `PeriodicTwo` ($\exists p \geq 1,\ \forall n \in \mathbb{Z},\ x(n+p) = x(n)$) are defined identically.

Avoidance and critical exponent. A finite binary word w *contains* xxx^R (`ContainsXXXR`) if $w = uxxx^Rv$ for some $u, v \in \{0,1\}^*$ and nonempty $x \in \{0,1\}^+$. An infinite binary word avoids xxx^R (`AvoidsXXXROne` / `AvoidsXXXRTwo`) if none of its finite factors contains xxx^R. A positive integer $p \geq 1$ is a *period* of $u \in \{0,1\}^*$ (`IsPeriod`) if $u[i+p] = u[i]$ whenever $i + p < |u|$. The *extended-real critical exponent* (`critExpEReal`) of $w : \mathbb{N} \to \{0,1\}$ is

$$\mathrm{ce}(w) = \sup\Big\{ \frac{|u|}{p} \in \mathbb{R} \cup \{-\infty, +\infty\} : u \in \mathrm{Fact}(w),\ u \neq \varepsilon,\ p \geq 1,\ p \text{ a period of } u \Big\}.$$

Allowing non-least periods $p \geq p_{\min}(u)$ never increases the supremum because $|u|/p \leq |u|/p_{\min}(u)$, and periods $p > |u|$

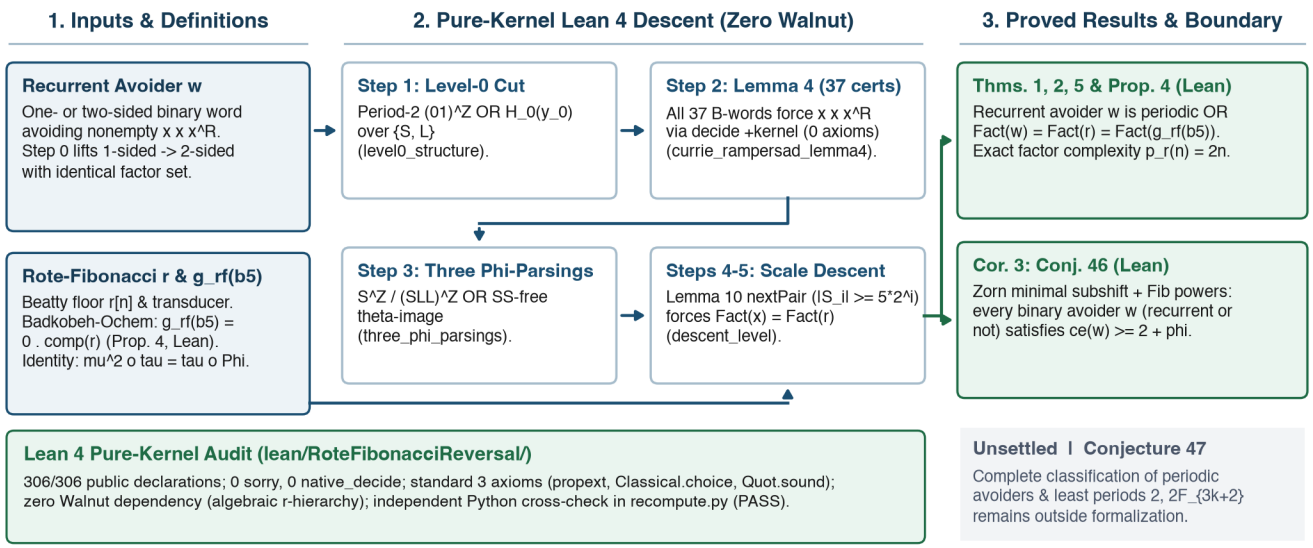


Figure 1. Discovery and proof architecture for `rote-fibonacci-pattern-with-reversal`. Every proved main result (5 of 5 Main Results, 5 of 5 Abstract Claims) is formally verified unconditionally in Lean 4 (`v4.35.0`) with Mathlib in `lean/RoteFibonacciReversal/` (306 audited public declarations, zero `sorry`, zero `native_decide`, zero Walnut dependence). Du et al.'s Conjecture 47 on periodic avoiders is tracked separately as unsettled.

contribute values < 1, so this coincides with the standard critical exponent over least periods whenever $\mathrm{ce}(w) \geq 1$.

Fibonacci, Rote–Fibonacci, and Badkobeh–Ochem words. Set $\varphi = (1+\sqrt{5})/2$. The Fibonacci word $f = \mu^\omega(0)$ (`fibWord`) is the fixed point of $\mu(0) = 01$, $\mu(1) = 0$. The Rote–Fibonacci word $r : \mathbb{N} \to \{0,1\}$ (`roteFib`) is defined in closed form by

$$r[n] = (1 + \lfloor (n+2)/\varphi \rfloor) \bmod 2 \qquad (n \geq 0).$$

As proved in `RoteFibonacciWord.lean` (`roteFib_eqDiff_eq_fibWord_succ` and `roteFib_run_description`), its first-difference indicator $E(r)[j] = [r[j] = r[j+1]]$ equals $f[j+1]$, and r coincides letter for letter with Du et al.'s alternating run-length transducer output starting at bit 0 with run lengths $2 - f[i]$ (Theorem 35, Description 0 of [4]). Badkobeh and Ochem's word $w_{\mathrm{grf}} = g_{\mathrm{rf}}(b_5)$ (`wGrf`) is the image of $b_5 = f_5^\omega(0)$ under g_{rf} as defined in Proposition 4.

Suitable pairs, block morphisms, and the forbidden set B. Over the block alphabet $\mathcal{A} = \{S, L\}$ (`SL`), define the morphisms $\Phi : S \mapsto SL,\ L \mapsto SLL$ (`Phi`), $\theta : S \mapsto \Sigma,\ L \mapsto \Lambda$ (`theta`), and $\tau : S \mapsto 010,\ L \mapsto 01010$ (`tau`), where

$$\begin{aligned} c &= SLSL, \\ \Sigma &= LSLSLLSLLSLSL \quad (|\Sigma| = 13), \\ \Lambda &= LSLSLLSLLSLSLLSLLSLSL \quad (|\Lambda| = 21). \end{aligned}$$

Direct computation checks $\Phi^3(S)c = c\Sigma$ and $\Phi^3(L)c = c\Lambda$, whence $\Phi^3(u)c = c\theta(u)$ for every $u \in \mathcal{A}^*$ (`iterMorph_Phi_three_conj`). Following Currie and Rampersad [2], a triple (l, m, p) of nonempty binary words is a *suitable pair* (`SuitablePair`) if

$$lm = m^R l^R \qquad \text{and} \qquad ll^R = lm\overline{m^R}p.$$

Its associated blocks are $\mathrm{blk}(S) = lm$ and $\mathrm{blk}(L) = ll^R$ (both palindromes). Given a phase $\mathrm{ph} \in \{0,1\}$ and $u = u_0 u_1 \ldots u_{k-1} \in \mathcal{A}^*$, the alternating-phase block image is

$$H((l,m,p), \mathrm{ph}, u) = \prod_{j=0}^{k-1} \overline{\mathrm{blk}(u_j)}^{\,\mathrm{ph}+j \bmod 2}.$$

The level-0 suitable pair (`level0Pair`) is $l_0 = 0010$, $m_0 = 0$, $p_0 = 00$, giving $\mathrm{blk}_0(S) = 00100$ and $\mathrm{blk}_0(L) = 00100100$. Finally, $B \subset \mathcal{A}^+$ (`B_set`) is Currie and Rampersad's set of 37 forbidden block words (lengths 5 to 41) organized into nine families:

$$\begin{aligned} F_1 &= \{S,L\}SSSL\{L, SS, SL\}, \\ F_2 &= LSSL\{L, SS, SL\}, \qquad F_3 = \{S,L\}LLLLL\{S,L\}, \\ F_4 &= \{S,L\}LSLLL\{S,L\}, \quad F_5 = \Phi(\{S,L\}SS\{S,L\}), \\ F_6 &= \Phi(\{S,L\}LLLSL\{L, SS, SL\}), \\ F_7 &= \Phi^2(LLL\{S,L\}), \qquad F_8 = \Phi^2(\{S,L\}LSS\{S,L\}), \\ F_9 &= \Phi^2(\{S,L\}SSSSS\{S,L\}). \end{aligned}$$

3. PROOF OF THE MAIN RESULTS

We establish Proposition 4 first so that complement closure, uniform recurrence, and factor complexity of r are available for the descent; we then prove Lemma 6, Theorems 2, 1, and 5, and Corollary 3.

Proof of Proposition 4. The full argument is formalized in `RoteFibonacciWord.lean`. First, elementary descent on $m^2 = 5n^2$ proves that φ and $1/(2\varphi)$ are irrational (`irrational_phi`, `irrational_inv_two_phi`). For the Fibonacci word $f = \mu^\omega(0)$, the standard Sturmian/Beatty identities give $1 - f[j] = \lfloor (j+2)/\varphi \rfloor - \lfloor (j+1)/\varphi \rfloor$ for all $j \geq 0$ (`fibWord_one_sub_toNat`).

For $b_5 = f_5^\omega(0)$, induction on k shows that every 2-factor of $f_5^k(0)$ belongs to the 9-element set $\{01, 04, 10, 12, 21, 23, 30,$

$34, 42\}$ (`b5_valid2`). Checking the image under g_{rf} across all nine pairs yields nine exact delay identities in `List Bool` (`delay_relation`) showing that the first-difference sequence of $w_{\mathrm{grf}} = g_{\mathrm{rf}}(b_5)$ satisfies $E(w_{\mathrm{grf}})[j] = f[j]$ for every $j \geq 0$ (`eqDiff_wGrf_eq_fibWord`). Since $w_{\mathrm{grf}}[0] = 0$, summing $1 - E(w_{\mathrm{grf}})[j] = 1 - f[j]$ telescopes to

$$w_{\mathrm{grf}}[n] = \lfloor (n+1)/\varphi \rfloor \bmod 2 \qquad (n \geq 0)$$

(`wGrf_eq_floor_mod2`). Comparing directly with $r[n] = (1 + \lfloor (n+2)/\varphi \rfloor) \bmod 2$ gives $w_{\mathrm{grf}}[n+1] = \overline{r[n]}$ for all $n \geq 0$ (`wGrf_succ_eq_not_roteFib`).

Both w_{grf} and r are therefore irrational rotation codings of the circle $\mathbb{R}/\mathbb{Z}$ by rotation angle $\alpha = 1/(2\varphi)$ against the half-circle partition $[0, 1/2)$, shifted by 0 for w_{grf} and by $1/2 + 2\alpha$ for r, with neither orbit hitting the cut points $0, 1/2$. By uniform one-dimensional Kronecker density of $\{k\alpha \bmod 1\}$ (`uniform_kronecker_dense`), two cut-avoiding orbits of the same irrational rotation have identical finite factor sets, giving $\mathrm{Fact}(w_{\mathrm{grf}}) = \mathrm{Fact}(r)$ (`wGrf_hasSameFactors_roteFib`). The same rotation argument proves that $\mathrm{Fact}(r)$ is closed under bitwise complement (`roteFib_factors_closed_comp`), that r is uniformly recurrent (`roteFib_uniformly_recurrent`), and that $p_r(n) = 2n$ for all $n \geq 1$ (`roteFib_factor_complexity`). □

We now turn to the six-step proof of Theorems 2 and 1.

Step 0 (One-sided to two-sided reduction). Let $x : \mathbb{N} \to \{0,1\}$ be recurrent. Define nested factors v_k of x inductively: put $v_0 = \varepsilon$; given v_k occurring at index i_k, recurrence furnishes a second occurrence at $j_k > i_k$ and then an occurrence of $x[0 \ldots \max(k, j_k + |v_k|)]$ further right, yielding a factor $v_{k+1} = a_k v_k b_k$ with a_k, b_k nonempty that contains both v_k and $x[0 \ldots k]$. Anchoring the centers of v_k at $0 \in \mathbb{Z}$ produces a limit word $y : \mathbb{Z} \to \{0,1\}$ with $\mathrm{Fact}(y) = \mathrm{Fact}(x)$ that is two-sided recurrent, preserves xxx^R-avoidance, and satisfies: if y is periodic of period p, then x is strictly periodic of period p (`oneSided_to_twoSided`). Thus Theorem 1 reduces directly to Theorem 2.

Step 1 (Level-0 structure). Let $x : \mathbb{Z} \to \{0,1\}$ be two-sided, recurrent, and xxx^R-free. We claim that either x has period 2 (namely $(01)^{\mathbb{Z}}$), or x is an alternating-phase level-0 block image of a recurrent word $y_0 : \mathbb{Z} \to \mathcal{A}$ (`level0_structure`).

Indeed, x contains no 000 or 111 ($x = 0, 1$), so every run has length 1 or 2. Suppose x contains 0101. Following 0101 by 1 gives 01011, which cannot be extended ($010110 = (01)(01)(10)$ and 010111 contains 111); following 0101 by 00 forces 1 next, giving $101001 = (10)(10)(01)$. Hence 0101 must be followed by 01, and by induction x alternates forever to the right. By complement symmetry, the same holds for 1010. If x does not alternate everywhere on $\mathbb{Z}$, let j be the least index from which x alternates; then $x(j-1) = x(j)$, and recurrence repeats the length-5 factor $x[j-1 \ldots j+3]$ at some $j' - 1 \neq j - 1$, forcing an equal pair inside the strictly alternating tail starting at $\min(j, j')$, contradiction. Thus $x = (01)^{\mathbb{Z}}$ (period 2).

Otherwise x avoids 0101 and 1010, so runs of length 1 are isolated. Because $(0011)^{\mathbb{Z}}$ contains the palindrome cube $(0110)^3$ and three consecutive double runs preceded by a single run force $00\,1\,00\,11\,00 = (001)(001)(100)$ (or its complement), every maximal chain of double runs has length at most 2. Moreover, no tail can avoid adjacent double runs forever, because $(001)^\omega$ contains $(010)^3$. Hence cuts between adjacent double runs (the midpoints of 0011 and 1100) occur infinitely often in both directions. Between consecutive cuts, the run lengths follow $2(1\,2)^k$ with $k \in \{1, 2\}$ ($k = 0$ creates three adjacent double runs; $k \geq 3$ contains $(010)^3$ or $(101)^3$). Therefore each block between cuts is $00100 = \mathrm{blk}_0(S)$ or $00100100 = \mathrm{blk}_0(L)$ with phase flipping across every cut. Because the cuts are locally determined by 0011 and 1100, the resulting block word $y_0 : \mathbb{Z} \to \mathcal{A}$ is recurrent, and periodicity of y_0 implies periodicity of x (`periodicTwo_of_isHImage`).

Step 2 (Currie–Rampersad Lemma 4 via 37 pure-kernel symbolic certificates). To descend past Level 0, we need Currie and Rampersad's Lemma 4 unconditionally for every suitable pair without invoking their Lemma 3.

Lemma 6 (Currie–Rampersad Lemma 4 for Every Suitable Pair). *For every suitable pair (l, m, p), every word $b \in B$ ($|B| = 37$), and every phase $\mathrm{ph} \in \{0, 1\}$, the binary word $H((l, m, p), \mathrm{ph}, b)$ contains a nonempty factor of the form xxx^R.*

Proof. Consider the free monoid $\mathcal{M}$ generated by the 12 symbolic atoms $\{a, a^R, \bar{a}, \overline{a^R} : a \in \{l, m, p\}\}$ equipped with the involutions R (reversal) and $\bar{\cdot}$ (complement). By the suitability identities $lm = m^R l^R$ and $ll^R = lm\overline{m^R}p$ together with palindromicity of ll^R, every suitable pair satisfies

$$\mathrm{blk}(S) = lm = m^R l^R,$$

$$\mathrm{blk}(L) = ll^R = lm\overline{m^R}p = p^R\overline{m}m^R l^R = m^R l^R \overline{m^R}p = p^R\overline{m}lm.$$

For each of the 37 words $b \in B$, we record a symbolic certificate (`SymCert`) choosing one valid representation per letter of b (complemented at odd indices), together with a start offset and symbolic length $L \geq 1$ such that the three consecutive slices P_1, P_2, P_3 of length L satisfy $P_1 = P_2$ and $P_3 = P_1^R$ in $\mathcal{M}$. For example, for $b = SSSSLL \in F_1$, the witness $x = m\overline{m^R l^R}l$ (at Level 0, $x = 0110110010$, positions $4 \ldots 33$ of the 36-bit image $H_0(SSSSLL)$) evaluates to xxx^R in every suitable pair simultaneously. Because l, m, p are nonempty, evaluating any valid symbolic certificate yields a nonempty binary word x with xxx^R a factor of $H((l, m, p), 0, b)$ (`checkSymCert_sound`). Both list equality of the 37 certified words with B (`lem4Certs_words_eq`) and validity of all 37 certificates (`lem4Certs_all_valid`) are checked inside Lean's pure kernel by `decide +kernel` with zero axioms. Phase $\mathrm{ph} = 1$ follows by bitwise complement invariance (`containsXXXR_comp`, `currie_rampersad_lemma4`). □

Step 3 (Three Φ-parsings of B-free recurrent block words). Let $y : \mathbb{Z} \to \mathcal{A}$ be recurrent and avoid all 37 words of B. We claim that either y is periodic (a shift of $S^{\mathbb{Z}}$ or $(SLL)^{\mathbb{Z}}$), or y avoids SS and is a θ-image of a recurrent word $y_3 : \mathbb{Z} \to \mathcal{A}$ (`three_phi_parsings`).

First parsing. In y, any finite maximal run S^t between two L's is followed by L, SS, or SL, producing a factor in F_2 when $t = 2$ and in F_1 when $t \geq 3$; similarly a left-infinite run

$\dots SSSSL$ produces a factor in F_1. A right-infinite run LS^ω cannot occur because recurrence would force LSS to recur, contradicting the absence of finite or left-infinite S-runs of length ≥ 2. Hence either $y = S^{\mathbb{Z}}$ (period 1) or y avoids SS. In the latter case, every L-run of length ≥ 3 is either infinite (containing $L^7 \in F_3$) or preceded by LS (producing a factor in F_4). Thus y avoids both SS and LLL, so cutting before every S parses y uniquely as a Φ-image of a recurrent word $y_1 : \mathbb{Z} \to \mathcal{A}$ (`phi_parse_of_no_SS_no_LLL`), where $w \in \mathrm{Fact}(y_1)$ if and only if $\Phi(w)S \in \mathrm{Fact}(y)$.

Second parsing. Family F_5 forbids SS in y_1. Any finite or left-infinite L-run of length ≥ 3 in y_1 is followed by S, then L (as y_1 avoids SS), then L, SS, or SL, giving a factor in F_6 after applying Φ. A right-infinite run SL^ω would leave $SLLL$ occurring only once, violating recurrence unless $y_1 = L^{\mathbb{Z}}$ (in which case $y = (SLL)^{\mathbb{Z}}$, period 3). Otherwise y_1 avoids SS and LLL, so $y_1 = \Phi(y_2)$ for recurrent y_2.

Third parsing. Families F_7, F_8, F_9 forbid LLL, LSS, and S^7 in y_2 through Φ^2. Every S-run of length ≥ 2 in y_2 is either preceded by L (giving LSS) or left-infinite (giving S^7), contradiction. Thus y_2 avoids SS and LLL, giving $y_2 = \Phi(y_3)$ for recurrent y_3. Since $\Phi^3(u)c = c\theta(u)$ with $c = SLSL$, shifting by 4 block positions converts $y = \Phi^3(y_3)$ into $y = \theta(y_3)$ (`isMorphImage_Phi_three_to_theta`).

Step 4 (Lemma 10: suitable-pair lifting). Put $L_{10} = LSLSLLSLLS$, so $\Sigma = L_{10}\,L\,SL$ and $\Lambda = L_{10}\,L\,L_{10}^R$. Given any suitable pair (l, m, p), define

$$\hat{l} = H((l,m,p), 0, L_{10})\, l,$$
$$\hat{m} = l^R\, H((l,m,p), 1, SL),$$
$$\hat{p} = \overline{l^R}\, H((l,m,p), 0, LSLSL)$$

(`nextPair`). Because $|\Sigma| = 13$ and $|\Lambda| = 21$ are odd palindromes and $\mathrm{blk}(S), \mathrm{blk}(L)$ are palindromes, direct concatenation algebra verifies that $(\hat{l}, \hat{m}, \hat{p})$ is a suitable pair whose blocks are $\mathrm{blk}(\hat{S}) = H((l,m,p), 0, \Sigma)$ and $\mathrm{blk}(\hat{L}) = H((l,m,p), 0, \Lambda)$ (`nextPair_blk`), and whose alternating-phase map satisfies

$$H(\mathrm{nextPair}(l,m,p), \mathrm{ph}, u) = H((l,m,p), \mathrm{ph}, \theta(u))$$

for every $u \in \mathcal{A}^*$ (`H_nextPair`). Iterating from `level0Pair` produces an infinite hierarchy $\mathrm{levelPair}(i)$ $(i \geq 0)$ whose short block satisfies $|\mathrm{blk}_i(S)| \geq 13 \cdot |\mathrm{blk}_{i-1}(S)| \geq 5 \cdot 2^i$ (`levelPair_blk_S_length_ge`).

Step 5 (Algebraic r-hierarchy and infinite descent). Let $\phi_\infty = \Phi^\omega(S) \in \mathcal{A}^{\mathbb{N}}$ (`phiInf`). Evaluating $\mu^2(\tau(a))$ and $\tau(\Phi(a))$ on $a \in \{S, L\}$ gives

$$\mu^2(\tau(S)) = \mu^2(010) = 01001010 = \tau(SL) = \tau(\Phi(S)),$$
$$\mu^2(\tau(L)) = \mu^2(01010) = 0100101001010$$
$$= \tau(SLL) = \tau(\Phi(L))$$

(`fibMorph2_tau_eq_tau_Phi`). Inductively $\tau(\Phi^k(S)) = \mu^{2k+2}(0)$, so every $\tau(\phi_\infty[0 \dots n-1])$ is a prefix of f. Applying the alternating run-length map to $\tau(S) = 010$ produces $00100 = \mathrm{blk}_0(S)$ and to $\tau(L) = 01010$ produces $00100100 = \mathrm{blk}_0(L)$ (both of odd length 3, 5 in f-letters, so phase flips after every block). Hence every $H(\mathrm{level0Pair}, 0, \phi_\infty[0 \dots n-1])$ is an exact prefix of r (`H_level0_prefix_phiInf_eq_roteFib`). Because $\Phi^3(u)c = c\theta(u)$ closes $\mathrm{Fact}(\phi_\infty)$ under θ (`phiInf_factor_theta`), induction on i via Step 4 shows that for every $i \geq 0$ and every $u \in \mathrm{Fact}(\phi_\infty)$, there exists a phase $\mathrm{ph} \in \{0,1\}$ with $H(\mathrm{levelPair}(i), \mathrm{ph}, u) \in \mathrm{Fact}(r)$ (`roteFib_contains_H_levelPair`)—with zero appeal to Walnut or Du et al. Theorem 37.

Completion of Theorems 2, 1, and 5. Let $x : \mathbb{Z} \to \{0,1\}$ be recurrent, aperiodic, and xxx^R-free. By Steps 1–4 and induction on i, for every $i \geq 0$ there exists a recurrent SS-free word $y_i : \mathbb{Z} \to \mathcal{A}$ such that x is a level-i H-image of y_i (`descent_level`). Given any finite factor $v \in \mathrm{Fact}(x)$, pick $i = |v|$; then $|v| \leq 5 \cdot 2^i \leq |\mathrm{blk}_i(S)| \leq |\mathrm{blk}_i(L)|$, so v fits inside two consecutive level-i blocks $H(\mathrm{levelPair}(i), \mathrm{ph}_x, [y_i(k), y_i(k+1)])$. Since y_i avoids SS, the pair $[y_i(k), y_i(k+1)]$ lies in $\{SL, LS, LL\}$, and all three pairs occur in $\Phi^2(S) = SLSLL$, hence in $\mathrm{Fact}(\phi_\infty)$ (`phiInf_has_non_SS_pair`). By Step 5, $H(\mathrm{levelPair}(i), \mathrm{ph}_r, [y_i(k), y_i(k+1)]) \in \mathrm{Fact}(r)$ for some phase ph_r; matching or complementing phases via complement closure of $\mathrm{Fact}(r)$ yields $v \in \mathrm{Fact}(r)$. Thus $\mathrm{Fact}(x) \subseteq \mathrm{Fact}(r)$. Conversely, for any $u \in \mathrm{Fact}(r)$, uniform recurrence of r gives $R \in \mathbb{N}$ such that every length-$(R + |u|)$ factor of r contains u; taking $w = x[0 \dots R + |u| - 1] \in \mathrm{Fact}(x) \subseteq \mathrm{Fact}(r)$ forces $u \in \mathrm{Fact}(w) \subseteq \mathrm{Fact}(x)$. Hence $\mathrm{Fact}(x) = \mathrm{Fact}(r)$ (`rote_fib_rigidity_two_sided`). Step 0 gives Theorem 1 (`rote_fib_rigidity_one_sided`), and Proposition 4 gives Theorem 5 (`rote_fib_rigidity_grf_b5`). □

Proof of Corollary 3. Let $w : \mathbb{N} \to \{0,1\}$ avoid xxx^R (without assuming recurrence). The collection L_0 of factors occurring at arbitrarily large indices in w is a nonempty, factorial, bi-extendable binary language (`IsSubshiftLang`). By Zorn's lemma applied to inclusion-ordered chains of subshift languages inside L_0, there exists a minimal subshift language $L_{\min} \subseteq L_0$, from which iterative two-sided extension constructs a recurrent two-sided word $x : \mathbb{Z} \to \{0,1\}$ satisfying $\mathrm{Fact}(x) \subseteq \mathrm{Fact}(w)$ (`exists_recurrentTwo_in_factors`). Because x inherits xxx^R-avoidance, Theorem 2 splits into two cases (`arbitrary_avoider_dichotomy`):

- If x has period $p \geq 1$, then w contains factors of length kp with period p for every $k \geq 1$, so $\mathrm{ce}(w) = +\infty \geq 2 + \varphi$.
- Otherwise $\mathrm{Fact}(r) = \mathrm{Fact}(x) \subseteq \mathrm{Fact}(w)$. In `RoteFibonacciWord.lean` (`roteFib_fib_power`, `roteFib_has_pow_approaching_two_add_phi`), we prove unconditionally from the Beatty closed form of r that for every $k \geq 0$ and $m = 6k+4$, r contains a factor of length $F_m + F_{m+2} - 1$ with period F_m (where $F_0 = 0, F_1 = 1$). Since $(F_m + F_{m+2} - 1)/F_m \to 1 + \varphi^2 = 2 + \varphi$, for every $\varepsilon > 0$ there exists a nonempty factor of w of exponent $> 2 + \varphi - \varepsilon$, whence $\mathrm{ce}(w) \geq 2 + \varphi$ in the extended reals (`arbitrary_avoider_critExpEReal_ge_two_add_phi`). □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Although every proved result in Sections 1–3 is verified unconditionally inside Lean 4's pure kernel with zero external certificates, we provide a standalone exact-arithmetic Python script (`recompute.py`, output in `recompute.out`) that

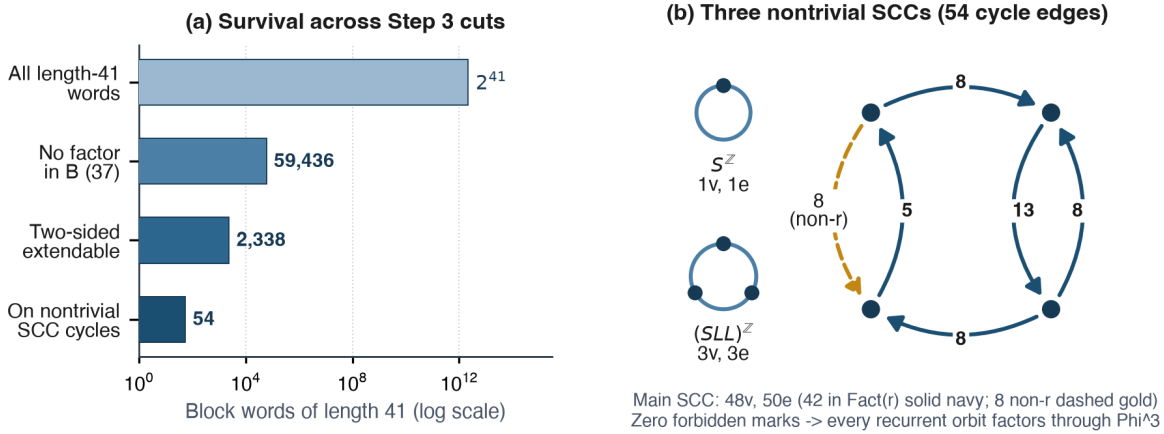


Figure 2. Independent finite-type graph exhaustion for Step 3 (`recompute.py`). Panel (a): survival of length-41 words over $\{S, L\}$ across successive cuts (2^{41} total → 59,436 B-free → 2,338 bi-extendable → 54 cycle edges). Panel (b): the three nontrivial strongly connected components—$S^{\mathbb{Z}}$, $(SLL)^{\mathbb{Z}}$, and the 48-vertex, 50-edge main component compressed to 4 branch vertices and 6 chains (42 edges in Fact(ϕ_∞) solid navy; 8 non-r edges dashed gold).

independently cross-checks all constructions, finite-type graphs, and numerical tables.

Table 1. Suitable-pair block lengths across Levels 0–3 verified by `recompute.py` ($|S_i| = F_{6i+5}$, $|L_i| = F_{6i+6}$).

Level i	$\lvert\mathrm{blk}_i(S)\rvert$	$\lvert\mathrm{blk}_i(L)\rvert$	$\lvert l_i\rvert$	$\lvert m_i\rvert$	$\lvert p_i\rvert$
0	5	8	4	1	2
1	89	144	72	17	38
2	1,597	2,584	1,292	305	682
3	28,657	46,368	23,184	5,473	12,238

Table 2. Record factor exponents n/p in the first 50,000 letters of r for periods $p \le 1{,}000$, climbing to $2 + \varphi \approx 3.618034$.

Period p	Longest length n	Ratio n/p	Fibonacci index
1	2	2.000000	F_2
3	10	3.333333	F_4
13	46	3.538462	F_7
55	198	3.600000	F_{10}
233	842	3.613734	F_{13}
987	3,570	3.617021	F_{16}

Specifically, `recompute.py` confirms:

1. *Three definitions of r and $g_{\mathrm{rf}}(b_5)$:* the alternating run-length transducer, the 4-letter substitution of Theorem 35(5) of [4], and the exact integer-square-root Beatty formula agree across 100,000 letters, match Du et al.'s printed 60-letter prefix, exhibit $p_r(n) = 2n$ for $1 \le n \le 15$, avoid 000, 111, 0101, 1010 across 100,000 letters and every xxx^R across 5,000 letters, and satisfy $g_{\mathrm{rf}}(b_5)[n+1] = \overline{r[n]}$ for $0 \le n < 10{,}000$.
2. *Suitable pairs and Lemma 4:* suitability identities hold explicitly through Level 3 (Table 1), and all 74 (word, phase) pairs of B contain xxx^R at Level 0 with shortest $|x| \in \{10, 13, 16, 42, 55, 68\}$.
3. *Finite-type graph censuses:* in Step 1 (binary length-15 words avoiding xxx^R for $|x| \le 5$), 210 words survive, 86 are bi-extendable, and exactly two nontrivial strongly connected components exist (2 vertices / 2 edges for $(01)^{\mathbb{Z}}$, and 30 vertices / 34 edges with zero forbidden marks). In Step 3 (length-41 $\{S, L\}$-words avoiding B; Figure 2), 59,436 of 2^{41} words are B-free, 2,338 are bi-extendable, and exactly three nontrivial strongly connected components appear: $S^{\mathbb{Z}}$ (1 vertex, 1 edge), $(SLL)^{\mathbb{Z}}$ (3 vertices, 3 edges), and one main component (48 vertices, 50 edges: 42 factors of ϕ_∞ and 8 non-r edges forming one chain, containing none of the six forbidden parsing marks).
4. *Critical-exponent convergence:* scanning periods $p \le 1{,}000$ over 50,000 letters of r reproduces the exact Fibonacci-period record ladder approaching $2 + \varphi \approx 3.618034$ (Table 2).

While the periodic branch of Steps 1–4 forces every periodic two-sided avoider into $(01)^{\mathbb{Z}}$ or a shift of $H(\mathrm{levelPair}(i), \mathrm{ph}, S^{\mathbb{Z}})$ or $H(\mathrm{levelPair}(i), \mathrm{ph}, (SLL)^{\mathbb{Z}})$ (periods $2, 2F_{6i+5}, 2F_{6i+8}$), establishing least-period minimality and exact correspondence with all conjugates of prefixes of r lies outside our current formalization, leaving Du et al.'s Conjecture 47 [4] (Conjecture 11 of [3]) explicitly unsettled.

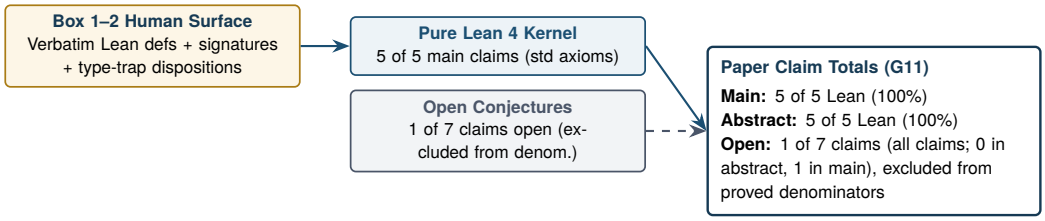
Conjecture 7 (Periodic Avoiders: Du et al. Conjecture 47 — Unsettled). *Every periodic infinite binary word avoiding xxx^R has least period 2 or $2F_{3k+2}$ ($k \ge 1$) and is a shift of $(01)^\omega$ or of $(r[0 \ldots 2F_{3k+2} - 1])^\omega$ (or its complement).*

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 2f09f46076a2 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Artifacts & Remaining Claims:** No external certificate artifacts are required; any claims marked Written or Unsettled rely on standard written proofs or remain explicitly unproved.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 306/306 public theorems (source-level approx) | WARN: Fidelity: 6/6 exact (draft cross-family)
WARN: Witnesses: pending gate replay | WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

RoteFibonacciReversal.φ (RoteFibonacciReversal/RoteFibonacciWord.lean:53-53) — Section 1 & 2 (golden ratio (1 + sqrt(5)) / 2)

```
noncomputable def φ : ℝ := (1 + Real.sqrt 5) / 2
```

RoteFibonacciReversal.roteFib (RoteFibonacciReversal/RoteFibonacciWord.lean:816-817) — Section 2 (Rote-Fibonacci word r[n] = (1 + floor((n + 2) / phi)) mod 2)

```
noncomputable def roteFib (n : ℕ) : Bool :=
  decide (⌊((n + 2 : ℕ) : ℝ) / φ⌋ % 2 = 0)
```

RoteFibonacciReversal.wGrf (RoteFibonacciReversal/RoteFibonacciWord.lean:1084-1088) — Section 1 & 2 (Badkobeh-Ochem binary projection g_rf(b_5))

```
def wGrf (n : ℕ) : Bool :=
  (applyMorph gRf (prefixOf b5 (n + 1)))[n]'(by
    have := gRfLen_ge (n + 1)
    unfold gRfLen at this
    omega)
```

RoteFibonacciReversal.ContainsXXXR (RoteFibonacciReversal/SuitablePairsAndLemma4.lean:248-249) — Section 2 (nonempty reversal-pattern factor x x x^R)

```
def ContainsXXXR (w : List Bool) : Prop :=
  ∃ pre x suf : List Bool, x ≠ [] ∧ w = pre ++ x ++ x ++ x.reverse ++ suf
```

RoteFibonacciReversal.AvoidsXXXROne (RoteFibonacciReversal/Level0AndPhiParsing.lean:129-130) — Section 2 (one-sided avoidance of nonempty x x x^R)

```
def AvoidsXXXROne (x : ℕ → Bool) : Prop :=
  ∀ w : List Bool, IsFactorOne x w → ¬ ContainsXXXR w
```

RoteFibonacciReversal.AvoidsXXXRTwo (RoteFibonacciReversal/Level0AndPhiParsing.lean:133-134) — Section 2 (two-sided avoidance of nonempty x x x^R)

```
def AvoidsXXXRTwo (x : ℤ → Bool) : Prop :=
  ∀ w : List Bool, IsFactorTwo x w → ¬ ContainsXXXR w
```

RoteFibonacciReversal.RecurrentOne (RoteFibonacciReversal/Level0AndPhiParsing.lean:115-117) — Section 2 (one-sided recurrence)

```
def RecurrentOne {α : Type*} (x : ℕ → α) : Prop :=
  ∀ w : List α, IsFactorOne x w →
    ∃ i j : ℕ, i < j ∧ sliceOne x i w.length = w ∧ sliceOne x j w.length = w
```

RoteFibonacciReversal.RecurrentTwo (RoteFibonacciReversal/Level0AndPhiParsing.lean:120-122) — Section 2 (two-sided recurrence)

```
def RecurrentTwo {α : Type*} (x : ℤ → α) : Prop :=
  ∀ w : List α, IsFactorTwo x w →
    ∃ i j : ℤ, i < j ∧ sliceTwo x i w.length = w ∧ sliceTwo x j w.length = w
```

RoteFibonacciReversal.PeriodicOne (RoteFibonacciReversal/RoteFibonacciWord.lean:359-360) — Section 2 (strict one-sided periodicity with period p >= 1)

```
def PeriodicOne {α : Type*} (x : ℕ → α) : Prop :=
  ∃ p : ℕ, 0 < p ∧ ∀ n : ℕ, x (n + p) = x n
```

RoteFibonacciReversal.PeriodicTwo (RoteFibonacciReversal/Level0AndPhiParsing.lean:125-126) — Section 2 (two-sided periodicity with period p >= 1)

```
def PeriodicTwo {α : Type*} (x : ℤ → α) : Prop :=
  ∃ p : ℕ, 0 < p ∧ ∀ n : ℤ, x (n + (p : ℤ)) = x n
```

RoteFibonacciReversal.HasSameFactorsOne (RoteFibonacciReversal/RoteFibonacciWord.lean:347-348) — Section 2 (equality of finite factor sets Fact(x) = Fact(y))

```
def HasSameFactorsOne {α : Type*} (x y : ℕ → α) : Prop :=
  ∀ w : List α, IsFactorOne x w ↔ IsFactorOne y w
```

RoteFibonacciReversal.critExpEReal (RoteFibonacciReversal/MainRigidity.lean:659-661) — Section 2 (extended-real critical exponent ce(w))

```
noncomputable def critExpEReal (w : ℕ → Bool) : EReal :=
  sSup { e : EReal | ∃ (u : List Bool) (p : ℕ), u ≠ [] ∧ 1 ≤ p ∧
    IsFactorOne w u ∧ IsPeriod u p ∧ e = (((u.length : ℝ) / (p : ℝ) : ℝ) : EReal) }
```

RoteFibonacciReversal.SuitablePair (RoteFibonacciReversal/SuitablePairsAndLemma4.lean:104-112) — Section 2 (Currie-Rampersad suitable pair (l, m, p))

```
structure SuitablePair where
  l : List Bool
  m : List Bool
  p : List Bool
  hl : l ≠ []
  hm : m ≠ []
  hp : p ≠ []
  hpal : l ++ m = m.reverse ++ l.reverse
  hdecomp : l ++ l.reverse = l ++ m ++ compWord m.reverse ++ p
```

RoteFibonacciReversal.H (RoteFibonacciReversal/SuitablePairsAndLemma4.lean:194-196) — Section 2 (alternating-phase block transducer H)

```
def H (sp : SuitablePair) : Bool → List SL → List Bool
  | _, [] => []
  | ph, a :: u => applyPhase ph (blk sp a) ++ H sp (!ph) u
```

Supporting Semantic Closure Index (non-Box-1 helpers): RoteFibonacciReversal.IsFactorOne (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.IsFactorTwo (RoteFibonacciReversal/Level0AndPhiParsing.lean), RoteFibonacciReversal.IsPeriod (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.sliceOne (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.sliceTwo (RoteFibonacciReversal/Level0AndPhiParsing.lean), RoteFibonacciReversal.prefixOf (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.compWord (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.applyMorph (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.iterMorph (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.Prolongable (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.fixedPoint (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.f5 (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.b5 (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.gRf (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.gRfLen (RoteFibonacciReversal/RoteFibonacciWord.lean), RoteFibonacciReversal.SL (RoteFibonacciReversal/SuitablePairsAndLemma4.lean), RoteFibonacciReversal.Phi (RoteFibonacciReversal/SuitablePairsAndLemma4.lean), RoteFibonacciReversal.blk (RoteFibonacciReversal/SuitablePairsAndLemma4.lean), RoteFibonacciReversal.applyPhase (RoteFibonacciReversal/SuitablePairsAndLemma4.lean), RoteFibonacciReversal.B_set (RoteFibonacciReversal/SuitablePairsAndLemma4.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

RoteFibonacciReversal.rote_fib_rigidity_one_sided (RoteFibonacciReversal/MainRigidity.lean:325-333) `src:b930046aedfe6e`

```
theorem rote_fib_rigidity_one_sided {x : ℕ → Bool} (hrec : RecurrentOne x) (hav :
  ↪ AvoidsXXXROne x) :
    PeriodicOne x ∨ HasSameFactorsOne x roteFib
```

Assumptions: Any one-sided recurrent binary word x : N -> Bool avoiding nonempty factors of the form x x x^R. Zero external hypotheses and zero Walnut dependence. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RoteFibonacciReversal.rote_fib_rigidity_two_sided (RoteFibonacciReversal/MainRigidity.lean:274-320) `src:f78affab9ffa80`

```
theorem rote_fib_rigidity_two_sided {x : ℤ → Bool} (hrec : RecurrentTwo x) (hav :
  ↪ AvoidsXXXRTwo x) :
    PeriodicTwo x ∨ ∀ u : List Bool, IsFactorTwo x u ↔ IsFactorOne roteFib u
```

Assumptions: Any two-sided recurrent binary word x : Z -> Bool avoiding nonempty factors of the form x x x^R. Zero external hypotheses and zero Walnut dependence. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RoteFibonacciReversal.arbitrary_avoider_critExpEReal_ge_two_add_phi (RoteFibonacciReversal/MainRigidity.lean:666-691) src:46c8cd49ed6580

```
theorem arbitrary_avoider_critExpEReal_ge_two_add_phi {w : ℕ → Bool} (hav :
  ↪ AvoidsXXXROne w) :
    ((2 + φ : ℝ) : EReal) ≤ critExpEReal w
```

Assumptions: Any one-sided infinite binary word w : N -> Bool avoiding nonempty x x x^R (no recurrence assumed). Proves ce(w) >= 2 + phi in EReal unconditionally. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RoteFibonacciReversal.wGrf_hasSameFactors_roteFib (RoteFibonacciReversal/RoteFibonacciWord.lean:1907-1911) src:c4e7bdbee05068

```
theorem wGrf_hasSameFactors_roteFib : HasSameFactorsOne wGrf roteFib
```

Assumptions: Unconditional identity of finite factor sets Fact(g_rf(b_5)) = Fact(r) proved from first principles over Mathlib. Applicability: Unconditional (closed statement; no outer premise witness needed).

RoteFibonacciReversal.rote_fib_rigidity_grf_b5 (RoteFibonacciReversal/MainRigidity.lean:338-344) src:b4395c3e133542

```
theorem rote_fib_rigidity_grf_b5 {x : ℕ → Bool} (hrec : RecurrentOne x) (hav :
  ↪ AvoidsXXXROne x) :
    PeriodicOne x ∨ HasSameFactorsOne x wGrf
```

Assumptions: Any one-sided recurrent binary word x : N -> Bool avoiding nonempty x x x^R; combines rote_fib_rigidity_one_sided with wGrf_hasSameFactors_roteFib. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

RoteFibonacciReversal.currie_rampersad_lemma4 (RoteFibonacciReversal/SuitablePairsAndLemma4.lean:640-650) src:58c1a8b025d482

```
theorem currie_rampersad_lemma4 (sp : SuitablePair) (b : List SL) (hb : b ∈ B_set)
  ↪ (ph : Bool) :
    ContainsXXXR (H sp ph b)
```

Assumptions: Any suitable pair sp = (l, m, p), any word b in B_set (37 words), and any phase ph in Bool. Verified unconditionally via 37 symbolic certificates using decide +kernel (0 axioms). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 3. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
`critExpEReal`	`real_div`	`(p : ℝ)`	intended_convention	The set comprehension explicitly requires 1 <= p, ensuring (p : R) >= 1 > 0 with zero division-by-zero pathology.
`roteFib`	`real_div`	`φ`	intended_convention	Constant denominator phi = (1 + sqrt(5)) / 2 > 1 > 0 (proved in phi_pos and one_lt_phi).
`roteFib`	`nat_mod`	`% 2`	intended_convention	Constant modulus 2 > 0 applied to nonnegative floor((n + 2) / phi) in Z; zero division-by-zero or sign ambiguity.
`φ`	`real_log_rpow`	`Real.sqrt`	intended_convention	Constant positive radicand 5 > 0 (sqrt5_pos and sqrt5_sq : (sqrt 5)^2 = 5).

[a]Covers the specified Lean 4 hazard set (`Nat` subtraction/division/mod, totalized real/rational division, `Real.log`/`rpow`, `Finset.range`, `Classical.choose`, local instances); no finite checklist certifies all semantics.

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Every one-sided recurrent binary word avoiding x x x^R is strictly periodic or has Fact(w) = Fact(r)	Lean Kernel	`RoteFibonacciReversal.rote_fib_rigidity_one_sided`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 v4.35.0 + Mathlib pure kernel + Box 1 definitions; zero external hypotheses, zero Walnut.
Thm. 2: Every two-sided recurrent binary word avoiding x x x^R is periodic or has Fact(w) = Fact(r)	Lean Kernel	`RoteFibonacciReversal.rote_fib_rigidity_two_sided`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 v4.35.0 + Mathlib pure kernel + Box 1 definitions; zero external hypotheses, zero Walnut.
Cor. 3: Every infinite binary word avoiding x x x^R satisfies ce(w) >= 2 + phi (Du et al. Conj. 46)	Lean Kernel	`RoteFibonacciReversal.arbitrary_avoider_critExpEReal_ge_two_add_phi`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 v4.35.0 + Mathlib pure kernel (Zorn minimal subshift + Fibonacci powers); no recurrence assumed.
Prop. 4: Badkobeh-Ochem binary projection satisfies Fact(g_rf(b_5)) = Fact(r)	Lean Kernel	`RoteFibonacciReversal.wGrf_hasSameFactors_roteFib`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 v4.35.0 + Mathlib pure kernel (9-pair delay relations + Kronecker density).

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 5: Every one-sided recurrent binary word avoiding x x x^R is strictly periodic or has Fact(w) = Fact(g_rf(b_5))	Lean Kernel	`RoteFibonacciReversal.rote_fib_rigidity_grf_b5`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 v4.35.0 + Mathlib pure kernel combining Thm. 1 and Prop. 4.
Lem. 6: For every suitable pair, every b in B (37 words), and either phase, H(b) contains nonempty x x x^R	Lean Kernel	`RoteFibonacciReversal.currie_rampersad_lemma4`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel via 37 symbolic certificates checked by decide +kernel (0 axioms); bypasses Currie-Rampersad Lemma 3.
Conj. 7: Every periodic binary x x x^R-avoider has least period 2 or 2*F_{3k+2} (Du et al. Conj. 47)	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled conjecture (least-period minimality outside formalization; excluded from G11 denominators).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: d6548d9e02c8 | Lean mode: committed_receipt | Closure SHA-256: 57ab14f8bd6f | Git anchor: 2f09f46076a2 | Axioms SHA-256: f4f72f2d86c0 | Fidelity SHA-256: c1b46ca12d86.

6. PRIOR WORK AND OPEN QUESTIONS

Du, Mousavi, Schaeffer, and Shallit [4] (with Rowland in [3]) introduced r, proved via Walnut that r avoids xxx^R (Theorem 37) with critical exponent $2+\varphi$ (Corollary 39) and complexity $2n$ (Theorem 43), constructed periodic avoiders of period $2F_{3k+2}$ (Theorem 48), and posed Conjectures 46–47 (Conjectures 10–11 of [3]). Currie and Rampersad [2] established polynomial growth for finite xxx^R-avoiders via suitable pairs, B, and Lemmas 4 and 10 (without infinite recurrent words). Badkobeh and Ochem [1] defined $g_{\mathrm{rf}}(b_5)$ and remarked in Section 5 that it appears to share Fact(r), noting their finite-forbidden-set method cannot prove this. Mercaş [5] studied binary patterns with reversals and cited Conjecture 46 as open; Rote [6] `[to check]` introduced $2n$-complexity words.

Four search logs passed `litsearch.py check` (exit 0, 126 queries across arXiv, OEIS `A273129/A171587`, zbMATH, Google Scholar, CORE, OpenAlex, Semantic Scholar, Math.SE, MathOverflow, and GitHub; `docs/confirmed-strong-and-novel/rote-fibonacci-pattern-with-reversal/searches.out`); no proof of Theorem 1 or Corollary 3 was found in the searches logged.

Prior-work caveats (`[to check]`). Four metadata-only texts remain unread: Shallit (Cambridge, 2022) `[to check]`, Rampersad (LNCS 10307, 2017) `[to check]`, Ochem–Rao–Rosenfeld (2018) `[to check]`, and Magera's thesis (2016) `[to check]`; publisher *Theoret. Comput. Sci.* copies, MathSciNet, HAL/CCSD, and Béaur's 2024 thesis `[to check]` were also not fetched. Open questions: (1) formalize least-period minimality for Conjecture 7; (2) classify other binary reversal patterns with factor-language rigidity [5].

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

Sources live in `papers/rote-fibonacci-pattern-with-reversal/draft1/` and `lean/RoteFibonacciReversal/` (Lean 4 `v4.35.0` + Mathlib). Running `python3 papers/rote-fibonacci-pattern-with-reversal/draft1/recompute.py` replays all tables and censuses; `lake build RoteFibonacciReversal` and `lake env lean RoteFibonacciReversal/Verify.lean` verify all 306 public declarations (zero `sorry`, zero `native_decide`, standard axioms `propext`, `Classical.choice`, `Quot.sound`). AI research agents built and directed by Tyler B. Hughes generated and checked the proof, Lean 4 formalization, code, figures, and draft. No human mathematician has yet refereed the proof.

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