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Working Draft

Real Point Sets with Few Lines Through Every Point: All $n \geq 8$ and Twelve Modulo-60 Families

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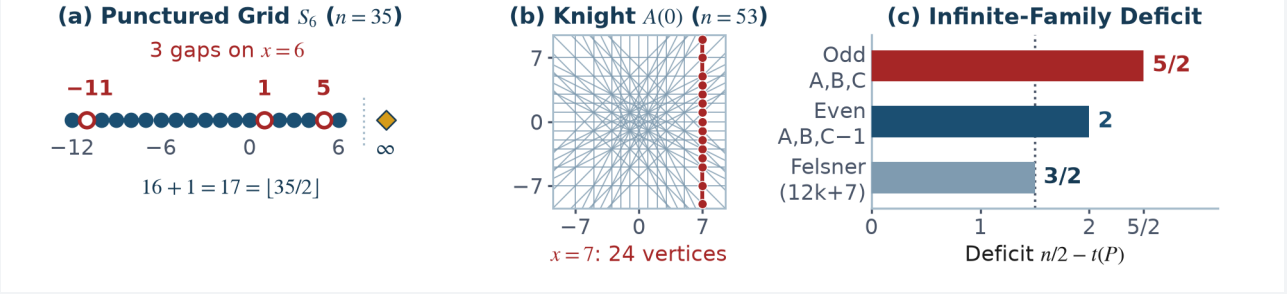
Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For every integer $n \geq 8$, we construct a noncollinear real n-point set with every point on at most $\lfloor n/2 \rfloor$ connecting lines, closing the remaining open class $n = 12j + 11$ ($j \geq 4$) of Akiyama, Ito, Kobayashi, and Nakamura via an explicit rational punctured triangular grid S_K attaining exact maximum $6j + 5$ ($j \geq 2$). A second eight-direction dual construction yields three infinite rational odd families ($n = 60s + 53$ and $60s + 69$ for $s \geq 0$; $60s + 61$ for $s \geq 1$) with exact maximum $(n - 5)/2 = G(n) - 2$ (deficit 5/2), where $G(n) = 2\lfloor n/4 \rfloor$ is Grünbaum's 2010 conjecture. Deleting a specified diagonal preserves the exact maximum at each adjacent even size, and three knight-deleted pairs attain exact $G(n) - 1$, refuting Grünbaum's conjecture across twelve classes modulo 60. All infinite upper bounds rest on finite interval-hull convexity certificates, and every proved result—all 12 modulo-60 rational maxima and all 36 hull certificates—is verified in the Lean 4 pure kernel. Whether an absolute constant bounds the straight-line deficit below $n/2$ remains open.

Verification Scope (G11): WARN: Human referee: pending
Main results: 7 of 7 in Lean Kernel (100%) · 1 open (not counted)
Abstract claims: 6 of 6 in Lean Kernel (100%) · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 52C30, 52C10, 05B35 **Keywords:** strong Dirac conjecture, real line arrangements, incidence geometry, Grünbaum conjecture, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Place n points in the real plane, not all on one line, and draw the straight line through every pair. Every triple of collinear points saves connecting lines; how small can the largest incidence count through a single point be? Write $t(P)$ for the maximum number of distinct connecting lines through any point of a finite noncollinear set $P \subset \mathbb{R}^2$, and set $t(n) = \min_{|P|=n} t(P)$. The shortfall (or deficit) $n/2 - t(P)$ measures how far the busiest point falls below half.

Two concrete questions stood open in the straight-line literature. First, Akiyama, Ito, Kobayashi, and Nakamura [1, 2] constructed noncollinear sets satisfying $t(P) \leq \lfloor n/2 \rfloor$ for every $n \geq 8$ except the progression $n = 12j + 11$ ($j \geq 4$), leaving that class open (as noted in subsequent surveys [6, 7, 3, 4]). Second, while sporadic small configurations attain deficit 7/2, the deepest infinite straight-line family in the sources consulted was Felsner's arrangement ($n = 6K + 7$, even K, deficit 3/2), and Grünbaum [5] conjectured that $t(n) = G(n) = 2\lfloor n/4 \rfloor$ for all sufficiently large n (a deficit of at most 3/2). Two explicit projective constructions resolve both questions unconditionally.

Theorem 1 (All-Size Half Bound and Rational Punctured Grid). *For every integer $n \geq 8$, there exists a noncollinear set $P \subset \mathbb{R}^2$ of n points satisfying $t(P) \leq \lfloor n/2 \rfloor$. Moreover, for every integer $j \geq 2$ ($n = 12j + 11$), an explicit rational noncollinear configuration $P \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ of $12j+11$ points attains the exact maximum $t(P) = 6j + 5 = (n - 1)/2$.*

In particular, every odd $n \geq 9$ admits $t(P) < \lceil n/2 \rceil$, closing the $12j + 11$ progression ($j \geq 4$) and providing a self-contained realization for every $n \geq 8$.

Theorem 2 (Twelve Modulo-60 Families Refuting Grünbaum's Conjecture). *Set* $G(n) = 2\lfloor n/4 \rfloor$.

1. *For every integer* $n \geq 68$ *with* $n \equiv 0, 1, 8, 9, 52, 53 \pmod{60}$, *there exists a rational noncollinear configuration* $P \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ *of* n *points attaining the exact maximum* $t(P) = G(n) - 2$.
2. *For every integer* $n \geq 124$ *with* $n \equiv 4, 5, 44, 45, 48, 49 \pmod{60}$, *there exists a rational noncollinear configuration* $P \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ *of* n *points attaining the exact maximum* $t(P) = G(n) - 1$.

For the smallest base family $A(s)$, the exact deficit-5/2 bound begins already at $s = 0$ ($n = 53$, $t(P) = 24 = G(53) - 2$), below the unified threshold $n \geq 68$ of Theorem 2.

Proposition 3 (Sharp-Start Deficit-5/2 Family A). *For every integer* $s \geq 0$, *there exists a rational noncollinear configuration* $P \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ *of* $n = 60s + 53$ *points attaining the exact maximum* $t(P) = 30s + 24 = (n - 5)/2$.

Table 1 lists the sharpest start range, odd and adjacent even sizes, and exact configuration maxima for all six pairs of eight-direction constructions (every row is proved in `lean/StrongDirac/ExactEqualities.lean`). For the top three odd rows the deficit is $n/2 - t(P) = 5/2$ (and 2 for the adjacent even deletions); for the lower three pairs the odd deficit is 3/2 and the even deficit is 1, beating $G(n)$ by 1 in every case. All twelve constructions provide upper bounds on the extremal function $t(n)$.

The standalone Python check below verifies the vertex count on the extremal dual line $x = 7$ of $A(0)$ ($n = 53$): each line $px + qy = c$ with $q \neq 0$ meets $x = 7$ at height $(c - 7p)/q$, yielding 23 distinct finite heights plus the point at infinity, so $r(x = 7) = 24 = (53 - 5)/2$. This snippet is executed verbatim by `recompute.py`.

Instant Witness Check (6 lines, verified verbatim by recompute.py)

```
from fractions import Fraction
def I(b): return range(-b, b + 1, 2)
bands = [(0, 1, 7), (0, 1, 2), (1, 1, 6), (1, -1, 6),
         (1, 2, 3), (1, -2, 3), (2, 1, 3), (2, -1, 3)]
heights = {Fraction(c - 7*p, q) for p, q, b in bands for c in I(b)}
assert len(heights) + 1 == 24
```

Every proved theorem, proposition, and lemma in this manuscript (Theorems 1–2, Proposition 3, and Lemmas 4–7) is formalized inside the pure Lean 4 kernel (`lean/StrongDirac/`, 231 audited declarations, Section 5). Figure 1 illustrates the smallest deficit-5/2 dual arrangement $A(0)$ and summarizes the proof pipeline.

2. PROJECTIVE DUALITY AND THE EIGHT-DIRECTION CONSTRUCTIONS

A nonzero triple $(x, y, z) \in \mathbb{Z}^3 \backslash \{0\}$ specifies a point $(x : y : z)$ in the real projective plane $\mathbb{RP}^2$. Two triples represent the same projective point if and only if their cross product vanishes, and three triples are collinear if and only if their 3×3 determinant $\det(p, q, r) = 0$. A finite set $X \subset \mathbb{Z}^3 \backslash \{0\}$ is *proper* when all its elements are nonzero and represent pairwise distinct projective points, and *noncollinear* when $\det(p, q, r) \neq 0$ for some $p, q, r \in X$. For $p \in X$, write $\mathrm{projLineCount}(X, p)$ for the number of distinct projective lines spanned by p and $X \backslash \{p\}$.

By planar projective duality, a triple $(p, q, -c)$ corresponds to the projective line $px + qy = c$ (with $(0, 0, 1)$ corresponding to the line at infinity ℓ_∞), and $\mathrm{projLineCount}(X, L)$ equals $r(L)$, the number of distinct intersection vertices of the arrangement $\mathcal{L}$ lying on L. Parallel affine lines share one vertex on ℓ_∞. Every finite proper integer projective set lifts back to a rational Euclidean configuration without changing any incidence count.

Lemma 4 (Rational Projective-to-Euclidean Embedding). *For every finite proper set* $X \subset \mathbb{Z}^3 \backslash \{0\}$, *there exists a map* $\varphi : \mathbb{Z}^3 \to \mathbb{R}^2$ *such that* $\varphi(p) \in \mathbb{Q}^2$ *for all* $p \in X$, φ *is injective on* X, $|\varphi(X)| = |X|$, *noncollinearity of* X *implies noncollinearity of* $\varphi(X)$ *in* $\mathbb{R}^2$, *and the Euclidean spanned line count satisfies* $t(\varphi(X), \varphi(p)) = \mathrm{projLineCount}(X, p)$ *for every* $p \in X$.

Proof. For each $p = (x, y, z) \in X$, the quadratic denominator $D(k, p) = x + ky + k^2 z$ is a nonzero polynomial of degree at most 2 in k, so it vanishes for at most 2 values of $k \in \{0, 1, \ldots, 2|X|\}$. By pigeonhole, some integer $k \in \{0, \ldots, 2|X|\}$ satisfies $D(k, p) \neq 0$ for every $p \in X$. Define $\varphi(p) = (y/D(k, p), z/D(k, p)) \in \mathbb{Q}^2$. Direct expansion of the 2×2 Euclidean signed area gives

$$\det\begin{pmatrix} \varphi(q) - \varphi(p) \\ \varphi(r) - \varphi(p) \end{pmatrix} = \frac{\det(p, q, r)}{D(k, p)\, D(k, q)\, D(k, r)},$$

which vanishes if and only if $\det(p, q, r) = 0$. Injectivity on X follows because equality $\varphi(p) = \varphi(q)$ forces $\mathrm{cross}(p, q) = 0$ and hence $p = q$ by properness. Therefore φ preserves cardinality, noncollinearity, and every point's exact spanned line count. □

On any line L of an n-line arrangement, let $m(v)$ be the number of arrangement lines through a vertex $v \in L$. Each other line intersects L once, yielding the double-counting identities

$$\sum_{v \in L} (m(v) - 1) = n - 1, \qquad \sum_{v \in L} (m(v) - 3) = n - 1 - 2r(L).$$

For $A(0)$ ($n = 53$), the 24 vertices on $x = 7$ have multiplicities $2, 3, 4, 12$ with frequencies $9, 10, 4, 1$ (Figure 1b). The 9 ordinary vertices contribute -9, while the 4 quadruple vertices and the twelve-fold vertex at infinity contribute $4 + 9 = 13$; the net excess $+4$ gives $r(x = 7) = (52 - 4)/2 = 24$.

The eight-direction knight-lattice arrangements. For any integer $d \geq 0$, write $I(d) = \{-d, -d + 2, \ldots, d\}$ for the arithmetic progression of step 2 and size $d + 1$. Given four nonnegative parameters (a, e, b, m) with $a \not\equiv e \pmod 2$, define the projective arrangement $\mathcal{K}(a, e, b, m)$ consisting of ℓ_∞ together with the lines $px + qy = c$ in eight parallel pencils:

$$\begin{aligned} \text{axes:} \quad & (p, q) \in \{(1, 0), (0, 1)\}, && c \in I(a) \cup I(e), \\ \text{diagonals:} \quad & (p, q) \in \{(1, 1), (1, -1)\}, && c \in I(b), \\ \text{knights:} \quad & (p, q) \in \{(1, \pm 2), (2, \pm 1)\}, && c \in I(m). \end{aligned}$$

Because $I(a)$ and $I(e)$ have opposite parities, they are disjoint, so $|\mathcal{K}(a, e, b, m)| = 2a + 2e + 2b + 4m + 11$. Family $A(s)$ takes $(a, e, b, m) = (9s + 7, 3s + 2, 8s + 6, 5s + 3)$ ($n = 60s + 53$), and Family $B(s)$ adds $(2, 0, 2, 2)$ to obtain $(9s + 9, 3s + 2, 8s + 8, 5s + 5)$ ($n = 60s + 69$). Let D_8 denote the eight second-outermost

Table 1. All six pairs of eight-direction rational constructions (twelve residue classes modulo 60) attaining exact maxima below Grünbaum's $G(n) = 2\lfloor n/4 \rfloor$. Every odd and adjacent even row is verified at its sharpest range in `lean/StrongDirac/ExactEqualities.lean`.

Family	Range	Odd n	Even $n-1$	Residue mod 60	Exact $t(P)$	Deficit (odd / even)
A	$s \geq 0$	$60s+53$	$60s+52$	52, 53	$30s+24 = G(n)-2$	5/2 / 2
B	$s \geq 0$	$60s+69$	$60s+68$	8, 9	$30s+32 = G(n)-2$	5/2 / 2
$C = B\backslash D_8$	$s \geq 1$	$60s+61$	$60s+60$	0, 1	$30s+28 = G(n)-2$	5/2 / 2
$B\backslash D_4$	$s \geq 1$	$60s+65$	$60s+64$	4, 5	$30s+31 = G(n)-1$	3/2 / 1
$A\backslash D_8$	$s \geq 2$	$60s+45$	$60s+44$	44, 45	$30s+21 = G(n)-1$	3/2 / 1
$A\backslash D_4$	$s \geq 2$	$60s+49$	$60s+48$	48, 49	$30s+23 = G(n)-1$	3/2 / 1

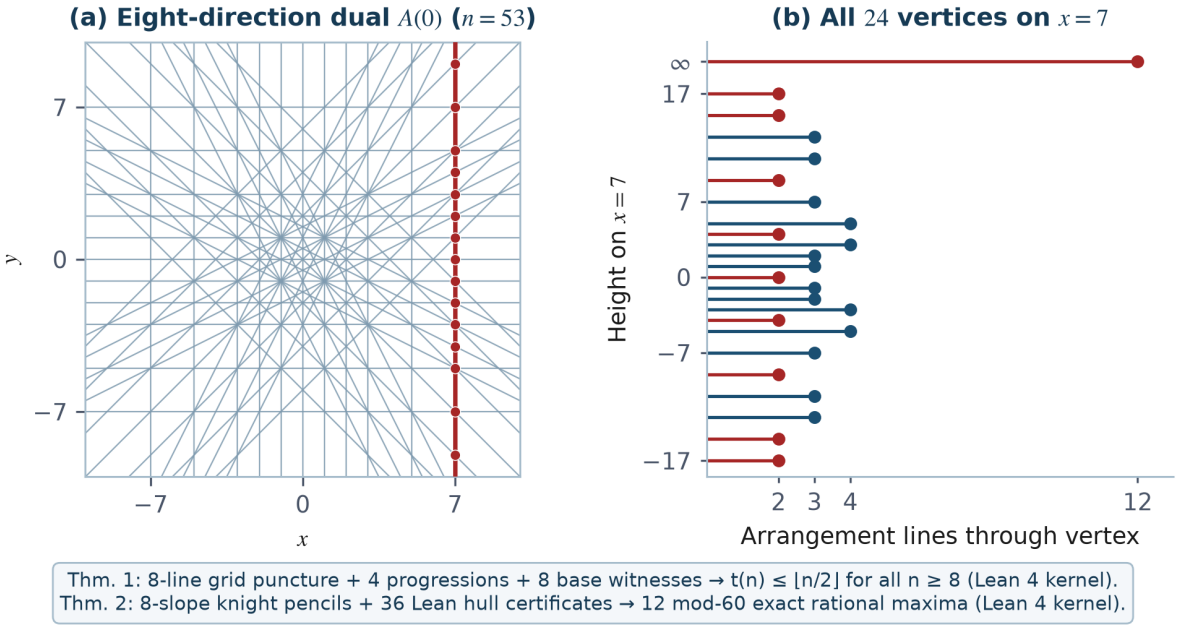


Figure 1. Smallest eight-direction witness $A(0)$ ($n = 53$, $t(P) = 24$, deficit 5/2). (a) The 52 finite dual lines in $[-10, 10]^2$ with the extremal vertical $x = 7$ highlighted in crimson and its visible vertices marked. (b) Exact incidence multiplicities for all 24 vertices on $x = 7$: 9 ordinary (multiplicity 2), 10 triple, 4 quadruple, and 1 twelve-fold vertex at infinity. Bottom: quantitative proof summary for Theorems 1 and 2, both formalized in the Lean 4 pure kernel.

knight lines with $c = \pm(m-2)$, and let $D_4 \subset D_8$ denote the four with normals $(1, \pm 2)$. Deleting D_8 or D_4 from $A(s)$ or $B(s)$ leaves both axis directions and ℓ_∞ intact and produces the four deleted odd rows of Table 1: $C = B\backslash D_8$, $B\backslash D_4$, $A\backslash D_8$, and $A\backslash D_4$.

3. PROOFS OF THE MAIN THEOREMS

The Punctured Triangular Grid and All $n \geq 8$. Let $j \geq 2$ and set $K = 2j+2 \geq 6$ (even). Starting from the triangular grid $x = i$, $y = i$, $x+y = i$ for $-K \leq i \leq K$, puncture $i = \pm(K-1)$ from all three pencils and $i = 1$ from the two axis pencils. Let $J_Z = [-K, K]\backslash\{\pm(K-1)\}$ and $J = J_Z\backslash\{1\}$. Adding the three medians $x-y = 0$, $x+2y = 0$, $2x+y = 0$ and the line at infinity ℓ_∞ defines S_K, of cardinality $2|J|+|J_Z|+4 = 2(2K-2)+(2K-1)+4 = 6K-1 = 12j+11$.

Lemma 5 (Punctured Triangular Grid S_K). *For every integer $j \geq 2$ ($K = 2j+2 \geq 6$), the punctured grid S_K yields an explicit rational noncollinear Euclidean configuration $P \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ of $12j+11$ points in which every point lies on at most $6j+5 = 3K-1$ connecting lines and some point (dual to $x = K$) attains exactly $6j+5$.*

Proof. By Lemma 4, it suffices to bound $r(L) \leq 3K-1$ for every $L \in S_K$ and prove $r(x = K) = 3K-1$. On a vertical line $x = i$ ($i \in J$), the finite vertex heights are $J \cup (J_Z - i) \cup \{i, -2i, -i/2\}$. On a diagonal $x+y = i$ ($i \in J_Z$), the finite abscissae are $J \cup (i-J) \cup \{i/2, 2i, -i\}$. In both cases every integral coordinate lies in an interval of $2K+1+|i|$ consecutive integers, with at most one additional half-integer (occurring only when i is odd) and one point at infinity. Thus $r(L) \leq 2K+1+|i|+2 \leq 3K-1$ whenever $|i| \leq K-4$.

At $|i| = K-3$ and $|i| = K-2$, the punctures remove both $\operatorname{sgn}(i)(K-1)$ from J and $-\operatorname{sgn}(i)(K-1)$ from J_Z, creating two integer holes inside the span; at $|i| = K-3$ the single possible half-integer adds at most 1 while at $|i| = K-2$ (even) all vertices are integral, giving at most $3K-1$ in both cases. At $|i| = K$, the span holds $3K+1$ integers (all coordinates integral since K is even) and omits three explicit holes:

	$i = K$	$i = -K$
$x = i$	$1,\ K-1,\ 1-2K$	$1,\ 1-K,\ 2K-1$
$x+y = i$	$1,\ 1-K,\ 2K-1$	$1,\ K-1,\ 1-2K$

leaving at most $(3K+1)-3+1 = 3K-1$ vertices. Horizontal lines follow by swapping x and y. The line at infinity has 6 parallel directions. On each median, scaling an affine coordinate (e.g., $2y$ on $x = y$ and $-2y$ on $x + 2y = 0$) embeds all finite vertices into $2J_Z \cup J_Z$; since every even element of J_Z already lies in $2J_Z$ and at most $K-2$ odd elements remain, each median has at most $(2K-1)+(K-2)+1 = 3K-2$ vertices.

For exact attainment on $x = K$ (Figure 2), the horizontals supply J ($2K-2$ heights), the lower diagonals supply $[-2K, -K-1]\backslash\{1-2K\}$ ($K-1$ heights), and the diagonal $x+y = 1$ supplies $y = 1-K$. These three integer sets are pairwise disjoint, and adding ℓ_∞ yields $(2K-2)+(K-1)+1+1 = 3K-1 = 6j+5$ distinct vertices. □

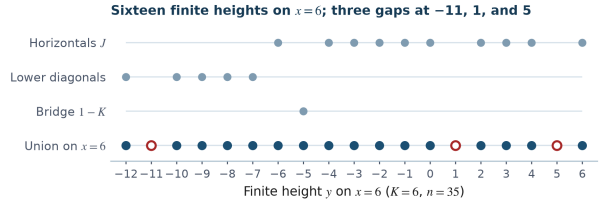


Figure 2. Complete height transversal on $x = 6$ in S_6 ($K = 6, n = 35$). The horizontals J, lower diagonals, and bridge $1-K$ cover 16 distinct finite heights in $[-12, 6]$, leaving the three punctured gaps $\{-11, 1, 5\}$ (open crimson circles). Including the point at infinity gives $17 = 3K-1 = (35-1)/2$.

Proof of Theorem 1. Deleting lines from an arrangement never raises $r(L)$ on any surviving line, and deleting a line meeting L at an ordinary vertex lowers $r(L)$ by 1. Pairing each construction below with its ℓ_∞-deletion (keeping two parallel lines and a transverse line so noncollinearity persists) covers every $n \geq 8$ outside eight small sizes:

Construction	Sizes	Range	Bound
Square grid + $y = \pm x$	$4u+4$, $4u+5$	$u \geq 1$	$2u+2$
Full triangular grid F_{2k}	$12k+6$, $12k+7$	$k \geq 1$	$6k+2$
F_{2k} minus 4 lines	$12k+2$, $12k+3$	$k \geq 3$	$6k+1$
Punctured grid S_K	$12j+10$, $12j+11$	$j \geq 2$	$6j+5$

For the square grid ($x = i$, $y = i$ for $|i| \leq u$, $y = \pm x$, ℓ_∞), every diagonal crossing with a grid line lies on another grid line, giving $r(L) \leq 2u+2$. For F_{2k} ($K = 2k$, unpunctured), the interval count in Lemma 5 gives $r(L) \leq 3K+1$ for $|i| \leq K-2$ and $r(L) \leq 3K+2$ at $|i| \in \{K-1, K\}$ and on the three medians. For $F_{2k}\backslash\{X, Y, Z, H\}$ ($k \geq 3$), removing $X\colon x = 3$, $Y\colon y = 3$, $Z\colon x+y = 3$, and $H\colon y = -3$ eliminates an ordinary vertex on every line that could attain $3K+2$: for $d \in \{K-1, K\}$, the partners $(x = d, Y)$, $(x = -d, H)$, $(y = d, X)$, $(y = -d, Z)$, $(x+y = d, H)$, $(x+y = -d, X)$, $(x = y, Z)$, $(x+2y = 0, X)$, and $(2x+y = 0, Y)$ each meet at a vertex whose unused grid coordinate either exceeds K in magnitude or is half-integral, ruling out every third arrangement line. Together with Lemma 5, these four pairs leave exactly eight sizes $n \in \{10, 11, 14, 15, 22, 23, 26, 27\}$, realized with exact maxima

$$(n, t(P)) \in \{(10,5),\ (11,5),\ (14,6),\ (15,6),\ (22,11),\ (23,10),\ (26,13),\ (27,13)\}$$

using triangular grids without medians ($n = 10, 22$), four- and five-line deletions of F_4 ($n = 27, 26$), golden-ratio configurations over $\mathbb{Z}[\varphi]$ ($n = 11, 14, 15$), and a two-point deletion of the 25-point octagon configuration over $\mathbb{Z}[\sqrt{2}]$ ($n = 23$), all verified in `lean/StrongDirac/TightnessAndMod12.lean`. □

Interval-Hull Convexity Certificates and Theorem 2. On a target line $L : x + q_0 y = c$ of a knight arrangement $\mathcal{K}(a, e, b, m)$, a non-parallel pencil $px + qy = k$ with determinant $D = q - q_0 p \neq 0$ meets L at $y = (k - pc)/D$. By $D_4 \times \{\pm 1\}$ symmetry (sign flips in y, z and (x, y)-swap), it suffices to bound $c \geq 0$ on the two axis bands ($q_0 = 0$), one diagonal ($q_0 = 1$), and one knight direction ($q_0 = 2$). Splitting s by parity writes every band radius linearly as $B_1 t + B_0$ with fixed parity $\rho \in \{0, 1\}$. Choosing $\Lambda \geq 1$ divisible by every $|D|$ and scaling $z = \Lambda y \in \mathbb{Z}$ turns each pencil into an arithmetic progression of step $\mu = 2\Lambda/|D|$ centered at $h_c c$ with radius $h_t t + h_0$, where $h_c = -\Lambda p/D$ and $(h_t, h_0) = \Lambda(B_1, B_0)/|D|$.

Set $M = 2\Lambda$. For each active residue $\sigma \pmod M$ hit by at least one pencil at offset c, let $L_\sigma(c, t)$ and $U_\sigma(c, t)$ be the smallest lower endpoint and largest upper endpoint among contributing pencils. The finite vertices on L in residue σ all lie in $[L_\sigma, U_\sigma]$, giving the interval-hull upper bound

$$H(c, t) = \sum_{\sigma \text{ active}} \left(\lfloor (U_\sigma - \sigma)/M \rfloor + \lfloor (\sigma - L_\sigma)/M \rfloor + 1 \right). \quad (1)$$

The Lean checker `certifyTarget` validates the divisibility header ($2 \mid Q_c$, $Q_c \mid vQ_t$, $M \mid h_c Q_c$, $M \mid h_t Q_t$) and checks, across every residue cell ($c_0 \bmod Q_c, t_0 \bmod Q_t$), that $H \leq N_1 t + N_0$ holds at both endpoints of the initial nonempty row and that both endpoint slopes along $(0, Q_t)$ and (vQ_t, Q_t) are at most $N_1 Q_t$.

Lemma 6 (Soundness of Interval-Hull Certificates). *If* `certifyTarget` *accepts* $(q_0, \Lambda, \mathcal{P}, \rho_0, v, w, Q_c, Q_t, N_1, N_0)$, *then for every* $t \in \mathbb{N}$, *every proper projective set* X *covered by the pencils* $\mathcal{P}$ *together with lines parallel to* $(1, q_0)$, *and every point* $(1, q_0, c) \in X$ *with* $0 \leq c \leq vt + w$ *and* $c \equiv \rho_0 \pmod 2$, *we have* $\mathrm{projLineCount}(X, (1, q_0, c)) \leq 1 + N_1 t + N_0$.

Proof. The header divisibilities freeze the set of pencils active in each residue $\sigma \pmod M$ throughout each cell ($c_0 \bmod Q_c, t_0 \bmod Q_t$) and ensure that every floor term in (1) shifts affinely along steps of size Q_c in c or Q_t in t. Because $H(c, t)$ sums maxima of affine functions in the offset index, H is convex across each row and attains its row maximum at the left or right cell endpoint. Along either boundary ray, the step increase of a maximum of affine functions is bounded by the maximum of the individual step increases, which `certifyTarget` bounds by $N_1 Q_t$. Induction over rows gives $H(c, t) \leq N_1 t + N_0$; adding 1 for the shared point at infinity proves the claim. □

Figure 3 depicts one representative cell from the Family A even-s diagonal certificate ($\Lambda = 6$, $M = Q_c = 12$, $Q_t = 3$, target $60t + 23$ finite heights). On $c \equiv 0 \pmod{12}$, $t \equiv 0 \pmod 3$, the active residues are $\{0, 2, 6, 10\}$, the base row $t = 0$ has $H(0, 0) = 13$, and the two boundary slope bounds are 98 and 180 (both $\leq 60 \cdot 3$). At $t = 3$, the five offsets $c \in$

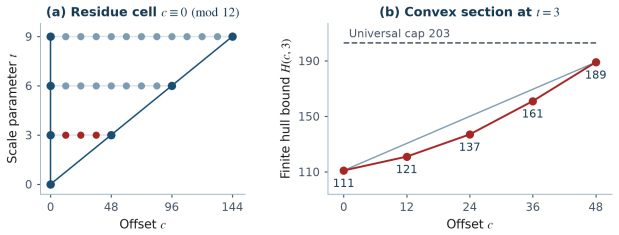


Figure 3. Representative residue cell $c \equiv 0 \pmod{12}$, $t \equiv 0 \pmod{3}$ for the Family A diagonal ($s = 2t$, $\Lambda = 6$). (a) Left and right cell boundaries advancing by $(0, 3)$ and $(48, 3)$. (b) Convex hull upper bound $H(c, 3) = [111, 121, 137, 161, 189]$ lying below its chord and below the universal cap 203.

$\{0, 12, 24, 36, 48\}$ give convex values $111, 121, 137, 161, 189$, all strictly below the universal cap 203.

To pass from each odd family to its adjacent even size while keeping the exact maximum intact, we use the following collinearity preservation lemma.

Lemma 7 (Collinear Deletion Preservation). *Let $X \subset \mathbb{Z}^3 \setminus \{0\}$ be a proper projective set and let $p, q, r \in X$ be pairwise distinct points with $\det(p, q, r) = 0$. Then* $\mathrm{projLineCount}(X \setminus \{q\}, p) = \mathrm{projLineCount}(X, p)$.

Proof. Erasing q cannot add spanned lines through p, so $\mathrm{projLineCount}(X \setminus \{q\}, p) \leq \mathrm{projLineCount}(X, p)$. Conversely, because $\det(p, q, r) = 0$ and p, q, r are projectively distinct, the line $\mathrm{projLine}(p, q)$ equals $\mathrm{projLine}(p, r)$, which remains spanned by the surviving point $r \in (X \setminus \{q\}) \setminus \{p\}$. □

Proof of Proposition 3 and Theorem 2. Step 1: Upper bounds via 36 kernel-checked hull certificates. For Families A and B, splitting $s = 2t$ and $s = 2t + 1$ across the four target bands yields 16 applications of Lemma 6 (2,984 cells total), proving $r(L) \leq 30s + 24$ on $A(s)$ and $r(L) \leq 30s + 32$ on $B(s)$ for all finite lines (while ℓ_∞ has 8 vertices). For the four deleted families ($C = B \setminus D_8$, $B \setminus D_4$, $A \setminus D_8$, $A \setminus D_4$), 20 shifted interior hull certificates (4,432 cells total) bound all interior offsets in B ($s \geq 1$) by $30s + 28$ (with $30s + 30$ at $|c| = e$ and $30s + 29$ at $|c| = m - 4$) and in A ($s \geq 2$) by $30s + 21$. On the remaining outer boundary lines, explicit ordinary-vertex deletions against D_8 or D_4 subtract enough distinct vertices to meet every row bound in Table 1: on $B \setminus D_8$, lines at $|c| \in \{a, b, m\}$ lose 4 ordinary vertices, $|c| = e$ loses 2, and $|c| = m - 4$ loses 1; on $A \setminus D_8$ ($s \geq 2$), each outer line loses at least 3; on $B \setminus D_4$ and $A \setminus D_4$, the maximal boundary lines and surviving second-outermost knights each lose at least 1. (At $s = 0$, $B(0) \setminus D_8$ has $n = 61$ and $t = 29$ rather than 28, so $s \geq 1$ is sharp.)

Step 2: Exact attainment for all six odd families. Write $\delta = 0$ for A and $\delta = 1$ for B. On the outer axis $x = a$, the horizontals supply $I(a)$ and $I(e)$, while the diagonals and knights supply disjoint step-2 intervals of both signs whose positive halves are:

Pencil	Positive height interval	Count/sign
$(1, \pm 1)$	$[a + 2,\ a + b]$	$4s + 3 + \delta$
$(2, \pm 1)$	$[a + b + 2,\ a + b + 6s + 4 + 2\delta]$	$3s + 2 + \delta$
$(1, \pm 2)$	$[e + 2,\ 7s + 4 + 2\delta]$	$2s + 1 + \delta$

Adding ℓ_∞ yields exactly $30s + 24 + 8\delta$ distinct vertices on $x = a$, establishing Proposition 3 (via Lemma 4) and exactness for A and B. On $C = B \setminus D_8$ ($s \geq 1$), exactly four lines of D_8 ($c = -(m-2)$) meet $x = a$ at ordinary vertices while the four with $c = m - 2$ hit existing vertices ($\pm(2s + 3) \in I(a) \cup I(e)$ and diagonals $-4s - 6 \in I(b)$), dropping $r(x = a)$ by exactly 4 to $30s + 28$.

For $B \setminus D_4$ ($s \geq 1$), $A \setminus D_8$ ($s \geq 2$), and $A \setminus D_4$ ($s \geq 2$), the outer knight line $L_m : x + 2y = m$ attains the exact maxima $30s + 31$, $30s + 21$, and $30s + 23$. Scaling heights by 60 via $z(q) = 60(q_z - mq_x)/(q_y - 2q_x)$ (every nonzero denominator $D \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$ divides 60), we exhibit explicit pairwise-disjoint witness subsets partitioned by residues modulo 60 (and modulo 120 on 0 (mod 60)) whose cardinalities sum to $30s + 30$, $30s + 20$, and $30s + 22$, respectively; adding ℓ_∞ completes the exact lower bounds (`familyBE4_knight_lineCount`, `familyAD8_knight_lineCount`, and `familyAD4_knight_lineCount` in `lean/StrongDirac/ExactEqualities.lean`).

Step 3: Adjacent even sizes via Lemma 7. In all six odd families, erase the outer diagonal $q = (1, 1, b)$ ($x + y = b$). In A, B, and C, q meets the maximizing axis $p = (1, 0, a)$ at height $y = b - a = -(s + 1) \in I(a)$, which lies on the surviving horizontal $r = (0, 1, -(s + 1))$. In $B \setminus D_4$, $A \setminus D_8$, and $A \setminus D_4$, q meets the maximizing knight $p = (1, 2, m)$ at height $y = m - b = -(3s + 3) \in I(a)$, which lies on the surviving horizontal $r = (0, 1, -(3s + 3))$. In both cases $\det(p, q, r) = 0$, so Lemma 7 preserves the exact maximum at size $n - 1$. Since every odd size satisfies $n \equiv 1 \pmod 4$, we have $G(n - 1) = G(n)$. Applying Lemma 4 to all twelve families yields rational Euclidean realizations covering every $n \geq 68$ in $\{0, 1, 8, 9, 52, 53\} \pmod{60}$ at $G(n) - 2$ and every $n \geq 124$ in $\{4, 5, 44, 45, 48, 49\} \pmod{60}$ at $G(n) - 1$. □

4. COMPUTATIONAL RECHECK AND DESIGN RAY

Independent Python replay. Although every theorem and lemma above is verified inside the Lean 4 kernel without external certificates, `recompute.py` provides a fast standalone standard-library audit that executes the Section 1 Python box verbatim, independently verifies all eight small exceptional configurations ($n \in \{10, 11, 14, 15, 22, 23, 26, 27\}$), all four grid progressions (S_K through $K = 18$), all twelve sharp-start knight rows through $s = 3$, all 36 hull certificates ($2{,}984 + 4{,}432 = 7{,}416$ cells, confirming that lowering the intercept by 1 on any of the four baseline outer-axis branches causes rejection), all boundary ordinary-vertex deletion tests, and the exact modulo-120 interval-union attainment proof on $x + 2y = m$ (period 12 in s, with permanent endpoint-order threshold $u_* \leq 1$).

Design ray $(9, 3, 8, 5)$ and leading-width chamber analysis. Why do Families A and B scale along $(a, e, b, m) \propto (9, 3, 8, 5)$? Consider positive leading width coefficients $0 < e < m < b < a$ with half-size rate $H_0 = a + e + b + 2m$ and integer parity phase $a_{\mathrm{int}} \equiv m_{\mathrm{int}} \not\equiv e_{\mathrm{int}} \pmod 2$, b_{int} even. Define the four linear forms

$$B_1 = 3a - 2e - 2b - m, \qquad B_2 = -a + 3e,$$
$$B_3 = -3a + 4b - m, \qquad B_4 = -15a - 10b + 43m,$$

and the three nonnegative interval-gap parameters $h = (a - 2e - m)_+$, $j = (a - b - m)_+$, $k = (b - 2m)_+$ (where $u_+ = \max(u, 0)$). Computing exact interval-union lengths in scaled coordinates ($\Lambda = 2, 2, 6, 60$) gives the leading boundary excesses $R - H_0$ on the four outer lines:

Outer line	Zero-gap rate	Gap correction
$x = a$	$B_1/2$	$-h/2 - j$
$x = e$	B_2	$+h$
$x + y = b$	$B_3/3$	$+j - k/3$
$x + 2y = m$	$B_4/60$	$+h/4 + j/2 + k/3$

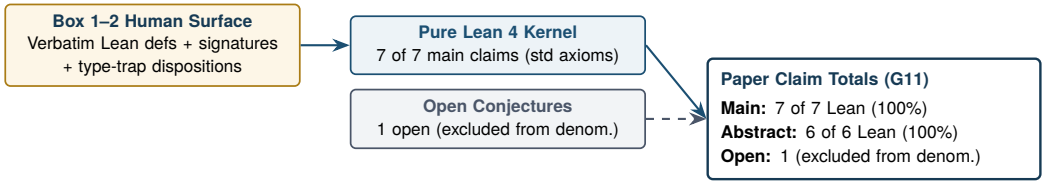
Because $27B_1 + 18B_2 + 16B_3 + B_4 \equiv 0$, weighting the four excesses by $(27, 9, 24, 30)$ gives $3h + 12j + 2k \geq 0$. Requiring $R \leq H_0$ on all four outer lines therefore forces $h = j = k = 0$ and $B_1 = B_2 = B_3 = B_4 = 0$, uniquely selecting the design ray $(a, e, b, m) = u(9, 3, 8, 5)$. As a separate explanatory check, `recompute.py` verifies this piecewise-linear excess table across all 154 chamber vertices ($18, 16, 20, 23$ per target in each parity phase) of the normalized tetrahedron $0 \leq e \leq m \leq b \leq a = 1$ (e.g., at $(41, 4, 20, 11)$, exact rates are $98, 80, 79, 459/5$, whereas dropping gap terms gives 119 on $x = a$). This ray calculation motivates the parameters of A and B and is never invoked by the formal proofs of Section 3.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 0533df5120e9 (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Artifacts & Remaining Claims:** No external certificate artifacts are required; any claims marked Written or Open rely on standard written proofs or remain explicitly unproved.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) · PASS: sorry/admit: 0 (per committed receipt) · PASS: native_decide: 0 (per committed receipt) · PASS: Custom axioms & meta: 0 (per committed receipt) · PASS: Coverage: 231/231 public theorems (source-level approx) · WARN: Fidelity: 7/7 exact (draft cross-family) · WARN: Witnesses: pending gate replay · WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

StrongDirac.Pt (StrongDirac/Projective.lean:34-34) — Section 2 (homogeneous integer coordinates in RP^2)

```
abbrev Pt : Type := ℤ × ℤ × ℤ
```

StrongDirac.Proper (StrongDirac/Projective.lean:57-58) — Section 2 (nonzero pairwise distinct projective points)

```
def Proper (X : Finset Pt) : Prop :=
  (∀ p ∈ X, Nonzero p) ∧ (∀ p ∈ X, ∀ q ∈ X, SamePt p q → p = q)
```

StrongDirac.NonCollinear (StrongDirac/Projective.lean:61-62) — Section 2 (projective noncollinearity via 3x3 det)

```
def NonCollinear (X : Finset Pt) : Prop :=
  ∃ p ∈ X, ∃ q ∈ X, ∃ r ∈ X, det p q r ≠ 0
```

StrongDirac.projLineCount (StrongDirac/Projective.lean:73-74) — Section 2 (exact spanned projective line count r(L))

```
noncomputable def projLineCount (X : Finset Pt) (p : Pt) : ℕ :=
  (spannedProjLines X p).card
```

StrongDirac.CollinearR2 (StrongDirac/Projective.lean:419-420) — Section 2 (Euclidean 2x2 signed-area collinearity)

```
def CollinearR2 (a b c : ℝ × ℝ) : Prop :=
  (b.1 - a.1) * (c.2 - a.2) - (b.2 - a.2) * (c.1 - a.1) = 0
```

StrongDirac.NonCollinearR2 (StrongDirac/Projective.lean:423-424) — Section 1 & 2 (noncollinear finite planar point set)

```
def NonCollinearR2 (P : Finset (ℝ × ℝ)) : Prop :=
  ∃ a ∈ P, ∃ b ∈ P, ∃ c ∈ P, ¬ CollinearR2 a b c
```

StrongDirac.lineCountR2 (StrongDirac/Projective.lean:435-436) — Section 1 & 2 (exact spanned Euclidean line count t(P, a))

```
noncomputable def lineCountR2 (P : Finset (ℝ × ℝ)) (a : ℝ × ℝ) : ℕ :=
  (spannedLinesR2 P a).card
```

StrongDirac.ExactEqualities.IsRationalSet (StrongDirac/ExactEqualities.lean:66-67) — Section 1 & 2 (rational point set P subset Q^2 subset R^2)

```
def IsRationalSet (P : Finset (ℝ × ℝ)) : Prop :=
  ∀ a ∈ P, IsRationalPt a
```

StrongDirac.R17.grunbaumVal (StrongDirac/R17_2_1.lean:1176-1176) — Section 1 (Grunbaum asymptotic formula G(n) = 2*floor(n/4))

```
def grunbaumVal (n : ℕ) : ℕ := 2 * (n / 4)
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): StrongDirac.R17.cert_b0_Xa (certifyTarget 0 2 (nonParallel 0 branch0Pencils) 1 18 7 4 2 60 23 = true); StrongDirac.R17.cert_b0_S (certifyTarget 1 6 (nonParallel 1 branch0Pencils) 0 16 6 12 3 60 23 = true); StrongDirac.TightnessAndMod12.cert_x11 (checkQSet 1 1 x11Pts 5 = true); StrongDirac.TightnessAndMod12.cert_o23 (checkQSet 0 2 o23Pts 10 = true).

Supporting Semantic Closure Index (non-Box-1 helpers): StrongDirac.Nonzero (StrongDirac/Projective.lean), StrongDirac.SamePt (StrongDirac/Projective.lean), StrongDirac.det (StrongDirac/Projective.lean), StrongDirac.projLine (StrongDirac/Projective.lean), StrongDirac.spannedProjLines (StrongDirac/Projective.lean), StrongDirac.lineR2 (StrongDirac/Projective.lean), StrongDirac.spannedLinesR2 (StrongDirac/Projective.lean), StrongDirac.ExactEqualities.IsRationalPt (StrongDirac/ExactEqualities.lean), StrongDirac.HullCert.Pencil (StrongDirac/HullCert.lean), StrongDirac.HullCert.Pencil.Mem (StrongDirac/HullCert.lean),

StrongDirac.HullCert.AffPencil (StrongDirac/HullCert.lean), StrongDirac.HullCert.Pencil.toAff (StrongDirac/HullCert.lean), StrongDirac.HullCert.AffPencil.isActive (StrongDirac/HullCert.lean), StrongDirac.HullCert.listMax (StrongDirac/HullCert.lean), StrongDirac.HullCert.candSigmas (StrongDirac/HullCert.lean), StrongDirac.HullCert.activeList (StrongDirac/HullCert.lean), StrongDirac.HullCert.hiTerm (StrongDirac/HullCert.lean), StrongDirac.HullCert.loTerm (StrongDirac/HullCert.lean), StrongDirac.HullCert.hullSum (StrongDirac/HullCert.lean), StrongDirac.HullCert.slopeSum (StrongDirac/HullCert.lean), StrongDirac.HullCert.firstStep (StrongDirac/HullCert.lean), StrongDirac.HullCert.checkC0 (StrongDirac/HullCert.lean), StrongDirac.HullCert.certifyHeader (StrongDirac/HullCert.lean), StrongDirac.HullCert.certifyTarget (StrongDirac/HullCert.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

StrongDirac.ExactEqualities.thm_half (StrongDirac/ExactEqualities.lean:205-216) `src:0bfeca5a4be58f`

```
theorem thm_half :
    (∀ n : ℕ, 8 ≤ n →
      ∃ P : Finset (ℝ × ℝ),
        P.card = n ∧ NonCollinearR2 P ∧ ∀ a ∈ P, lineCountR2 P a ≤ n / 2) ∧
    (∀ j : ℕ, 2 ≤ j →
      ∃ P : Finset (ℝ × ℝ),
        IsRationalSet P ∧
        P.card = 12 * j + 11 ∧
        NonCollinearR2 P ∧
        (∀ a ∈ P, lineCountR2 P a ≤ 6 * j + 5) ∧
        (∃ a ∈ P, lineCountR2 P a = 6 * j + 5))
```

Assumptions: None beyond Box 1 definitions (all mod-12 classes, 8 base witnesses, rational S_K). Applicability: Unconditional (closed statement; no outer premise witness needed).

StrongDirac.ExactEqualities.thm_knights (StrongDirac/ExactEqualities.lean:1428-1477) `src:03b0f7f96f7f54`

```
theorem thm_knights :
    (∀ n : ℕ, 68 ≤ n →
      (n % 60 = 0 ∨ n % 60 = 1 ∨ n % 60 = 8 ∨ n % 60 = 9 ∨ n % 60 = 52 ∨ n % 60 = 53)
        ↪ →
      ∃ P : Finset (ℝ × ℝ),
        IsRationalSet P ∧ P.card = n ∧ NonCollinearR2 P ∧
        (∀ a ∈ P, lineCountR2 P a ≤ grunbaumVal n - 2) ∧
        (∃ a ∈ P, lineCountR2 P a = grunbaumVal n - 2)) ∧
    (∀ n : ℕ, 124 ≤ n →
      (n % 60 = 4 ∨ n % 60 = 5 ∨ n % 60 = 44 ∨ n % 60 = 45 ∨ n % 60 = 48 ∨ n % 60 =
        ↪ 49) →
      ∃ P : Finset (ℝ × ℝ),
        IsRationalSet P ∧ P.card = n ∧ NonCollinearR2 P ∧
        (∀ a ∈ P, lineCountR2 P a ≤ grunbaumVal n - 1) ∧
        (∃ a ∈ P, lineCountR2 P a = grunbaumVal n - 1))
```

Assumptions: None beyond Box 1 definitions (36 hull certificates, algebraic lower bounds, rational lift). Applicability: Unconditional (closed statement; no outer premise witness needed).

StrongDirac.ExactEqualities.familyA_odd_rational_exact (StrongDirac/ExactEqualities.lean:1142-1152) `src:3a5e980e0dc6ad`

```
theorem familyA_odd_rational_exact (s : ℕ) :
    ∃ P : Finset (ℝ × ℝ),
      IsRationalSet P ∧ P.card = 60 * s + 53 ∧ NonCollinearR2 P ∧
      (∀ a ∈ P, lineCountR2 P a ≤ 30 * s + 24) ∧
      (∃ a ∈ P, lineCountR2 P a = 30 * s + 24)
```

Assumptions: Any integer s >= 0 (starts at n = 53, exact maximum 24). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

StrongDirac.ExactEqualities.exists_rational_euclidean_embedding (StrongDirac/ExactEqualities.lean:138-158) `src:fd1ab3e63e6d34`

```
theorem exists_rational_euclidean_embedding (X : Finset Pt) (hX : Proper X) :
    ∃ φ : Pt → ℝ × ℝ,
      (∀ p ∈ X, IsRationalPt (φ p)) ∧
      Set.InjOn φ X ∧
      (X.image φ).card = X.card ∧
      (NonCollinear X → NonCollinearR2 (X.image φ)) ∧
      (∀ p ∈ X, lineCountR2 (X.image φ) (φ p) = projLineCount X p)
```

Assumptions: Any finite Proper integer projective set X. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

StrongDirac.ExactEqualities.r15_2_1_rational_tight (StrongDirac/ExactEqualities.lean:187-198) src:abcd06251f0182

```
theorem r15_2_1_rational_tight {j : ℕ} (hj : 2 ≤ j) :
    ∃ P : Finset (ℝ × ℝ),
      IsRationalSet P ∧
      P.card = 12 * j + 11 ∧
      NonCollinearR2 P ∧
      (∀ a ∈ P, lineCountR2 P a ≤ 6 * j + 5) ∧
      (∃ a ∈ P, lineCountR2 P a = 6 * j + 5)
```

Assumptions: Any integer j >= 2 (K = 2j + 2 >= 6, n = 12j + 11). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

StrongDirac.HullCert.projLineCount_le_of_certifyTarget (StrongDirac/HullCert.lean:920-1126) src:e4c1dade564f28

```
theorem projLineCount_le_of_certifyTarget
    {q0 : ℤ} {Lam : ℕ} {pencils : List Pencil}
    {rho0 TB1 TB0 Qc Qt N1 N0 : ℕ}
    (hcert : certifyTarget q0 Lam pencils rho0 TB1 TB0 Qc Qt N1 N0 = true)
    {t : ℕ} {X : Finset Pt} (hX : Proper X)
    (hcover : ∀ x ∈ X, x.2.1 - q0 * x.1 = 0 ∨ ∃ P ∈ pencils, P.Mem t x)
    {c : ℤ} (hp : (1, q0, c) ∈ X)
    (hc0 : 0 ≤ c) (hc_hi : c ≤ ((TB1 * t + TB0 : ℕ) : ℤ))
    (hc_par : c % 2 = (rho0 : ℤ)) :
    projLineCount X (1, q0, c) ≤ 1 + N1 * t + N0
```

Assumptions: Passing certifyTarget, Proper X covered by pencils, base point (1, q0, c) in X with 0 <= c <= TB1*t + TB0 and c mod 2 = rho0. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

StrongDirac.ExactEqualities.projLineCount_erase_eq_of_collinear (StrongDirac/ExactEqualities.lean:222-239) src:b9bf8a17796b5b

```
theorem projLineCount_erase_eq_of_collinear {X : Finset Pt} (hX : Proper X)
    {p q r : Pt} (hp : p ∈ X) (hq : q ∈ X) (hr : r ∈ X)
    (hqp : q ≠ p) (hrp : r ≠ p) (hrq : r ≠ q) (hdet : det p q r = 0) :
    projLineCount (X.erase q) p = projLineCount X p
```

Assumptions: Proper X, pairwise distinct p, q, r in X, and det p q r = 0. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
thm_half	nat_div	/ 2	intended_convention	Nat division n / 2 is literally floor(n/2).
thm_knights	nat_mod	% 60	intended_convention	Standard Nat residue mod 60.
thm_knights	nat_sub	- 2	intended_convention	Hypothesis 68 <= n forces G(n) >= 34 > 2 (zero Nat underflow).
thm_knights	nat_sub	- 1	intended_convention	Hypothesis 124 <= n forces G(n) >= 62 > 1 (zero Nat underflow).
grunbaumVal	nat_div	/ 4	intended_convention	Nat division n / 4 is literally floor(n/4).
isActive	nat_mod	% (ap.mu : ℤ)	intended_convention	Signed Int modulo by positive step mu = 2*Lam/|D|.
Mem	nat_mod	% 2	intended_convention	Signed Int parity residue mod 2.
toAff	nat_div	/ D.natAbs	intended_convention	Header checks D != 0 and Lam % D = 0 so division is exact.
candSigmas	nat_mod	% (ap.mu : ℤ)	intended_convention	Signed Int residue mod positive step ap.mu.
candSigmas	nat_div	/ ap.mu	intended_convention	Exact quotient M / ap.mu = |D| for valid header.
certifyHeader	nat_div	/ Qc	intended_convention	Header enforces Qc > 0 and Qc divides TB1*Qt.
certifyHeader	nat_mod	% 2	intended_convention	Checks evenness of Qc and parity of rho0.
certifyHeader	nat_mod	% Qc	intended_convention	Divisibility guard (TB1 * Qt) % Qc = 0.
certifyHeader	nat_sub	- q0	intended_convention	Signed Int subtraction in Z (no Nat underflow).
certifyHeader	nat_mod	% D	intended_convention	Short-circuit header guard checks D != 0 first.
certifyHeader	nat_mod	% (M : ℤ)	intended_convention	Divisibility guard for scaled periods mod M.
certifyTarget	nat_div	/ Qc	intended_convention	Exact cell step ratio after header check.
certifyTarget	nat_div	/ 2	intended_convention	Qc % 2 = 0 checked in header so Qc / 2 is exact.
checkC0	nat_div	/ (Qc : ℤ)	intended_convention	Euclidean floor division by positive period Qc.
firstStep	nat_sub	- 1	intended_convention	Evaluated in branch c0 > A0 with A1 > 0 (no underflow).
firstStep	nat_div	/ A1	intended_convention	Ceiling division by positive step A1.

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
hiTerm	nat_div	/ (M : ℤ)	intended_convention	Euclidean floor division by positive M in Eq. (1).
loTerm	nat_sub	- ap.ht	intended_convention	Signed Int subtraction in Z.
loTerm	nat_sub	- ap.h0	intended_convention	Signed Int subtraction in Z.
loTerm	nat_div	/ (M : ℤ)	intended_convention	Euclidean floor division by positive M in Eq. (1).
projLineCount_le_of_certifyTarget	nat_sub	- q0	intended_convention	Signed Int subtraction in Z.
projLineCount_le_of_certifyTarget	nat_mod	% 2	intended_convention	Signed Int parity residue mod 2.
slopeSum	nat_div	/ (M : ℤ)	intended_convention	Exact Int division by positive M.
slopeSum	nat_sub	- ap.lc	intended_convention	Signed Int subtraction in Z.

[a]Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: All n >= 8 satisfy t(n) <= floor(n/2); rational S_K gives exact t(P) = 6j+5 for n = 12j+11 (j >= 2)	Lean Kernel	StrongDirac.ExactEqualities.thm_half	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (all 12 mod-12 classes, 8 small witnesses, and rational punctured grid S_K).
Thm. 2: Rational exact maxima G(n)-2 (6 classes mod 60, n >= 68) and G(n)-1 (6 classes mod 60, n >= 124)	Lean Kernel	StrongDirac.ExactEqualities.thm_knights	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs only (36 hull certificates, ordinary-vertex deletions, algebraic lower bounds, and rational lift).
Prop. 3: Sharp-start Family A(s) (s >= 0, n = 60s+53) attains exact rational t(P) = 30s+24 (deficit 5/2)	Lean Kernel	StrongDirac.ExactEqualities.familyA_odd_rational_exact	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (any integer s >= 0).
Lem. 4: Every finite Proper integer projective set embeds into Q^2 subset R^2 preserving exact line counts	Lean Kernel	StrongDirac.ExactEqualities.exists_rational_euclidean_embedding	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (any finite Proper set X in Z^3 \ {0}).
Lem. 5: Punctured triangular grid S_K (j >= 2, n = 12j+11) attains exact rational maximum 6j+5	Lean Kernel	StrongDirac.ExactEqualities.r15_2_1_rational_tight	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (any integer j >= 2, K = 2j + 2).
Lem. 6: Passing finite interval-hull certificate certifyTarget proves projLineCount <= 1 + N1*t + N0	Lean Kernel	StrongDirac.HullCert.projLineCount_le_of_certifyTarget	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (passing certifyTarget, Proper X, pencil cover, 0 <= c <= TB1*t + TB0, c mod 2 = rho0).
Lem. 7: Erasing q collinear with p and surviving r preserves exact projLine Count at p	Lean Kernel	StrongDirac.ExactEqualities.projLineCount_erase_eq_of_collinear	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs (Proper X, pairwise distinct p, q, r in X, det p q r = 0).
Conj. 8: Bounded-deficit Strong Dirac Conjecture t(n) >= n/2 - c for some absolute c > 0	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open conjecture (excluded from G11 denominators; Theorem 2 forces c >= 5/2).

[a]Allowed pure-kernel axioms: propext, Classical.choice, Quot.sound. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 586c679d66de | Lean mode: committed_receipt | Closure SHA-256: f2e7f1af71b4 | Git anchor: 0533df5120e9 | Axioms SHA-256: 886d13415a85 | Fidelity SHA-256: af7ff13c84fe. External validation receipts: Pinned commit | Palomar record.

6. PRIOR WORK AND OPEN QUESTIONS

Akiyama, Ito, Kobayashi, and Nakamura [1, 2] posed the all-size $\lfloor n/2 \rfloor$ problem; their two-page conference paper [1] (read in full) presents the earlier constructions, and the journal abstract [2] identifies the remaining open progression $n = 12k+11$ ($k \geq 4$) (paywalled pp. 323–326 of [2] remain unread [to check]). Grünbaum [5] (full eight-page author manuscript read) discusses Felsner's even-parameter family ($n = 12k + 7$, $t \leq 6k + 2$, deficit 3/2) and states his eventual twelve-residue

conjecture $G(n) = 2\lfloor n/4 \rfloor$. Lund, Purdy, and Smith [6] (read in full) construct $18j+7$ pseudolines with at most $8j+2$ vertices per pseudoline (unbounded deficit $j + 3/2$) while noting that no infinite straight-line family with deficit exceeding $3/2$ was known. Subsequent surveys and papers by Trok [7], Dumitrescu [3], and Florek [4] (all read in full via arXiv preprints) confirm that $n = 12k + 11$ ($k \geq 4$) and straight-line infinite families past deficit $3/2$ remained open in the open literature consulted; Florek [4] proves $t(P) \geq \lceil n/2 \rceil$ when all points lie on three concurrent lines.

Logged prior-work searches across arXiv, CORE, and Google Scholar found neither the eight-line punctured grid S_K nor an infinite straight-line family with deficit exceeding $3/2$. Print monographs and paywalled pages 323–326 of [2] [to check] remain unsearched, so no exhaustive priority claim is made.

Conjecture 8 (Bounded-Deficit Strong Dirac Conjecture [1, 6]). *There exists an absolute constant $c > 0$ such that every noncollinear set $P \subset \mathbb{R}^2$ of $n \geq 3$ points satisfies $t(P) \geq n/2 - c$.*

Theorem 2 and Proposition 3 force any such constant to satisfy $c \geq 5/2$. Two concrete open questions follow immediately: can an infinite straight-line family attain deficit strictly larger than $5/2$ (matching or exceeding the sporadic $7/2$ arrangements), and can straight-line deficits grow without bound?

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure scripts, and ledgers live in `papers/strong-dirac/draft4/`, and the 231-declaration Lean 4 formalization lives in `lean/StrongDirac/` (published at `mathdiscoveries/strong-dirac` and registered in the Palomar Registry as `PALOMAR-2026-11-22-000027` v2, which performs mechanical Lean 4 checking plus automated editorial review). Running `python3 papers/strong-dirac/draft4/recompute.py` executes the Section 1 Python box verbatim and replays all finite witnesses, all 36 hull certificates, and the 154-vertex design-ray check. AI research agents built and directed by Tyler B. Hughes discovered the constructions, proved and formalized all results in Lean 4, generated the figures, and drafted this manuscript. No human mathematician has yet refereed the proof.

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