An 81/29 bound for triangle covers from packings of one-red triangles

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Abstract

How many edges must one delete from a finite simple graph to destroy every triangle? A packing of pairwise edge-disjoint triangles gives a lower bound on that number. Tuza conjectured that twice the packing number always suffices. We prove that the minimum size of a triangle cover is at most 81/29 times the maximum size of a triangle packing. The proof starts with the packing decomposition of Haxell and the one-red triangle families used by Sichen Wang to obtain the factor 165/59. A red edge common to two pages of a book and the edge opposite it can each enter a random cover. Tracing where the opposite edge is itself a book spine bounds their total contribution. A separate packing exchange removes the need to buy certain opposite edges at all. Combining six cover inequalities gives the stated bound, with explicit slack. A constant vertex coloring makes the bound strict on every graph containing a triangle. The rational relaxation obtained by iterating the colored covers has a unique extremal ray. It attains 81/29 even with a factor-two bound imposed on the residual family. Lean checks the non-strict graph inequality and the theorem on the relaxation. Whether twice the packing number suffices for every graph remains open.

MSC 2020 05C70, 05C35, 05D40. **Keywords** triangle packing, triangle transversal, packing exchange.

1 Triangle packings and covers

A set of edges whose deletion destroys every triangle is a *triangle cover*, also called an edge transversal. Write $\tau(G)$ for the smallest size of such a set, and $\nu(G)$ for the largest number of pairwise edge-disjoint triangles in G. Each packed triangle needs a cover edge of its own. Conversely, the union of all edges in a maximum packing meets every triangle, since a missed triangle could be added. Thus $\nu(G) \le \tau(G) \le 3\nu(G)$. Tuza's conjecture asks whether the upper factor can always be reduced to 2 [3].

Theorem 1. *Every finite simple graph G satisfies*

$$29\,\tau(G) \;\le\; 81\,\nu(G).$$

The theorem has a kernel-checked proof. A short argument with the same random cover sharpens it on graphs containing a triangle.

Corollary 2. *If $\nu(G) > 0$, then $29\,\tau(G) \le 81\,\nu(G) - 1$.*

Both sides of the theorem vanish if G is triangle-free. When $\nu(G) > 0$, the improvement in the ratio over Wang's 165/59 is 6/1711. Wang colors original packing edges blue and uses families containing *all* triangles with exactly one red edge. This completeness makes a useful exchange possible. If a red edge would be needed to cover two triangles on the same book, one can ask whether replacing packed triangles by its neighbors would have given a larger packing in the first place. We use this principle twice. Its first use improves a random cut estimate. Its second gives a deterministic cover with fewer opposite edges. Wang's common-spine lemma and colored cover constructions supply the other inequalities [4].

An entire packing on four vertices. In K_4, choose the triangle 012 as the packing and the triangle 013 as its other page. They share the edge 01. The only triangles missing 01 are 023 and 123, both containing the opposite edge 23. This produces the cover $\{01, 23\}$. Every two triangles in K_4 share an edge, and each single edge misses some triangle. Hence $(\nu, \tau) = (1, 2)$. The following Python box checks those two lower-bound arguments and the proposed cover. Its output gives the triangle count, packing number and cover number in that order.

A triangle and its opposite edge

```
from itertools import combinations
E = set(combinations(range(4), 2))
T = [set(combinations(v, 2)) for v in combinations(range(4), 3)]
C = {(0, 1), (2, 3)}
assert all(A & B for A, B in combinations(T, 2))
assert all(C & A for A in T)
assert all(any(e not in A for A in T) for e in E)
print(len(T), 1, len(C))
```

The output is `4 1 2`. Figure 1 follows the same shared-edge geometry through the proof for general graphs.

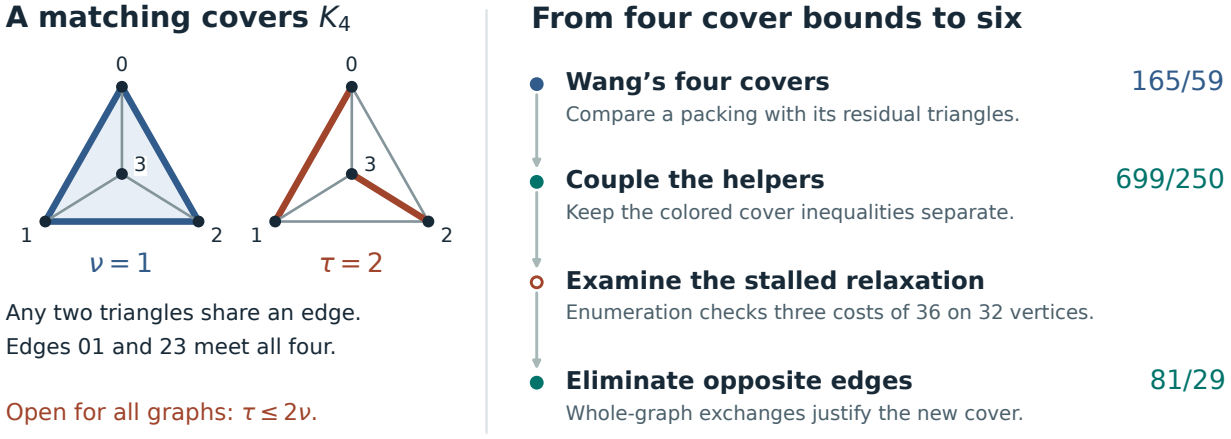

Figure 1: A maximum packing and a minimum cover in K_4, followed by the steps from Wang's cover estimates to Theorem 1. The illustrated graph attains the conjectured factor 2. The estimate 81/29 applies to all finite simple graphs.

Two further conclusions describe the reach of the method. The extremal statement uses the rational system specified in Section 4.

Theorem 3. *At positive packing scale, the supremum in the infinite system of iterated colored cover inequalities is* 81/29*. Precisely one ray attains it. On the six cover inequalities used in the proof, the normalized optimal multipliers are also unique.*

For actual graphs, the same argument retains explicit slack when a secondary packing is small or when its red edges are private. We give the two slack inequalities in Section 3.3 and the conditional bounds in Section 4.1. The relaxation attains its optimum exactly. Corollary 2 excludes exact attainment of that value by a finite graph at positive scale.

2 Packings, colors and books

We identify a triangle with its three undirected edges and use uvx to denote the triangle on three distinct vertices u, v, x. Two triangles meet when they share an edge. For a set P of triangles write $E(P)$ for its edge union. For a blue-red edge coloring, $B(P)$ denotes its blue edges. A *page* is a triangle containing a prescribed *spine* uv. Two pages uvx, uvy form a book with *opposite pair* $\{x, y\}$. The pair may fail to be an edge. Colors describe edges throughout, except in Section 3.1, where independent vertex bits determine a cut.

Wang's book lemma. Suppose books have edge-disjoint supports and at most three pages each. Assume that choosing one page from each book always gives a maximal packing in a given triangle family. Take every edge of a single-page book and the spine of every larger book. A triangle missed by these edges must meet every page of some book, since otherwise one could choose a disjoint page in each book and add the missed triangle. A triangle missing uv can meet uvx and uvy only as uxy or vxy, since two of its edges share a vertex. Neither triangle meets a third page uvz. Thus adding the opposite edges that exist for double books gives a cover. A book with j pages costs at most $4 - j$ edges [4].

The Haxell–Wang construction. Start with a maximum packing P and call the edges of $E(P)$ old. Write $n = |P|$ and $t = \tau(G)$. Every triangle contains an old edge. Choose a maximum packing A among triangles containing precisely one old edge, and set $a = |A|$. Delete $E(A)$ to obtain G'. Its remaining old edges form the blue set O' of size $3n - a$. Color all other remaining edges red. A triangle of G' has at least two blue edges, since a surviving triangle with just one old edge could have enlarged A.

Choose a packing P' in G' by first maximizing its number b of triangles with exactly two blue edges, and then maximizing its total size m. Let B^* be the b one-red members of P'. Their red edges form a set N of size b. We have

$$a + m \le n, \qquad b \le m, \qquad |B(P')| = 3m - b.$$

Every triangle surviving the deletion of $B(P')$ has one red edge in N. An all-blue survivor could enlarge P' without changing b, and a one-red survivor missing N could enlarge B^*. Denote the family of survivors by $\mathcal{S}$. Choose a maximum packing Q in $\mathcal{S}$ of size c. Each page in Q shares its red edge with exactly one member of B^*. Index the resulting double books by i and write

$$T_i = u_i v_i x_i \in B^*, \qquad Q_i^* = u_i v_i y_i \in Q, \qquad e_i = u_i v_i, \qquad f_i = \{x_i, y_i\}.$$

There are c doubles and $b - c$ singles among the B^* books. All books have pairwise disjoint edge supports. Vertices from different books may coincide.

A red edge is *private in a family* when exactly one triangle in that family contains it. Write h for the number of Q pages with private spine in $\mathcal{S}$. Delete $B(Q)$ as well, choose a maximum packing Q_1 in the remaining family, and put $d = |Q_1|$. Let h_1 count its private spines in this new family. Every page in Q_1 uses a distinct nonprivate spine of Q, so $h_1 \le d \le c - h$.

Call a double book a *root* when f_i is a red edge outside N. Write R for the roots and $\widehat{h} = |R|$. Each root has a private spine in $\mathcal{S}$. Indeed, a further page $u_i v_i z$ surviving $B(P')$ would give two disjoint one-red triangles $u_i v_i z$ and $u_i x_i y_i$ in place of T_i in B^*. All other B^* pages avoid the blue spokes and the new red edge. This is Wang's private-spine exchange [4]. Consequently $h = \widehat{h} + h_0$, where h_0 counts the private pages whose books are not roots.

For the K_4 example above, $P = \{012\}$ and $A = \{013\}$. Deleting the edges of 013 leaves a star centered at vertex 2. Thus $n = a = 1$ and all residual packing counts vanish. The opposite edge 23 in the *initial* book already gives the optimal transversal in this example. The larger constructions below are needed when residual triangles survive.

3 Seven cover inequalities

Set $L = 3(a + m) - b$. We use Wang's row numbers so that the retained inequalities can be compared with his four original covers. The primes distinguish the two refinements proved below.

$$t \le 3n - a, \tag{3}$$

$$4t \le 6n + 10a + 6b - 4c + \widehat{h}, \tag{4''}$$

$$t \le 3(a + m) - c - d, \tag{5}$$

$$t \le 3(a + m) - 2c + \widehat{h} + 2h_0, \tag{5''}$$

$$t \le L + 3c - 2d + h_1 - h, \tag{6a}$$

$$t \le L + 2c + 3d - h_1, \tag{6b}$$

$$3t \le 3L + 6c + 8d - 2h_1. \tag{6c}$$

Rows (3) and (5) are the Haxell and Wang book covers appearing in Wang's proof. The next paragraph proves both, keeping $a + m$ wherever the construction counts it. The colored estimates in (6a)–(6c) likewise come from Wang. The two new graph inequalities are (4″) and (5″).

Pair each page of A with the member of P containing its old edge. Two A pages cannot pair with the same member, since replacing it by both pages would enlarge P. Every choice of one page per book is again a packing of size n. With a doubles and $n - a$ singles, the book lemma gives (3). In G', give each of the m members of P' its page in Q, if present, and each such book its page in Q_1, if present. Distinct books have disjoint supports, because later pages avoid all earlier blue spokes and their red edges belong to distinct spines. Every choice has the same lexicographic counts (b, m). There are $m + c + d$ pages among the m books, so the book lemma costs at most $3m - c - d$. Adding $E(A)$ proves (5).

Two constructions in Wang's colored-cover proof. In a blue-red graph consider the family $\mathcal{F}$ of *all* one-red triangles. Let p be the size of its maximum packing, u the number of its private spines, and q, r the corresponding counts after deleting the blue edges of that packing. The proof of Lemma 2.3 in [4] gives the two bounds

$$\tau(\mathcal{F}) \le 3p - 2q + r - u, \qquad \tau(\mathcal{F}) \le 2p + 3q - r.$$

For the first, take all edges of each unpaired triangle and the q double spines. A missed one-red triangle uses an outside red opposite edge, so Wang's exchange assigns it to one of the r private residual spines. Add these opposite edges and omit the u original private red edges. Each omitted edge occurs only on a single page whose blue edges are already included. For the second, take the $2p$ original blue edges and all residual packing edges except the r private red ones. A missed triangle would lie in the residual family, meet a deleted private red edge, and hence equal a residual packed page already hit in blue.

Twice the first bound plus the second gives $3\tau(\mathcal{F}) \le 8p-2u$ because $r \le q$. Paying $2p$ blue edges and applying this merged bound one round later gives $3\tau(\mathcal{F}) \le 6p+8q-2r$. Finally $E(A)\cup B(P')$ costs L and leaves exactly $\mathcal{S}$. Applying the three bounds with $(p,q,u,r) = (c,d,h,h_1)$ proves the colored rows. Retaining the original two bounds separately allows the private-page terms to cancel against the two graph refinements.

3.1 A random vertex cut and the spines it can use

Let W contain the two blue edges of each single-page book of B^*, so $|W| = 2(b-c)$. Give the vertices independent, fair bits. Take W and the monochromatic blue edges outside W. Every all-blue triangle is hit. For each one-red triangle still missed, take its red edge and call the set of these edges D. Together with $E(A)$ this is a cover for every choice of bits. The expected number of selected blue edges is

$$|W| + \tfrac{1}{2}(|O'| - |W|) = \tfrac{3}{2}n - \tfrac{1}{2}a + b - c.$$

Consider $f \in D \setminus N$. Its triangle avoids $W \cup N$. The book lemma for the b books of B^*, Q identifies the triangle as a cross triangle $u_i x_i y_i$ or $v_i x_i y_i$ on a root, with $f = f_i$. It can survive only if x_i, y_i have equal bits and at least one of u_i, v_i has the opposite bit. This has probability $3/8$. Allowing different roots to name the same edge only improves the upper bound. Thus $\mathbb{E}|D \setminus N| \le 3\widehat{h}/8$. Wang charged at most $1/2$ separately for each spine in N, since a missed triangle on it requires its endpoints to have equal bits. We now save at least $1/8$ per root in that second charge.

Draw an arrow $j \to i$ if double book j has $f_j = e_i$. The spines are distinct, so each book has at most one outgoing arrow. Roots have none. All vertices whose paths reach a particular root form its *basin*. Different basins are disjoint. A basin consists of double books, and all incoming neighbors of any of its vertices belong to it. The four-book basin in Figure 2 illustrates this orientation.

Lemma 4. *For a book i in a root basin, let S_i be the set of vertices z such that $u_i v_i z$ is one-red and both blue edges avoid W. If k_i arrows enter i, then $|S_i| \le 2 + 2k_i$.*

Proof. Fix $z \in S_i \setminus \{x_i, y_i\}$ and $Z = u_i v_i z$. We will show that z is an endpoint of a spine e_j with $f_j = e_i$. Follow the path from i to its root. If there are r books on this path, their opposite edges are the root's outside red edge and the spines of the $r-1$ parents along the path. They are distinct and none is e_i.

At each path book j choose one of $u_j x_j y_j, v_j x_j y_j$. Their respective blue spoke pairs partition the four blue edges of that book. At least one pair avoids $B(Z)$. Indeed, to hit both pairs with $u_i z, v_i z$, the common endpoint z would have to be x_j or y_j, and $e_i = e_j$ would follow. For $j = i$ the choice of z rules this out, and for $j \ne i$ the spines differ. The chosen triangles and Z now form an edge-disjoint packing of $r+1$ one-red triangles, with the distinct red edges just described.

Only Z can meet an original packed page outside the path. Its blue edges avoid W, so any such page belongs to a double book. If Z meets no outside page, the $r+1$ triangles replace the path and enlarge B^*. If it meets two outside pages, each of its two blue edges lies on one of those

original pages. Both blue edges then avoid $B(Q)$. Switch both outside books to their Q pages and again the packing grows. Thus Z meets exactly one outside page, say T_j. It must also meet Q_j^*, or we could switch just that book. As Z misses e_j, the two-page geometry forces Z to be a cross triangle of j. Its red edge is $f_j = e_i$, and its third vertex z is an endpoint of e_j. Each incoming book accounts for at most two third vertices, which proves the claim. □

If $e_i \in D$, the bits on its endpoints agree and at least one member of S_i has the other bit. When both endpoints and all of S_i have the same bit the spine is saved. Lemma 4 therefore gives

$$\Pr(e_i \in D) \le \tfrac{1}{2} - 2^{-|S_i|-1} \le \tfrac{1}{2} - \tfrac{1}{8}\,4^{-k_i}.$$

For integer $k \ge 0$, $\frac{1}{8}\,4^{-k} \ge \frac{1}{8} - \frac{3}{32}k$ because equality holds for $k = 0, 1$ and the right side is negative for $k \ge 2$. A basin with s vertices has $s - 1$ arrows. Summing the inequality gives a saving of at least $(s+3)/32 \ge 1/8$ in this basin. Summing over the disjoint basins gives $\mathbb{E}|D \cap N| \le b/2 - \widehat{h}/8$, by linearity of expectation. Adding the outside reds, the selected blues and the $3a$ original edges gives

$$\mathbb{E}\,|\text{cover}| \le \tfrac{3}{2}n + \tfrac{5}{2}a + \tfrac{3}{2}b - c + \tfrac{1}{4}\widehat{h}.$$

A coloring at most as costly as the average proves $(4'')$.

Strictness for graph counts. If $n > 0$, then $|O'| = 3n - a \ge 2n > 0$. A constant vertex coloring takes exactly $E(A) \cup O'$ and has $D = \varnothing$. Choose any $e \in O'$. Since every triangle of G' has at least two blue edges, $E(A) \cup (O' \setminus \{e\})$ still covers G. The cover from this coloring therefore costs at least $t + 1$, whereas every coloring costs at least t. The constant coloring has positive probability. Consequently the mean is strictly greater than t. The integer version of the random bound thus retains a unit of slack, giving

$$4t + 1 \le 6n + 10a + 6b - 4c + \widehat{h} \qquad (n > 0). \tag{3.1}$$

3.2 A cover that omits the nonprivate opposite edges

Keep the full original page of each private nonroot. Call the remaining double books *active*. They consist of the nonprivate books and the roots. If I is their index set, $|I| = c - h_0$. Put

$$C = E(A) \;\cup\; E\big(P' \setminus \{T_i : i \in I\}\big) \;\cup\; \{e_i : i \in I\} \;\cup\; \{f_i : i \in R\}.$$

The number of edges bought is at most $3a + 3(m - c + h_0) + (c - h_0) + \widehat{h}$, the right side of $(5'')$. We prove that C meets every triangle. The argument allows arbitrary identifications of vertices from different books.

Suppose there were a missed triangle. In G' let all inactive packing pages form singles and vary the page in each active double. Every choice has counts (b, m) and is maximal. The book lemma identifies the missed triangle as $U_i^\alpha = \alpha x_i y_i$ with $\alpha \in \{u_i, v_i\}$ for some active i. Its edge f_i was omitted. Hence i cannot be a root, and we can choose a further page $Z_i = u_i v_i z_i$ in $\mathcal{S}$ different from Q_i^*. The vertices u_i, v_i, x_i, y_i, z_i are all distinct. The edge f_i must be blue, since a red f_i in N was already taken and a red f_i outside N would make i a root. Both cross triangles of i are therefore all-blue. Figure 2 illustrates the first exchange. If f_i lay outside $B(P')$, we could replace T_i by $Z_i, U_i^{u_i}$ to obtain $(b, m+1)$. It must instead belong to $B(T_k)$ for another active book k, since all other original pages were paid in full. There are two possibilities for k.

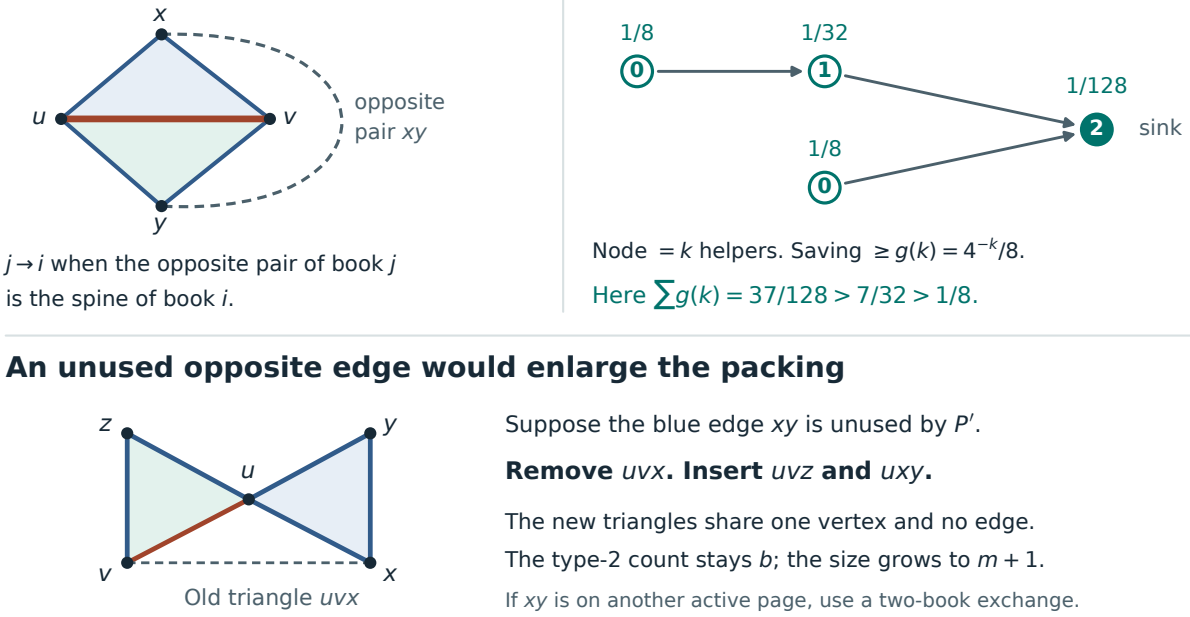

Figure 2: The same opposite pair can incur a red-edge charge or allow a packing exchange. In the illustrated basin the indegrees are $2, 1, 0, 0$. The sum of the four terms $4^{-k_i}/8$ is $37/128$, and the linear lower estimate gives $7/32$. Both are guaranteed savings in the spine estimate.

The book k is nonprivate. Choose $Z_k = u_k v_k z_k$ in $\mathcal{S}$ different from Q_k^*. Its blue edges are disjoint from $B(Q_k^*)$. The triangle $U_i^{u_i}$ avoids Q_k^* because its blue edges come from T_i, Q_i^*, T_k, and the blue edges of Q_k^* avoid all three. If Z_i also missed Q_k^*, we could replace T_i, T_k by $U_i^{u_i}, Z_i, Q_k^*$, increasing m at fixed b. Thus Z_i meets Q_k^*. Likewise, choose α so that $\alpha y_i \notin B(Z_k)$. If both choices lay in $B(Z_k)$, their common endpoint y_i would be z_k and $e_i = e_k$. The other two edges of U_i^{α} lie in $B(P')$ and avoid Z_k. The same exchange with Z_k in place of Q_k^* forces Z_i to meet Z_k.

These two intersections are blue and use different edges of Z_i. Their common endpoint z_i therefore lies on e_k, and their other endpoints give $\{u_i, v_i\} = \{y_k, z_k\}$. In particular y_k is an endpoint of e_i. Write $f_i = t' x_k$, where t' is an endpoint of e_k. If $t' = x_i$, then $t' y_k$ belongs to both $B(T_i)$ and $B(Q_k^*)$. If $t' = y_i$, it belongs to both $B(Q_i^*)$ and $B(Q_k^*)$. Both intersections are empty.

The book k is a root. Interchange u_k, v_k if necessary to write $f_i = u_k x_k$. The red edge $f_k = x_k y_k$ gives the one-red triangle $D_k = v_k x_k y_k$. It avoids $U_i^{u_i}$. Indeed $v_k x_k$ is a different edge of T_k from f_i, and $v_k y_k$ belongs to Q_k^*. Both avoid the respective blue spokes of T_i, Q_i^*. The red edge of D_k cannot lie in the all-blue $U_i^{u_i}$. Thus replacing T_i, T_k by $U_i^{u_i}, Z_i, D_k$ would increase m unless Z_i meets D_k. Their red edges differ and Z_i avoids $B(P')$, so the common edge has to be $v_k y_k = \alpha z_i$ for some endpoint α of e_i.

The two equalities of unordered pairs $\{u_k, x_k\} = \{x_i, y_i\}$ and $\{v_k, y_k\} = \{\alpha, z_i\}$ leave four choices. If $\alpha = v_k$, then αy_i lies in $B(Q_i^*)$. According as $y_i = x_k$ or u_k, it also lies in $B(T_k)$ or equals the red spine e_k. If $\alpha = y_k$, the edge αu_k lies in $B(Q_k^*)$. According as $u_k = x_i$ or y_i, it also belongs to $B(T_i)$ or $B(Q_i^*)$. Each alternative contradicts the packing or the colors. This exhausts the possible locations of f_i, proving that C is a cover and establishing $(5'')$.

3.3 Combining the covers

Put $\Delta = n - a - m \geq 0$. Multiply (3), (4″), (5), (5″), (6a), (6b), as printed, by $10, 1, 8, 1, 2, 4$. The second row already has $4t$ on its left. The coefficients of $b, c, d, \widehat{h}, h_0$ sum to zero, using $h = \widehat{h} + h_0$. The right-hand side is $36n + 45(a + m) - 2h_1$. Thus

$$29t + 45\Delta + 2h_1 \leq 81n. \tag{3.2}$$

This proves Theorem 1. If each row is first divided by its coefficient of t, the six normalized weights form the vector

$$\frac{1}{29}(10, 4, 8, 1, 2, 4).$$

Figure 3 shows the coefficient cancellation. Replacing (6b) by (6c) and taking integer multipliers $30, 3, 24, 3, 6, 4$ on the printed rows gives a second inequality,

$$87t + 135\Delta + 4d + 2h_1 \leq 243n. \tag{3.3}$$

For graph counts with $n > 0$, the integer multiplier of the strengthened random row (3.1) is one in (3.2). It follows that

$$29t + 45\Delta + 2h_1 \leq 81n - 1. \tag{3.4}$$

This proves Corollary 2. It saves a full cover edge over rounding the non-strict bound down whenever 29 divides n. The fixed additive slack does not give a smaller asymptotic coefficient.

One can see how much the omitted opposite edges matter by leaving out (5″) and replacing $\widehat{h}$ by h in (4″). On rows (3), the weakened (4″), (5), (6a) and (6c), each now written with t on the left, the multipliers $(85, 34, 80, 12, 39)$ give

$$500t + 7h + 28h_1 \leq 1398n.$$

This is already a graph bound of 699/250. Its associated relaxation has an extremal vector quite different from the one for 81/29.

4 What the linear programs can certify

The rational hierarchy. Let p_r, u_r be the packing size and the number of private packed spines at the rth deletion of blue edges from $\mathcal{S}$, starting with Q. For graph counts $(p_0, u_0) = (c, h)$ and $(p_1, u_1) = (d, h_1)$. Put $H_r = 2\sum_{j<r} p_j$. To define the relaxation, take all variables to be nonnegative rationals. Impose the first four rows of Section 3, with these substitutions, together with

$$a + m \leq n, \qquad b \leq m, \qquad p_0 \leq b, \qquad u_0 = \widehat{h} + h_0,$$
$$p_{r+1} + u_r \leq p_r, \qquad a + 3m - b + H_k \leq 3n \quad (k \geq 0), \qquad n - a \leq 3m - b - 2\widehat{h}.$$

The second line counts survivors of a packing, distinct blue edges, and hits on old triangles, respectively. For the last constraint, there are $n - a$ original packed triangles untouched by A. Such an all-blue triangle cannot hit both pages of a root, since it avoids the red spine and the red opposite edge. If it misses every original P' page outside the roots, switch each root page it hits to its residual partner. The old triangle could then be added at fixed b. The $n - a$ edge-disjoint old triangles must therefore meet the $3m - b - 2\widehat{h}$ blue edges on the other original pages. This argument also explains the stronger finite model used in the numerical comparisons below.

At every depth $r \geq 0$, add the four upper bounds

$$t \leq L + H_r + \min \begin{Bmatrix} 3p_r - u_r, & (8p_r - 2u_r)/3, \\ 3p_r - 2p_{r+1} + u_{r+1} - u_r, & \\ 3p_r - p_{r+1} - p_{r+2} - u_r & \end{Bmatrix}. \tag{4.1}$$

The first bound takes the packed edges and discards the private red spines. The second and third are Wang's merged and book estimates. For the last one, promote the next two packings to second and third pages. The private spines at depth r are unpaired and can be removed from single-page books whose blue edges are retained. In fact the third entry is at most the fourth by $p_{r+2} + u_{r+1} \leq p_{r+1}$. These inequalities specify the infinite system in Theorem 3. Finiteness of an input graph allows its packing sequence to be continued by zeros.

Proof of Theorem 3, primal statement. The first, second and third entries of (4.1), at the appropriate depths, include all the colored bounds used in (3.2) and (3.3). Hence $29t \leq 81n$. Conversely, the point

$$(n, t, a, m, b, \widehat{h}, h_0) = (29, 81, 6, 23, 18, 6, 0), \qquad p = u = (6, 0, 0, \ldots) \tag{*}$$

satisfies every constraint. Here $L = 69$. The blue-edge budget uses 57 at depth zero and 69 thereafter, out of 87, and the old triangles require $23 \leq 39$ hits. The four depth expressions all give 81: at the initial level the remaining cost is 12, and thereafter that cost is zero with $H_r = 12$. All six normalized cover right-hand sides evaluate to 81.

Suppose now that $t = 81n/29$. Equation (3.3) forces $\Delta = p_1 = u_1 = 0$. Since the multipliers in (3.2) are all positive, all six row bounds are equalities. Successively (3), $a + m = n$, (5) and (6b) fix $(a, m, p_0, b) = (6, 23, 6, 18)n/29$. Row $(4'')$ then fixes $\widehat{h} = 6n/29$, and $(5'')$ gives $h_0 = 0$. The inequalities $p_{r+1} + u_r \leq p_r$ force every later p_r, u_r to vanish. This proves uniqueness up to scale. Since the relaxation uses the homogeneous random bound, the spare unit in (3.1) is invisible to this extremal vector. □

Proof of Theorem 3, six-row dual statement. Write (x, y, z, w, v, s) for the multipliers of the six rows in the order of (3.2), with coefficient one on each left-hand side. In this dual problem $a + m$ has been bounded by n. The weights sum to one, and eliminating a, b, c, d, h_0 by their nonnegativity requires respectively

$$A = x - \tfrac{5}{2}y, \quad B = v + s - \tfrac{3}{2}y, \quad C = y + z + 2w - 3v - 2s, \quad D = z + 2v - 3s, \quad H = v - 2w$$

to be nonnegative. The coefficient of n is $3 - 3y/2$, so achieving $81/29$ requires $y \geq 4/29$. The identity

$$10(1 - 29y/4) = 10A + 31B + 9C + D + 4H$$

now gives $y = 4/29$ and forces all five slacks to vanish. Their common solution is

$$(x, y, z, w, v, s) = (10, 4, 8, 1, 2, 4)/29.$$

Equation (3.2) verifies the remaining sign conditions. The six specified rows matter here. For instance, (3.3) certifies the same universal ratio using the recursive row (6c) in place of (6b). □

Six covers give one inequality — Write each cover as $t \le C_i$ before weighting.

Cover	Weight	n	a	m	b	c	d	$\hat{h}$	h_0	h_1	C_i at (*)
(3)	10	+30	−10	·	·	·	·	·	·	·	81
(4″)	4	+6	+10	·	+6	−4	·	+1	·	·	81
(5)	8	·	+24	+24	·	−8	−8	·	·	·	81
(5″)	1	·	+3	+3	·	−2	·	+1	+2	·	81
(6a)	2	·	+6	+6	−2	+6	−4	−2	−2	+2	81
(6b)	4	·	+12	+12	−4	+8	+12	·	·	−4	81
Sum	**29**	**36**	**45**	**45**	**0**	**0**	**0**	**0**	**0**	**−2**	

$$29t \le 36n + 45(a+m) - 2h_1 \le 81n$$

The maximizing LP point (*)

$n = 29,\ t = 81,\ a = c = \hat{h} = 6,\ m = 23,\ b = 18.$

$d = h_0 = h_1 = 0$; every higher layer is empty.

At (*), a graph's constant-color cover would cost $99 > 81$.

Removing one row from seven

(3) or (4″)	3	(5″)	699/250
(5)	37/13	(6b) or (6c)	81/29
(6a)	45/16		

Figure 3: The weighted six-row certificate. Dividing each row by the coefficient of t makes its right-hand side equal to 81 at the point (*). The signed coefficients show how the six rows sum. The lower comparison removes one row at a time from the seven-row finite system, including the alternative recursive row.

Earlier equality points. Wang gives an equality point for his four printed rows. In units of $n/59$ it is $(a, m, b, c, d, h) = (12, 39, 39, 12, 0, 0)$, where all four bounds give 165. The extra information $3(a+m) - c - d$ instead gives $3(12+39) - 12 = 141$ at this point. This illustrates the role of keeping $a + m$ in a cover until the final combination [4].

For the earlier hierarchy use (3), (5), the weakened random row with h in place of $\widehat{h}$, the packing and successor constraints, and (4.1). In this system the 699/250 bound from Section 3.3 is exact at all depths. A witness is

$$(n, t, a, m, b) = (250, 699, 51, 199, 159), \qquad p = (42, 9, 3, 0, \ldots), \quad u = (0, 0, 3, 0, \ldots).$$

Indeed $L = 591$, the successive H_r are $0, 84, 102, 108$ and then remain at 108. Substitution in (4.1) gives a minimum of 699 at every depth, and the other three cover rows give 699 or more. This point and (*) describe limits on the named inequalities. To see what is missing even at (*), suppose one could supplement the full hierarchy with $\tau(\mathcal{S}) \le 2\nu(\mathcal{S})$. The resulting row $t \le L + 2p_0$ reads $81 \le 69 + 12$ at the same point. A proof using this hierarchy that improves on 81/29 must exclude the ray in another way.

4.1 Bounds under additional packing hypotheses

Two further conic combinations localize the obstruction. At the scale of the printed inequalities, the multipliers $70, 7, 56, 6, 12, 10$ on (3), (4″), (5), (5″), (6a), (6c) give the first line below. The multipliers $10, 1, 6, 4, 2$ on (3), (4″), (5″), (6a), (6b) give the second:

$$202t + 312\Delta + 8h_1 \le 564n + \widehat{h},$$
$$26t + 36\Delta + 2(d - h_1) \le 72n + 3\widehat{h} + 8h_0.$$

In particular any graph with $t > (282/101)n$ requires a root in *every* choice of the construction. If $h = 0$, both $\widehat{h}, h_0$ vanish and $t \le (36/13)n$ follows. The following are graph implications: it suffices to exhibit a tuple in G satisfying the indicated hypothesis.

Hypothesis	Bound on t/n	Multipliers in the seven-row order
$\widehat{h} = 0$	$282/101$	$(35, 14, 28, 3, 6, 0, 15)$
$d = \widehat{h} = 0$	$39/14$	$(5, 2, 4, 0, 0, 3, 0)$
$h = 0$	$36/13$	$(5, 2, 0, 3, 2, 1, 0)$
$d = h = 0$	$11/4$	$(5, 2, 0, 2, 0, 3, 0)$
$c = 0$	$27/10$	$(5, 2, 0, 0, 3, 0, 0)$
$a = 0$	$51/19$	$(0, 4, 8, 1, 2, 4, 0)$
$a = h = 0$	$21/8$	$(0, 2, 0, 3, 2, 1, 0)$
$b = 0$	$18/7$	$(5, 2, 0, 0, 0, 0, 0)$

Here $n > 0$ for the displayed ratios. The multipliers apply to the rows of Section 3 after division by their coefficient of t. Using $a + m \le n$ and $h_1 \le d \le c - h \le c \le b$ in each sum gives the listed bound. Each sum has positive weight on $(4'')$; the paper proof of (3.1) therefore makes every listed inequality strict for $n > 0$. Figure 4 locates these estimates alongside the changes in the unconditional proof. The conditions $d = 0$, $h_0 = 0$ and $h_1 = 0$ leave the full extremal ray feasible, even if imposed simultaneously.

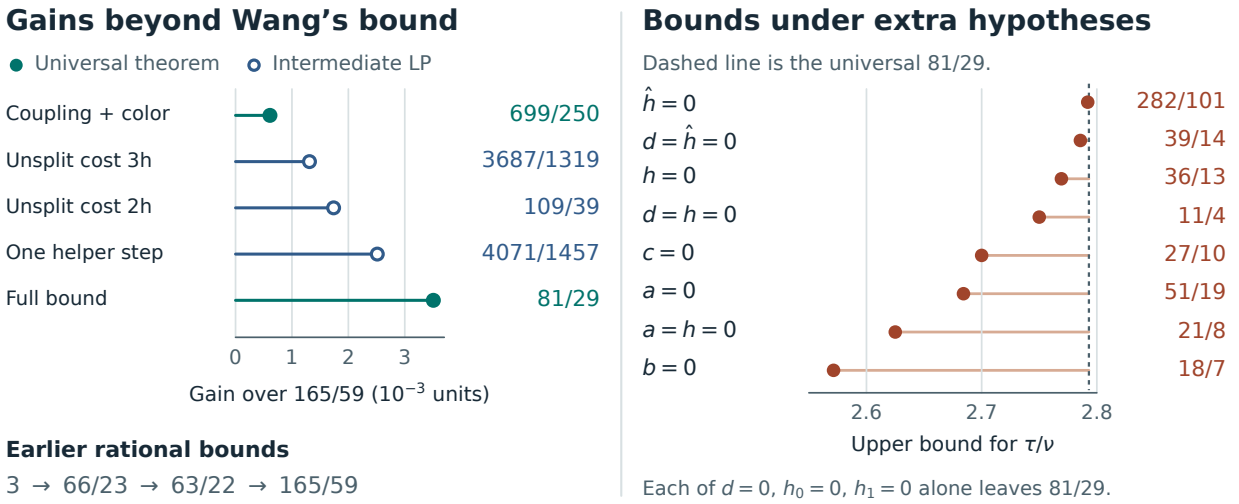

Figure 4: Left, reductions from Wang's 165/59 as more information about the books is used in the graph bound. Right, upper bounds conditional on the specified counts in the packing tuple. The points show guarantees from the cover inequalities.

4.2 Finite calculations and examples

The finite linear programs fix $n = 1$ and keep $a, m, b, c, d, \widehat{h}, h_0, h_1$ and t. For the conditional comparisons the checker uses the seven rows of Section 3, the packing and privacy conditions, the old-triangle hit constraint, and the blue budget through both c and d. An exact rational simplex calculation supplies primal and dual certificates. On the optimal face, minimizing and maximizing

each coordinate separately recovers the singleton at the normalization (*). The analogous calculation on the six-row dual recovers its unique multipliers. The unbounded-depth conclusion follows from Theorem 3.

Three comparisons in Figure 4 unpack the improvement. Adding a weaker version of (5″) that charges $3h$ for private pages gives $3687/1319$ in the earlier finite system. Charging $2h$ in the same row gives $109/39$. Using the split $\widehat{h}, h_0$ in the opposite row but tracing helper arrows just one step gives $4071/1457$ in the corresponding finite system. These are rational-program calculations. In the seven-row system, removing (5″) recovers $699/250$; omitting either (6b) or (6c) singly leaves the optimum at $81/29$. Thus a row can be essential in one particular certificate while dispensable once another row is admitted.

A residual example shows why a layer matching the packing parameters can still admit a much smaller cover. The colored graph $H_{14,3,1}$ in the accompanying checker has 32 vertices and 62 edges, of which 14 are red. All 32 triangles are one-red. The checker exhibits a sequence of maximum packings with sizes $(14, 3, 1)$ and private counts $(0, 0, 1)$. Three ways of combining the residual bounds each give an allowance of 36 edges per block. Yet the first packing has 14 members and the 14 red edges form a transversal, so $\nu = \tau = 14$ on this graph. Its parameters model a layer of the earlier relaxation. It supplies no graph extremizer for either universal coefficient.

The two-book exchange proof also admits finite checks of all vertex identifications of the local roles. In the nonprivate case there are 115975 set partitions of the ten roles. Of these, 28 obey the preliminary local conditions, and none survives all of them. For a root there are 21147 partitions of nine roles, of which 12 pass the preliminary conditions and none is an obstruction. Lemma 4, by contrast, needs arbitrarily long paths and is proved for them directly. The finite checker also tests a small census of rooted trees and enumerates the connected unlabeled graphs on three through six vertices.

Examples at the conjectured factor. The exact pair $(\nu, \tau) = (2, 4)$ for K_5 is classical. Vertices $0, 1, 2$ and $0, 3, 4$ give two edge-disjoint triangles. Three packed triangles would use nine of the ten edges. The unused edge would leave an odd used degree at each endpoint there, whereas a union of edge-disjoint triangles has only even degrees. For the cover, delete the edges inside the parts of a bipartition of sizes two and three. This deletes four edges. Conversely a triangle-free graph on five vertices has at most six edges. Indeed for every edge uv its endpoint degrees sum to at most five, since the endpoint neighborhoods are disjoint. For e edges, summing these inequalities and applying Cauchy–Schwarz to the degrees gives $4e^2 \le 25e$, so $e \le 6$. At least four edges of K_5 must be deleted.

Every triangle of a disjoint union lies wholly in one component. Restricting a packing or cover to the components gives one inequality for each invariant. Unions of componentwise optimizers give the reverse inequalities. Thus every disjoint union of $k \ge 1$ copies of K_4 has $(\nu, \tau) = (k, 2k)$, and for K_5 the pair is $(2k, 4k)$. These are actual graphs of unbounded packing size attaining the proposed constant 2. More generally, if G has q connected components containing triangles, applying Corollary 2 to each gives $29\tau(G) \le 81\nu(G) - q$.

5 The kernel-checked statements

The `TuzaBound` library represents edges by unordered vertex pairs. Its graph theorem quantifies over every finite vertex type and every `SimpleGraph` on it, with the packing and transversal numbers defined from the graph's triangles. The export `TuzaBound.tuza_bound_81_29_nat` proves

the natural-number statement in Theorem 1. A companion export proves existence of a transversal of the asserted size. These results construct the packings and prove both exchange arguments. They require no externally computed graph, list of partitions, or linear-programming certificate as input. Corollary 2, including the componentwise strengthening, is an additional paper proof. Its spare-unit assertion is absent from the pinned Lean declarations.

The relaxation has different quantifiers. The file `TuzaBound/ExtremalAndDuality.lean` defines the rational hierarchy, including the old-packing hit condition as a hypothesis, and proves its exact optimum, unique primal ray and six-row dual uniqueness. The same file proves the conditional graph estimates and disjoint-copy identities. The separate `TuzaBound/FixedPoint.lean` proves the earlier 699/250 optimum. The hand proof of the old-triangle hit condition in Section 4 explains its graph interpretation. Neither the graph theorem nor the hierarchy's upper-bound certificate uses that condition to obtain 81/29.

The formalization builds on Sichen Wang's MIT-licensed Lean development [5]. Of the 29 local source files, 14 preserve upstream mathematical contents after mechanical porting, 8 adapt upstream files, and 7 were added here, including the verification and import modules. Wang's license remains in the distribution. The pinned source and audit form the repository snapshot `9823019d`. Lean and Mathlib version `4.35.0` build the library. The source census records 604 public declarations. The selected 164 axiom reports contain only `propext`, `Classical.choice` and `Quot.sound`, with no proof placeholders or native decision oracle in the library. The proof map names the declarations supporting the mathematical statements in this manuscript.

6 Prior work and the remaining graph problem

Tuza asked whether $\tau \le 2\nu$ at the Eger colloquium. The conjecture appears on page 888 of its proceedings [3]. The examples K_4 and K_5 already require that factor. Baron and Kahn obtained dense examples: for every fixed $\alpha > 0$ there are arbitrarily large graphs of positive density with $\tau > (1 - o(1))|E|/2$ and $\nu < (1 + \alpha)|E|/4$. Thus sparse disjoint unions do not account for all obstructions to improving the factor two [1].

Haxell proved the general bound 66/23 and developed the decomposition into triangles with one, two or three old edges [2] [to check]. Yi retained the red-blue structure of a residual family, giving 63/22 and the sharper $(162+4\sqrt{3})/59$. In Haxell's four-inequality argument, even assigning the conjectured factor to the residual family leaves 54/19 [6]. Wang used the fact that the residual family contains every one-red triangle in the remaining graph. His private-spine exchange, common-spine rule and two colored transversals yielded 165/59 and were accompanied by his Lean proof [4, 5]. The new estimates here concern what can be saved on root spines and on opposite edges when those same families are used in the outer graph.

The references ledger identifies the pages checked in the original Tuza, Yi, Wang, and Baron–Kahn sources and the upstream Lean code. For Haxell's article, the publisher abstract and bibliographic record were checked, and the mathematical comparison was cross-checked in Yi and Wang. The original full proof remains an author check.

Corollary 2 excludes equality on every finite graph with a triangle. Could a sequence of such graphs nevertheless have $\tau/\nu \to 81/29$? The first-layer counts normalized by n are bounded. Every limit point of a sequence approaching 81/29 must satisfy the six equalities in the primal proof, hence agree with (*) up to scale. There nearly all residual packed pages would have private spines and red outside opposite edges, and d/n would tend to zero. Ruling out those proportions in the original maximum packing, or proving a positive proportional saving from repeated edges in the random cover, would address the asymptotic gap. Tuza's proposed factor 2 remains open.

7 Data, reproducibility and use of AI

The companion `checker.zip` runs with the Python standard library. After extraction, `python3 recompute.py` checks the printed K_4 example, the finite rational programs, the local equality patterns and small graph examples, together with the fingerprints of the supplied Lean sources and archived axiom output. In the manuscript directory the same command also compares the example against the printed source. To repeat the kernel build separately, the repository's `lean` directory provides the pinned toolchain and Lake manifest. From there run `lake build TuzaBound TuzaBound.Verify`. The accompanying `README.md` gives the axiom-audit command and figure rebuilds. The Python examples and rational programs illustrate and check parts of the argument. Lean checks the two theorems against their formal definitions.

AI systems drafted the exposition, produced the figures and assisted with the formalization and finite checks. The title-page author line and the presentation are subject to human review. The references ledger records source access separately for each citation.

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