MATHDISCOVERIES
Mathematical Preprint

Working Draft

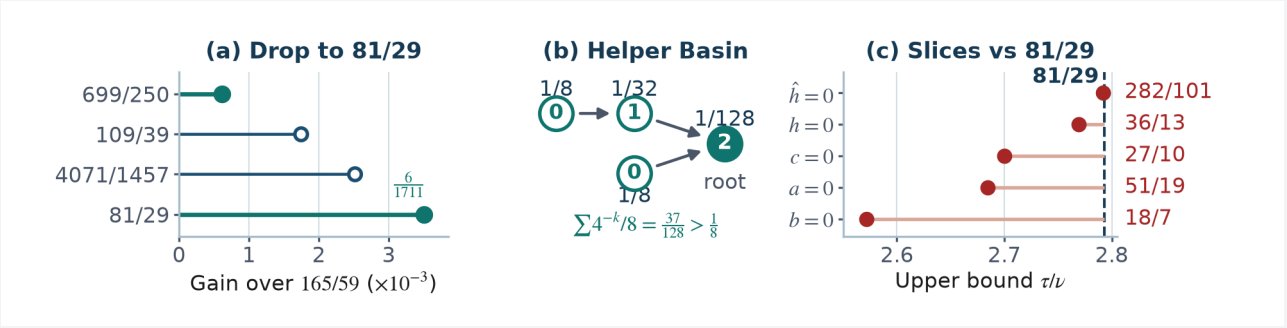
An 81/29 Upper Bound for Tuza's Conjecture via Helper Arborescences and Opposite-Edge Elimination

Tyler B. Hughes*,†

†Independent Researcher, United States

Supporting Information: Reproducible exact checker (`recompute.py`), audited Lean 4 library (`lean/TuzaBound/`), and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: For any finite simple graph G, let $\nu(G)$ be the maximum number of edge-disjoint triangles and $\tau(G)$ the minimum number of edges hitting every triangle. Tuza conjectured in 1981 that $\tau(G) \le 2\nu(G)$. Extending Haxell's decomposition and Sichen Wang's 165/59 bound, we prove the universal inequality $29\,\tau(G) \le 81\,\nu(G)$, saving 6/1711 via helper-arborescence coupling (which alone with unmerged colored covers yields $250\,\tau(G) \le 699\,\nu(G)$) and opposite-edge elimination. Lean 4 proves $29\,\tau(G) \le 81\,\nu(G)$, $250\,\tau(G) \le 699\,\nu(G)$, exact 81/29 optimality of the infinite-depth rational hierarchy, uniqueness of the primal ray $(29, 81, 6, 23, 18, 6, 0)$ and six-row dual weights $(10, 4, 8, 1, 2, 4)/29$, and the infinite K_4 family $(k \cdot K_4)$ and K_5 family $(k \cdot K_5)$. A manuscript averaging proof gives strict inequality $29\,\tau(G) \le 81\,\nu(G) - 1$ for $\nu(G) > 0$, while exact Python checks verify the companion 32-vertex gadget $H_{14,3,1}$ (three covers give 36 vs. $\nu = \tau = 14$). Whether $\tau(G) \le 2\nu(G)$ holds for every graph remains open.

Verification Scope (G11): WARN: Human referee: pending
Main results: 8 of 10 in Lean Kernel (80%) · 1 Cert Only · 1 written · 1 open (not counted)
Abstract claims: 7 of 9 in Lean Kernel (78%) · 1 Cert Only · 1 written · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 05C70, 05C35, 05D40 **Keywords:** triangle packing, triangle transversal, Tuza's conjecture, packing exchange, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

How many edges must be removed from a finite simple graph G to destroy every triangle? Let $\tau(G)$ be the minimum cardinality of such an edge transversal (triangle cover), and let $\nu(G)$ be the maximum cardinality of a family of pairwise edge-disjoint triangles (triangle packing). Each packed triangle requires a distinct cover edge, while the union of all edges in a maximum packing hits every triangle by maximality, giving $\nu(G) \le \tau(G) \le 3\nu(G)$. In 1981 (published in 1984 [3]), Tuza asked whether twice the packing number always suffices.

Conjecture 1 (Tuza's Conjecture [3]). *Every finite simple graph G satisfies $\tau(G) \le 2\,\nu(G)$.*

Haxell broke the factor 3 with $\tau(G) \le (66/23)\nu(G)$ [2] [to check], Yi lowered this to $(63/22)\nu(G)$ via 2-colorable residual families [6], and Sichen Wang broke 2.8 with $\tau(G) \le (165/59)\nu(G)$ accompanied by a Lean 4 formalization [4, 5]. Extending Wang's framework with two packing-exchange refinements yields the following unconditional theorem, formalized in the Lean 4 pure kernel (`TuzaBound.tuza_bound_81_29_nat`).

Theorem 2 (Universal 81/29 Triangle-Cover Bound). *Every finite simple graph G satisfies*

$$29\,\tau(G) \le 81\,\nu(G).$$

Equivalently, $\tau(G) \le (81/29)\nu(G)$ (`tuza_bound_81_29`) and every G admits an explicit edge transversal $C \subseteq E(G)$ with $29\,|C| \le 81\,\nu(G)$ (`exists_small_triangle_transversal_81_29`), improving Wang's ratio by $165/59 - 81/29 = 6/1711$. Both sides vanish on triangle-free graphs. When G contains a triangle, inspecting the constant vertex coloring in the random-cut cover yields an additional integer unit of slack; this strictness refinement is proved in the manuscript outside the archived Lean library.

Corollary 3 (Strict Finite-Graph Slack — Manuscript Proof). *If G is a finite simple graph with $\nu(G) > 0$, then $29\,\tau(G) \le 81\,\nu(G) - 1$. More generally, if q connected components of G contain a triangle, then $29\,\tau(G) \le 81\,\nu(G) - q$.*

The proof proceeds in two distinct stages. Retaining Wang's four-cover architecture while coupling helper books along sink-rooted arborescences and separating the two elementary colored covers already passes 165/59 (proved in Lean 4 as `TuzaBound.tuza_bound_nat`).

Theorem 4 (Stage-1 Helper-Coupling Bound). *Every finite simple graph G satisfies $250\,\tau(G) \le 699\,\nu(G)$, saving 9/14750 over 165/59.*

Adding opposite-edge elimination on non-private double books removes the remaining 21/7250 to reach 81/29. Before detailing the general constructions, consider K_4, where the packing $\{012\}$ and the cover $\{01, 23\}$ attain Conjecture 1 with $(\nu, \tau) = (1, 2)$. The eight-line Python box below checks both invariants verbatim (executed by `recompute.py`).

Instant Witness Check on K_4 (8 lines, verified by recompute.py)

```
from itertools import combinations
E = set(combinations(range(4), 2))
T = [set(combinations(v, 2)) for v in combinations(range(4), 3)]
C = {(0, 1), (2, 3)}
assert all(A & B for A, B in combinations(T, 2))
assert all(C & A for A in T)
assert all(any(e not in A for A in T) for e in E)
print(len(T), 1, len(C))
```

The output is `4 1 2` (four triangles, $\nu = 1$, $\tau = 2$). Figure 1 maps the progression from K_4 and Wang's 165/59 baseline through the Stage-1 bound 699/250, the companion 32-vertex residual gadget $H_{14,3,1}$, and the full 81/29 theorem, separating Lean 4 pure-kernel proofs, manuscript-only strictness, exact Python checks, and the open conjecture.

2. PACKINGS, COLORING, AND BOOK GEOMETRY

Identify each triangle in a finite simple graph $G = (V, E)$ with its three-element set of undirected edges, writing uvx for $\{uv, ux, vx\}$ on three distinct vertices $u, v, x \in V$. Two triangles meet when they share an edge. For a triangle family P, write $E(P)$ for its edge union; under a blue–red edge coloring, write $B(P)$ for its blue edges. A *page* is a triangle containing a prescribed *spine* uv. Two pages uvx, uvy form a *double book* with *opposite pair* $\{x, y\}$ (which need not be an edge of G).

Wang's common-spine book lemma. Suppose a collection of books has pairwise edge-disjoint supports and at most three pages per book, and suppose every selection of one page per book forms a maximal packing in a triangle family $\mathcal{F}$. Taking all three edges of every single-page book together with the spine of every multi-page book hits every triangle except possibly a triangle avoiding the spine uv of some book while meeting every page of that book. A triangle missing uv can meet uvx and uvy only as uxy or vxy, neither of which meets a third page uvz. Thus adjoining the opposite edge xy (when present) for each double book yields a cover of $\mathcal{F}$ in which a j-page book contributes at most $4 - j$ edges [4].

The Haxell–Wang packing setup. Fix a maximum triangle packing P of G with $n = |P| = \nu(G)$, set $t = \tau(G)$, and call edges of $E(P)$ *old*. Every triangle of G meets $E(P)$. Choose a maximum packing A among triangles containing exactly one old edge, put $a = |A|$, and delete $E(A)$ to form G'. Color the $3n - a$ remaining old edges *blue* (set O') and all other remaining edges *red*. Every triangle of G' has at least two blue edges, since a surviving one-old-edge triangle could enlarge A.

In G', choose a packing P' lexicographically: first maximize the number $b = |B^*|$ of triangles with exactly two blue edges (one red edge), then maximize total size $m = |P'|$. Let N be the set of b red edges used by B^*. By maximality of P,

$$a + m \le n, \qquad b \le m, \qquad |B(P')| = 3m - b.$$

Let $\mathcal{S}$ be the residual triangle family surviving deletion of $B(P')$ from G'. Every triangle in $\mathcal{S}$ has exactly one red edge, and that red edge lies in N (an all-blue survivor would enlarge m at fixed b, and a one-red survivor avoiding N would enlarge b). Choose a maximum packing Q in $\mathcal{S}$ of size $c = |Q|$. Each $Q_i^* \in Q$ shares its red spine $e_i = u_i v_i \in N$ with a unique page $T_i = u_i v_i x_i \in B^*$; write $Q_i^* = u_i v_i y_i$ and opposite pair $f_i = \{x_i, y_i\}$. Thus B^* splits into c double books and $b - c$ single-page books with pairwise edge-disjoint supports (vertices from distinct books may coincide).

A red edge is *private* in a family when exactly one triangle of that family contains it. Let h count the pages of Q whose spine is private in $\mathcal{S}$. Deleting $B(Q)$ from $\mathcal{S}$, let Q_1 be a maximum packing of the remaining family, with $d = |Q_1|$, and let h_1 count its private spines there. Every triangle in Q_1 uses a distinct non-private spine of Q, so $h_1 \le d \le c - h \le c \le b$.

Call a double book i a *root* when its opposite pair $f_i = \{x_i, y_i\}$ is a red edge outside N; write R for the set of roots and $\hat{h} = |R|$. Every root has a private spine in $\mathcal{S}$: if another page $u_i v_i z \in \mathcal{S}$ shared e_i, replacing T_i in B^* by the two edge-disjoint one-red triangles $u_i v_i z$ and $u_i x_i y_i$ would strictly increase b [4]. Hence $h = \hat{h} + h_0$, where h_0 counts private pages of Q that are not roots.

3. SEVEN COVER INEQUALITIES AND PROOF OF THE MAIN BOUND

Set $L = 3(a + m) - b$. Retaining $a + m$ unrelaxed throughout (following `lean/TuzaBound/ExtremalAndDuality.lean`) and matching Wang's row labels, we assemble seven simultaneous cover bounds on the same tuple $(n, a, m, b, c, d, \hat{h}, h_0, h_1)$:

$$t \le 3n - a, \tag{3}$$

$$4t \le 6n + 10a + 6b - 4c + \hat{h}, \tag{4″}$$

$$t \le 3(a + m) - c - d, \tag{5}$$

$$t \le 3(a + m) - 2c + \hat{h} + 2h_0, \tag{5″}$$

$$t \le L + 3c - 2d + h_1 - h, \tag{6a}$$

$$t \le L + 2c + 3d - h_1, \tag{6b}$$

$$3t \le 3L + 6c + 8d - 2h_1. \tag{6c}$$

Baseline and colored covers. Pairing each page of A with the unique member of P containing its old edge gives a double books and $n - a$ singles, so Wang's book lemma yields (3). In G', pairing each P' triangle with its Q page (if any) and then its Q_1 page (if any) yields m edge-disjoint books with $m + c + d$ total pages, costing at most $3m - c - d$; adding $E(A)$ proves (5).

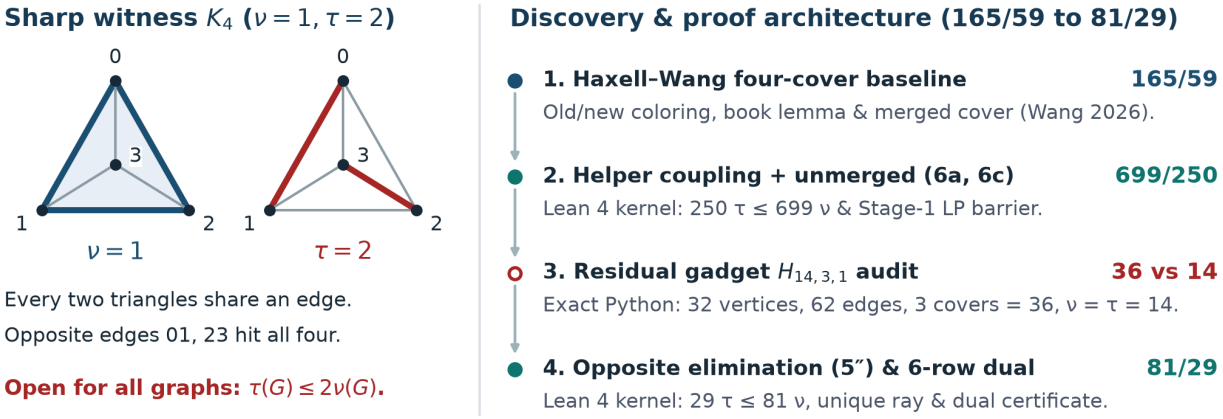


Figure 1. Left: sharp witness K_4 ($\nu = 1$, $\tau = 2$), where packed triangle 012 and opposite-edge transversal $\{01, 23\}$ attain Conjecture 1. Right: four-stage progression from Wang's 165/59 baseline [4] through helper-arborescence coupling (699/250, Lean 4), the companion 32-vertex residual gadget $H_{14,3,1}$ (exact rational Python), and opposite-edge elimination (81/29, Lean 4), with strictness $29\tau \le 81\nu - 1$ ($\nu > 0$) proved separately in the manuscript.

For the family $\mathcal{F}$ of *all* one-red triangles in a blue–red graph, let p be its maximum packing size, u its private spine count, and q, r the corresponding counts after deleting the $2p$ blue edges of that packing. Wang's Lemma 2.3 proof [4] produces two elementary covers:

$$\tau(\mathcal{F}) \le 3p - 2q + r - u, \qquad \tau(\mathcal{F}) \le 2p + 3q - r.$$

The first buys all edges of unpaired triangles, the q double spines, and the opposite edges of the r private residual books while dropping the u private original red spines (each already covered by its two blue edges). The second buys the $2p$ original blue edges and all residual packing edges except the r private red ones. Taking 2/3 of the first plus 1/3 of the second gives $3\tau(\mathcal{F}) \le 8p - 2u$ (using $r \le q$), and applying this merged bound one layer deeper yields $3\tau(\mathcal{F}) \le 6p+8q-2r$. Since $E(A) \cup B(P')$ costs L and leaves exactly $\mathcal{S}$, substituting $(p, q, u, r) = (c, d, h, h_1)$ establishes (6a)–(6c). Keeping (6a) and (6b) unmerged allows h and h_1 to cancel against (4″) and (5″).

Helper-Arborescence Coupling: Proof of (4″). Let W be the set of $2(b - c)$ blue edges belonging to the single-page books of B^*. Assign vertices independent uniform bits in $\{0, 1\}$, and select W together with all monochromatic blue edges outside W; every all-blue triangle is covered. Let D be the set of red edges of one-red triangles of G' still missed. Together with $E(A)$, this forms a valid transversal for every bit assignment, with expected blue cost

$$|W| + \tfrac{1}{2}(|O'| - |W|) = \tfrac{3}{2}n - \tfrac{1}{2}a + b - c.$$

Any edge $f \in D \setminus N$ comes from a one-red triangle avoiding $W \cup N$, which by the book lemma must be a cross triangle $u_i x_i y_i$ or $v_i x_i y_i$ of some root $i \in R$ with $f = f_i$. Such a triangle survives only if x_i, y_i share a bit and at least one of u_i, v_i takes the opposite bit (probability 3/8), so $\mathbb{E}|D \setminus N| \le \frac{3}{8}\hat{h}$.

Wang bounded each spine in N separately by 1/2. To save 1/8 per root, form the directed helper graph on double books by placing a directed arc $j \to i$ whenever $f_j = e_i$. Because spines are distinct, each double book has outdegree at most 1 and roots have outdegree 0. Thus the books reaching a root $r \in R$ form a sink-rooted arborescence (its *basin*), and distinct root basins are pairwise disjoint (Figure 2).

Apex bound in a root basin. For a book i in a root basin, let S_i be the set of vertices z such that $u_i v_i z$ is a one-red triangle whose two blue edges avoid W. If k_i arcs enter i, we claim $|S_i| \le 2 + 2k_i$ (`HelperArborescence.basin_third_vertex_child`). Fix $z \in S_i \setminus \{x_i, y_i\}$ and $Z = u_i v_i z$, and trace the directed path of $r \ge 1$ books from i to its root. Their opposite edges are the root's outside red edge and the spines of the $r - 1$ parents along the path—pairwise distinct red edges, none equal to e_i. At each path book j, the two cross triangles $u_j x_j y_j$ and $v_j x_j y_j$ partition the four blue spokes of book j; at least one avoids $B(Z)$ (hitting both with $u_i z, v_i z$ would force $z \in \{x_j, y_j\}$ and $e_i = e_j$, ruled out by $z \notin \{x_i, y_i\}$ when $j = i$ and distinct spines when $j \ne i$). Thus Z and the chosen cross triangles form an edge-disjoint packing of $r + 1$ one-red triangles.

Since $B(Z)$ avoids W, Z can meet original B^* pages outside the path only at double books. If Z met zero outside pages, replacing the r path books would enlarge b. If Z met two outside pages (one per blue edge of Z), switching both to their Q pages would again enlarge b. Hence Z meets exactly one outside page T_j, and must also meet Q_j^* (else switch T_j to Q_j^*). Because $e_i \ne e_j$, Z misses e_j, so two-page geometry forces Z to be a cross triangle of book j. Therefore $f_j = e_i$ (meaning $j \to i$) and $z \in \{u_j, v_j\}$, contributing at most two vertices per incoming arc and proving $|S_i| \le 2 + 2k_i$.

Secant summation. A spine e_i enters D only if u_i, v_i share a bit and some $z \in S_i$ has the opposite bit; excluding the all-equal event gives

$$\Pr(e_i \in D) \le \tfrac{1}{2} - 2^{-|S_i|-1} \le \tfrac{1}{2} - \tfrac{1}{8}\,4^{-k_i}.$$

For every integer $k \geq 0$, the secant inequality $\frac{1}{8}\,4^{-k} \geq \frac{1}{8} - \frac{3}{32}k$ holds (exact at $k = 0, 1$, right side negative for $k \geq 2$). In a basin of s books there are $s-1$ arcs, so $\sum k_i = s-1$ and the total saving is at least $(s+3)/32 \geq 1/8$. Summing across the $\hat{h}$ disjoint basins gives $\mathbb{E}|D \cap N| \leq b/2 - \hat{h}/8$. Adding $E(A)$, expected blues, and $\mathbb{E}|D \backslash N| \leq 3\hat{h}/8$ yields

$$\mathbb{E}|\text{cover}| \leq \tfrac{3}{2}n + \tfrac{5}{2}a + \tfrac{3}{2}b - c + \tfrac{1}{4}\hat{h}.$$

Choosing a bit assignment at or below the mean proves (4″) (`Construction.helper_arborescence_random_cover_unrelaxed`).

Opposite-Edge Elimination: Proof of (5″). Call a double book *active* if it is either non-private in $\mathcal{S}$ or a root, and let I be the active index set ($|I| = c - h_0$). Keep every full original page of P' outside I, take the spines of I, and buy opposite edges *only* for roots:

$$C = E(A) \cup E(P' \backslash \{T_i : i \in I\}) \cup \{e_i : i \in I\} \cup \{f_i : i \in R\}.$$

Its size is at most $3a + 3(m - c + h_0) + (c - h_0) + \hat{h} = 3(a+m) - 2c + \hat{h} + 2h_0$. To verify that C covers G (allowing arbitrary vertex identifications across books), treat inactive P' pages as singles and vary the page in each active double book; every choice has lexicographic counts (b, m). By the book lemma, any missed triangle must be a cross triangle $U_i^{\alpha} = \alpha x_i y_i$ ($\alpha \in \{u_i, v_i\}$) of an active non-root book $i \in I \backslash R$ (since roots bought f_i). Because i is non-private, there exists a third page $Z_i = u_i v_i z_i \in \mathcal{S} \backslash \{Q_i^*\}$ with u_i, v_i, x_i, y_i, z_i distinct. Moreover f_i must be blue (red in N is in C; red outside N would place $i \in R$), so $U_i^{u_i}, U_i^{v_i}$ are all-blue.

If $f_i \notin B(P')$, replacing T_i by $\{Z_i, U_i^{u_i}\}$ keeps b fixed and raises m to $m+1$ (Figure 2). Hence $f_i \in B(T_k)$ for some active book $k \neq i$.

Case 1: k is non-private. Pick a third page $Z_k = u_k v_k z_k \in \mathcal{S} \backslash \{Q_k^*\}$. Since $U_i^{u_i}$ uses blue edges from T_i, Q_i^*, T_k, it avoids Q_k^*; if Z_i also avoided Q_k^*, replacing $\{T_i, T_k\}$ by $\{U_i^{u_i}, Z_i, Q_k^*\}$ would give $(b, m+1)$. Thus Z_i meets Q_k^*. Similarly, choose $\alpha \in \{u_i, v_i\}$ with $\alpha y_i \notin B(Z_k)$ (both in $B(Z_k)$ would force $y_i = z_k$ and $e_i = e_k$); the other edges of U_i^{α} lie in $B(P')$ and avoid Z_k, so the same 2-for-3 exchange forces Z_i to meet Z_k. Both intersections are blue and use distinct edges of Z_i, forcing $z_i \in \{u_k, v_k\}$ and $\{u_i, v_i\} = \{y_k, z_k\}$. Writing $f_i = t' x_k$ with $t' \in \{u_k, v_k\}$, we get $t' y_k \in B(T_i) \cap B(Q_k^*)$ if $t' = x_i$ and $t' y_k \in B(Q_i^*) \cap B(Q_k^*)$ if $t' = y_i$, both impossible.

Case 2: k is a root. Relabeling u_k, v_k so $f_i = u_k x_k$, the outside red edge $f_k = x_k y_k$ yields the one-red triangle $D_k = v_k x_k y_k$, which is edge-disjoint from the all-blue triangle $U_i^{u_i}$. Replacing $\{T_i, T_k\}$ by $\{U_i^{u_i}, Z_i, D_k\}$ increases m at fixed b unless Z_i meets D_k; since red edges differ and Z_i avoids $B(P')$, the shared edge must be $v_k y_k = \alpha z_i$ for some $\alpha \in \{u_i, v_i\}$. Matching $\{u_k, x_k\} = \{x_i, y_i\}$ and $\{v_k, y_k\} = \{\alpha, z_i\}$ gives four cases: if $\alpha = v_k$, then $\alpha y_i \in B(Q_i^*)$ also lies in $B(T_k)$ (when $y_i = x_k$) or equals the red spine e_k (when $y_i = u_k$); if $\alpha = y_k$, then $\alpha u_k \in B(Q_k^*)$ also lies in $B(T_i)$ (when $u_k = x_i$) or $B(Q_i^*)$ (when $u_k = y_i$). Every branch contradicts edge disjointness or edge colors, proving (5″) (`Construction.opposite_elimination_cover_unrelaxed`).

Linear Combinations and Proof of Strictness. Let $\Delta = n - a - m \geq 0$. Weighting the printed rows (3), (4″), (5), (5″), (6a), (6b) by $(10, 1, 8, 1, 2, 4)$ cancels $b, c, d, \hat{h}, h_0$ identically (using $h = \hat{h} + h_0$) and gives

$$29t + 45\Delta + 2h_1 \leq 81n, \tag{1}$$

proving Theorem 2 (normalized weights $(10, 4, 8, 1, 2, 4)/29$, Figure 3; formalized as `bound_81_29_full_slack`). Replacing (6b) by (6c) with integer multipliers $(30, 3, 24, 3, 6, 4)$ yields

$$87t + 135\Delta + 4d + 2h_1 \leq 243n \tag{2}$$

(`bound_81_29_with_d_slack`). Dropping (5″) and relaxing $\hat{h} \leq h$ in (4″), weights $(85, 34, 80, 12, 39)$ on normalized (3), relaxed (4″), (5), (6a), (6c) give $500t + 7h + 28h_1 \leq 1398n$, proving Theorem 4 (`bound_with_private_slack`).

Proof of Corollary 3 (Manuscript Only). Suppose $n = \nu(G) > 0$, so $|O'| = 3n - a \geq 2n > 0$. In Section 3.1, the constant bit assignment selects exactly $E(A) \cup O'$ ($D = \emptyset$, cost $3n + 2a$). Pick any $e \in O'$; because every triangle of G' contains at least two blue edges, $E(A) \cup (O' \backslash \{e\})$ still covers G. Hence the constant assignment produces a cover of size at least $t + 1$, while every assignment covers G (cost $\geq t$) and occurs with positive probability. Therefore the expected cover size strictly exceeds t, so clearing denominators in (4″) leaves at least one integer unit of slack:

$$4t + 1 \leq 6n + 10a + 6b - 4c + \hat{h} \qquad (n > 0). \tag{3}$$

Because the cleared row (4″) enters (1) with multiplier 1, we obtain $29t + 45\Delta + 2h_1 \leq 81n - 1$, giving $29\tau(G) \leq 81\nu(G) - 1$ (saving a full edge over $\lfloor 81n/29 \rfloor$ whenever $29 \mid n$). Applying this bound separately to each of q triangle-bearing connected components and summing yields $29\tau(G) \leq 81\nu(G) - q$. □

4. INFINITE-DEPTH LP BARRIERS, CONDITIONAL SLICES, AND FINITE WITNESSES

Let (p_r, u_r) denote the maximum packing size and private-spine count at depth $r \geq 0$ of iterated blue-edge deletions in $\mathcal{S}$, so $(p_0, u_0) = (c, h)$ and $(p_1, u_1) = (d, h_1)$, and set $H_r = 2\sum_{j<r} p_j$. Consider the infinite-depth rational relaxation `IsValidInfiniteHierarchyPoint8129` in nonnegative rationals $(n, t, a, m, b, \hat{h}, h_0, p, u)$ enforcing (3)–(5″) together with

$$a + m \leq n, \quad b \leq m, \quad p_0 \leq b, \quad u_0 = \hat{h} + h_0,$$
$$p_{r+1} + u_r \leq p_r, \qquad a + 3m - b + H_K \leq 3n,$$
$$n - a \leq 3m - b - 2\hat{h},$$

for all $r, K \geq 0$, and at every depth $r \geq 0$ all four colored bounds

$$t \leq L + H_r + \min \begin{Bmatrix} 3p_r - u_r, & (8p_r - 2u_r)/3, \\ 3p_r - 2p_{r+1} + u_{r+1} - u_r, \\ 3p_r - p_{r+1} - p_{r+2} - u_r \end{Bmatrix}. \tag{4}$$

(In graphs, the old-packing hit inequality $n - a \leq 3m - b - 2\hat{h}$ holds because each of the $n - a$ untouched all-blue triangles in P cannot hit both pages of a root and must therefore meet one of the $3m - b - 2\hat{h}$ blue edges of P' outside the roots.) The Lean 4 kernel proves that 81/29 is the exact optimum of

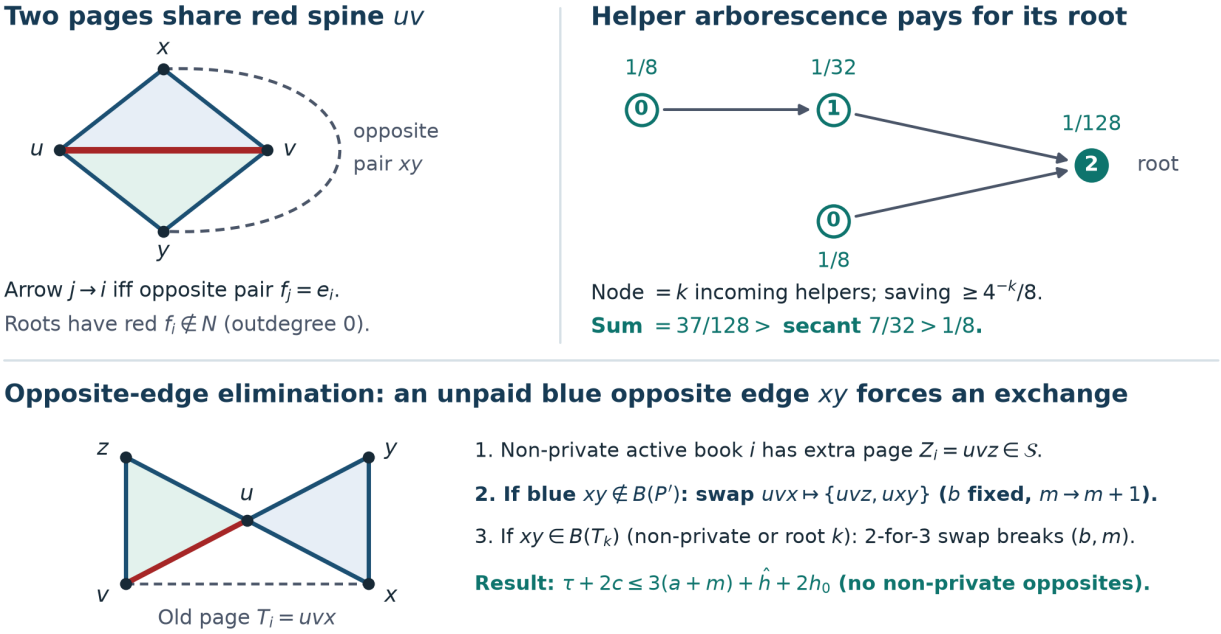


Figure 2. Top left: two-page book on red spine uv with opposite pair xy, defining helper arcs $j \to i$ if and only if $f_j = e_i$. Top right: four-book helper arborescence with indegrees $(0,1,0,2)$, yielding local savings $\sum 4^{-k}/8 = 37/128 > 7/32 > 1/8$. Bottom: opposite-edge elimination when an unpaid blue opposite edge xy lies outside $B(P')$, replacing uvx by edge-disjoint $\{uvz, uxy\}$ at fixed b and size $m+1$.

this entire infinite-depth hierarchy, with unique primal ray and unique six-row dual certificate.

Theorem 5 (Exact 81/29 Infinite-Depth LP Barrier). *Every feasible point of the infinite-depth* 81/29 *hierarchy satisfies* $t \le (81/29)n$, *and there exists a positive-scale feasible point attaining* $t = (81/29)n$, *namely*

$$(n,t,a,m,b,\hat{h},h_0) = (29,81,6,23,18,6,0),\quad p = u = (6,0,0,\dots). \tag{5}$$

Proof. Upper bound (1) holds from (3)–(5″) and the $r = 0,1$ instances of (4). Direct substitution verifies (5) across all $r, K \ge 0$: $L = 69$, $H_0 = 0$, $H_{k+1} = 12$, blue budget uses 57 ($K = 0$) and 69 ($K \ge 1$) against 87, hit-degree reads $23 \le 39$, and every cover bound evaluates to 81. Even imposing the hypothetical Tuza-on-$\mathcal{S}$ bound $t \le L + 2p_0$ gives $81 = 69 + 12$ (`primal8129_satisfies_tuza_S`). □

Theorem 6 (Primal Extreme-Ray Uniqueness). *Every feasible point of the infinite-depth* 81/29 *hierarchy attaining* $t = (81/29)n$ *satisfies*

$$(a,m,b,p_0,\hat{h},h_0,u_0) = (6,23,18,6,6,0,6)\,\frac{n}{29}$$

and $p_{k+1} = u_{k+1} = 0$ *for every* $k \ge 0$.

Proof. Setting $t = 81n/29$ in (2) forces $\Delta = p_1 = u_1 = 0$. Because all six weights in (1) are strictly positive, all six rows hold with equality, solving uniquely for $(a,m,p_0,b,\hat{h},h_0,u_0)$. Induction on $p_{k+2} + u_{k+1} \le p_{k+1}$ forces the entire tail $p_{k+1} = u_{k+1} = 0$. □

Theorem 7 (Six-Row Dual Certificate Uniqueness). *Any rational vector* $(y_3, y_4, y_5, y_{5''}, y_{6a}, y_{6b})$ *with sum* 1, *satisfying the five elimination inequalities for* a,b,c,d,h_0 *on normalized rows* (3)–(6b) *and objective* $3 - \frac{3}{2}y_4 \le 81/29$, *equals* $(10,4,8,1,2,4)/29$.

Proof. Write the five nonnegative elimination slacks as $A = y_3 - \frac{5}{2}y_4$, $B = y_{6a} + y_{6b} - \frac{3}{2}y_4$, $C = y_4 + y_5 + 2y_{5''} - 3y_{6a} - 2y_{6b}$, $D = y_5 + 2y_{6a} - 3y_{6b}$, and $H = y_{6a} - 2y_{5''}$. The objective forces $y_4 \ge 4/29$, while the exact identity

$$10(1 - \tfrac{29}{4}y_4) = 10A + 31B + 9C + D + 4H \ge 0$$

forces $y_4 = 4/29$ and $A = B = C = D = H = 0$, solving uniquely to $(10,4,8,1,2,4)/29$. □

Before opposite-edge elimination, the Stage-1 hierarchy (`IsValidInfiniteHierarchyPoint`, omitting (5″) and relaxing $\hat{h} \le h$) stalls similarly at 699/250 across every depth (`infinite_depth_lp_exact_699_250`).

Proposition 8 (Pre-Elimination Stage-1 LP Barrier). *Every feasible point of the pre-elimination infinite-depth hierarchy satisfies* $t \le (699/250)n$, *attained at positive scale by* $(n,t,a,m,b) = (250,699,51,199,159)$ *with* $p = (42,9,3,0,\dots)$ *and* $u = (0,0,3,0,\dots)$.

Conditional graph slices and tightness families. Two further conic combinations on arbitrary graphs (both formalized in `ExtremalAndDuality.lean`) show that surpassing 282/101 requires roots ($\hat{h} > 0$) and surpassing 36/13 requires private spines ($h > 0$):

$$202t + 312\Delta + 8h_1 \le 564n + \hat{h},$$
$$26t + 36\Delta + 2(d - h_1) \le 72n + 3\hat{h} + 8h_0.$$

Table 1 lists all eight conditional graph bounds proved in `ExtremalAndDuality.lean` (each becomes strict for $n > 0$ via (3)

Weighted six-row certificate (each row normalized to $t \le C_i$) Evaluated at (*): $n = 29, t = 81$

Cover	Weight	n	a	m	b	c	d	$\hat{h}$	h_0	h_1	$C_i(*)$
(3)	**10**	+30	−10	·	·	·	·	·	·	·	81
(4″)	**4**	+6	+10	·	+6	−4	·	+1	·	·	81
(5)	**8**	·	+24	+24	·	−8	−8	·	·	·	81
(5″)	**1**	·	+3	+3	·	−2	·	+1	+2	·	81
(6a)	**2**	·	+6	+6	−2	+6	−4	−2	−2	+2	81
(6b)	**4**	·	+12	+12	−4	+8	+12	·	·	−4	81
Sum	**29**	**36**	**45**	**45**	**0**	**0**	**0**	**0**	**0**	**−2**	**81/29**

Combined identity: $29t + 45(n - a - m) + 2h_1 \le 81n$ **(unique 6-row dual** $(10, 4, 8, 1, 2, 4)/29$**).**

Unique primal ray (*) & one-row deletions

$(n, t, a, m, b, \hat{h}, h_0) = (29, 81, 6, 23, 18, 6, 0)$; $p = u = (6, 0, \dots)$

All 6 covers equal 81; even $\tau(S) \le 2\nu(S)$ gives $81 = 69 + 12$.

Constant vertex coloring costs $3n + 2a = 99 > 81$ (strict).

Drop (3) or (4″)	3
Drop (5)	37/13 ≈ 2.8462
Drop (6a)	45/16 = 2.8125
Drop (5″)	699/250 = 2.7960
Drop (6b) or (6c) alone	**81/29 ≈ 2.7931**

Conditional graph bounds (all 8 slices)

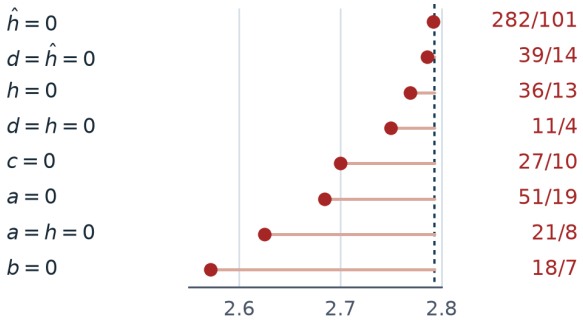


Figure 3. Top: weighted six-row certificate demonstrating exact cancellation of $b, c, d, \hat{h}, h_0$ and value 81 on every normalized cover at the unique extremal ray (*). Bottom left: extremal ray (*), constant-coloring cost $99 > 81$, and exact one-row deletion optima from the seven-row system. Bottom right: all eight conditional graph bounds under zero-coordinate packing hypotheses (all proved directly on arbitrary graphs in `ExtremalAndDuality.lean`).

because every certificate places positive weight on (4″); by contrast, setting $d = 0$, $h_0 = 0$, or $h_1 = 0$ leaves (5) feasible).

Table 1. Conditional graph upper bounds ($n > 0$) proved in `ExtremalAndDuality.lean`, with normalized multipliers in seven-row order (3, 4″, 5, 5″, 6a, 6b, 6c).

Hypothesis	Bound	Multipliers
$\hat{h} = 0$	282/101	(35, 14, 28, 3, 6, 0, 15)
$d = \hat{h} = 0$	39/14	(5, 2, 4, 0, 0, 3, 0)
$h = 0$	36/13	(5, 2, 0, 3, 2, 1, 0)
$d = h = 0$	11/4	(5, 2, 0, 2, 0, 3, 0)
$c = 0$	27/10	(5, 2, 0, 0, 3, 0, 0)
$a = 0$	51/19	(0, 4, 8, 1, 2, 4, 0)
$a = h = 0$	21/8	(0, 2, 0, 3, 2, 1, 0)
$b = 0$	18/7	(5, 2, 0, 0, 0, 0, 0)

On the lower-bound side, componentwise additivity (`nu_disjointCopy`, `tau_disjointCopy`) together with K_4 ($\nu = 1, \tau = 2$) and K_5 ($\nu = 2$ by degree parity; $\tau = 4$ via Mantel's theorem on 5 vertices, $4e^2 \le 25e \implies e \le 6$) gives two infinite families attaining Conjecture 1 in Lean 4.

Proposition 9 (Infinite K_4 Sharpness Family). *For every $k \in \mathbb{N}$, the disjoint union $k \cdot K_4$ satisfies $\tau(k \cdot K_4) = 2\,\nu(k \cdot K_4)$ (with $\nu = k$ and $\tau = 2k$).*

Proposition 10 (Infinite K_5 Sharpness Family). *For every $k \in \mathbb{N}$, the disjoint union $k \cdot K_5$ satisfies $\tau(k \cdot K_5) = 2\,\nu(k \cdot K_5)$ (with $\nu = 2k$ and $\tau = 4k$).*

Finally, exact rational arithmetic in `recompute.py` certifies the companion finite examples, LP ablation ladder, and exhaustive local two-book partition checks (none of which is needed for Theorem 2).

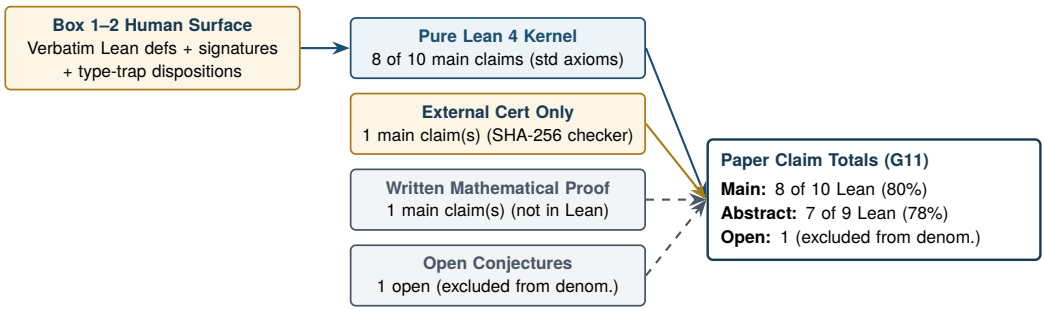
Proposition 11 (Companion 32-Vertex Gadget and Finite Checks — Cert Only). *Exact rational verification in `recompute.py` establishes: (i) the colored graph $H_{14,3,1}$ on 32 vertices and 62 edges (14 red, 48 blue, 32 one-red triangles) admits a three-layer maximum packing sequence of sizes (14, 3, 1) and private counts (0, 0, 1) whose three local cover formulas all equal 36, whereas $\nu = \tau = 14$; (ii) primal–dual simplex solves confirm the ablation ladder (165/59 → 699/250 → 3687/1319 → 109/39 → 4071/1457 → 81/29), all seven one-row deletions, and face-width uniqueness on optimal primal and dual faces; and (iii) exhaustive enumeration of all 115,975 set partitions of 10 roles (non-private Case 1: 28 admissible, 0 obstructions) and all 21,147 partitions of 9 roles (root Case 2: 12 admissible, 0 obstructions) independently confirms the two-book exchange analysis, alongside Wang's point (12, 39, 39, 12, 0, 0)/59 (165 on Wang's rows vs. 141 on unrelaxed (5)).*

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 36f5eec0342e (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Open in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) · PASS: sorry/admit: 0 (per committed receipt) · PASS: native_decide: 0 (per committed receipt) · PASS: Custom axioms & meta: 0 (per committed receipt) · PASS: Coverage: 160/510 public theorems (source-level approx) · WARN: Fidelity: 8/8 exact (draft cross-family) · WARN: Witnesses: pending gate replay · WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

TuzaBound.nu (TuzaBound/Basic/Graph.lean:110-110) — Section 1 (triangle packing number nu(G))

Context binders in scope: `variable {V : Type*} [Fintype V] [DecidableEq V]`

```
noncomputable def nu (G : SimpleGraph V) : ℕ := packingNumber (graphTriangles G)
```

TuzaBound.tau (TuzaBound/Basic/Graph.lean:113-113) — Section 1 (triangle transversal number tau(G))

Context binders in scope: `variable {V : Type*} [Fintype V] [DecidableEq V]`

```
noncomputable def tau (G : SimpleGraph V) : ℕ := transversalNumber (graphTriangles G)
```

TuzaBound.graphTriangles (TuzaBound/Basic/Graph.lean:75-76) — Section 2 (triangle family of a simple graph)

Context binders in scope: `variable {V : Type*} [Fintype V] [DecidableEq V]`

```
noncomputable def graphTriangles (G : SimpleGraph V) : Finset (Finset (Sym2 V)) :=
  trianglesOn (graphEdges G)
```

TuzaBound.IsPacking (TuzaBound/Basic/Packing.lean:45-46) — Section 2 (edge-disjoint triangle packing)

Context binders in scope: `variable {E : Type*} [DecidableEq E]`

```
def IsPacking (F P : Finset (Finset E)) : Prop :=
  P ⊆ F ∧ (P : Set (Finset E)).Pairwise Disjoint
```

TuzaBound.IsCover (TuzaBound/Basic/Packing.lean:53-54) — Section 2 (edge transversal predicate)

Context binders in scope: `variable {E : Type*} [DecidableEq E]`

```
def IsCover (F : Finset (Finset E)) (C : Finset E) : Prop :=
  ∀ T ∈ F, (T ∩ C).Nonempty
```

TuzaBound.IsValidInfiniteHierarchyPoint8129 (TuzaBound/ExtremalAndDuality.lean:58-84) — Section 4 (infinite-depth 81/29 LP hierarchy)

```
structure IsValidInfiniteHierarchyPoint8129
    (n t a m b h_hat h0 : ℚ) (p u : ℕ → ℚ) : Prop where
  nonneg_n : 0 ≤ n
  nonneg_t : 0 ≤ t
  nonneg_a : 0 ≤ a
  nonneg_m : 0 ≤ m
  nonneg_b : 0 ≤ b
  nonneg_h_hat : 0 ≤ h_hat
  nonneg_h0 : 0 ≤ h0
  nonneg_p : ∀ r, 0 ≤ p r
```

```
  nonneg_u : ∀ r, 0 ≤ u r
  pack_am : a + m ≤ n
  pack_bm : b ≤ m
  pack_p0 : p 0 ≤ b
  split_u0 : h_hat + h0 = u 0
  disj_pu : ∀ r, p (r + 1) + u r ≤ p r
  blue_budget : ∀ K, a + 3 * m - b + bluePrefix p K ≤ 3 * n
  hit_degree : n - a ≤ 3 * m - b - 2 * h_hat
  bound_3 : t ≤ 3 * n - a
  bound_4'' : t ≤ (3 / 2 : ℚ) * n + (5 / 2 : ℚ) * a + (3 / 2 : ℚ) * b - p 0 +
    (1 / 4 : ℚ) * h_hat
  bound_5 : t ≤ 3 * a + 3 * m - p 0 - p 1
  bound_5'' : t ≤ 3 * a + 3 * m - 2 * p 0 + h_hat + 2 * h0
  triv_r : ∀ r, t ≤ 3 * a + 3 * m - b + bluePrefix p r + 3 * p r - u r
  col_r : ∀ r, t ≤ 3 * a + 3 * m - b + bluePrefix p r + (8 / 3 : ℚ) * p r - (2 / 3 :
    ↪ ℚ) * u r
  book2_r : ∀ r, t ≤ 3 * a + 3 * m - b + bluePrefix p r + 3 * p r - 2 * p (r + 1) + u
    ↪ (r + 1) - u r
  book3_r : ∀ r, t ≤ 3 * a + 3 * m - b + bluePrefix p r + 3 * p r - p (r + 1) - p (r
    ↪ + 2) - u r
```

TuzaBound.disjointCopy (TuzaBound/ExtremalAndDuality.lean:687-690) — Section 4 (disjoint union k * G)

Context binders in scope: `variable {W : Type*} [Fintype W] [DecidableEq W]`

```
def disjointCopy (k : ℕ) (G : SimpleGraph W) : SimpleGraph (Fin k × W) where
  Adj x y := x.1 = y.1 ∧ G.Adj x.2 y.2
  symm := ⟨fun _ _ ⟨h1, h2⟩ => ⟨h1.symm, h2.symm⟩⟩
  loopless := ⟨fun x ⟨_, h2⟩ => G.loopless.irrefl x.2 h2⟩
```

Selected Concrete Sanity Examples (checked in Lean; examples only, not global identity proofs): TuzaBound.nu_complete4 (nu (τ : SimpleGraph (Fin 4)) = 1); TuzaBound.tau_complete4 (tau (τ : SimpleGraph (Fin 4)) = 2); TuzaBound.nu_complete5 (nu (τ : SimpleGraph (Fin 5)) = 2); TuzaBound.tau_complete5 (tau (τ : SimpleGraph (Fin 5)) = 4).

Supporting Semantic Closure Index (non-Box-1 helpers): TuzaBound.IsTriangle (TuzaBound/Basic/Geometry.lean), TuzaBound.triangleEdges (TuzaBound/Basic/Geometry.lean), TuzaBound.trianglesOn (TuzaBound/Basic/Graph.lean), TuzaBound.graphEdges (TuzaBound/Basic/Graph.lean), TuzaBound.IsMaximumPacking (TuzaBound/Basic/Packing.lean), TuzaBound.exists_maximumPacking (TuzaBound/Basic/Packing.lean), TuzaBound.maximumPacking (TuzaBound/Basic/Packing.lean), TuzaBound.packingNumber (TuzaBound/Basic/Packing.lean), TuzaBound.transversalNumber (TuzaBound/Basic/Packing.lean), TuzaBound.bluePrefix (TuzaBound/FixedPoint.lean), TuzaBound.IsValidInfiniteHierarchyPoint (TuzaBound/FixedPoint.lean), TuzaBound.disjointK4 (TuzaBound/ExtremalAndDuality.lean), TuzaBound.disjointK5 (TuzaBound/ExtremalAndDuality.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

TuzaBound.tuza_bound_81_29_nat (TuzaBound/Main.lean:165-166) `src:7f25a13bbc7271`

Context binders in scope: `variable {V : Type*} [Fintype V] [DecidableEq V]`

```
theorem tuza_bound_81_29_nat (G : SimpleGraph V) : 29 * tau G ≤ 81 * nu G
```

Assumptions: Arbitrary finite vertex type V with Fintype/DecidableEq and arbitrary SimpleGraph V. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TuzaBound.tuza_bound_nat (TuzaBound/Main.lean:201-202) `src:b3b299ae4195d5`

Context binders in scope: `variable {V : Type*} [Fintype V] [DecidableEq V]`

```
theorem tuza_bound_nat (G : SimpleGraph V) : 250 * tau G ≤ 699 * nu G
```

Assumptions: Arbitrary finite vertex type V with Fintype/DecidableEq and arbitrary SimpleGraph V. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TuzaBound.infinite_depth_lp_81_29_exact (TuzaBound/ExtremalAndDuality.lean:210-219) `src:e815d07a5e2b05`

```
theorem infinite_depth_lp_81_29_exact :
    (∀ n t a m b h_hat h0 p u,
      IsValidInfiniteHierarchyPoint8129 n t a m b h_hat h0 p u →
        t ≤ (81 / 29 : ℚ) * n) ∧
    (∃ n t a m b h_hat h0 p u,
      0 < n ∧ IsValidInfiniteHierarchyPoint8129 n t a m b h_hat h0 p u ∧
        t = (81 / 29 : ℚ) * n)
```

Assumptions: None beyond Box 1 hierarchy definition (proves both global <= 81/29 bound and positive-scale attainment at n = 29). Applicability: Unconditional (closed statement; no outer premise witness needed).

TuzaBound.infinite_depth_lp_81_29_primal_unique (TuzaBound/ExtremalAndDuality.lean:224-280) src:a6dc43e6a8f118

```
theorem infinite_depth_lp_81_29_primal_unique
    {n t a m b h_hat h0 : ℚ} {p u : ℕ → ℚ}
    (h : IsValidInfiniteHierarchyPoint8129 n t a m b h_hat h0 p u)
    (ht : t = (81 / 29 : ℚ) * n) :
    a = (6 / 29 : ℚ) * n ∧
    m = (23 / 29 : ℚ) * n ∧
    b = (18 / 29 : ℚ) * n ∧
    p 0 = (6 / 29 : ℚ) * n ∧
    h_hat = (6 / 29 : ℚ) * n ∧
    h0 = 0 ∧
    u 0 = (6 / 29 : ℚ) * n ∧
    ∀ k : ℕ, p (k + 1) = 0 ∧ u (k + 1) = 0
```

Assumptions: Feasible point of IsValidInfiniteHierarchyPoint8129 attaining t = (81/29) * n. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TuzaBound.dual_certificate_81_29_unique (TuzaBound/ExtremalAndDuality.lean:287-315) src:3ac8ddef9b84ca

```
theorem dual_certificate_81_29_unique
    {y3 y4 y5 y5p y6a y6b : ℚ}
    (hsum : y3 + y4 + y5 + y5p + y6a + y6b = 1)
    (h_a : -y3 + (5 / 2 : ℚ) * y4 ≤ 0)
    (h_b : (3 / 2 : ℚ) * y4 - y6a - y6b ≤ 0)
    (h_c : -y4 - y5 - 2 * y5p + 3 * y6a + 2 * y6b ≤ 0)
    (h_d : -y5 - 2 * y6a + 3 * y6b ≤ 0)
    (h_h0 : 2 * y5p - y6a ≤ 0)
    (h_obj : 3 * y3 + (3 / 2 : ℚ) * y4 + 3 * y5 + 3 * y5p + 3 * y6a + 3 * y6b ≤ 81 /
      ↪ 29) :
    y3 = 10 / 29 ∧ y4 = 4 / 29 ∧ y5 = 8 / 29 ∧
      y5p = 1 / 29 ∧ y6a = 2 / 29 ∧ y6b = 4 / 29
```

Assumptions: Normalized six-row multipliers satisfying the five elimination inequalities and objective <= 81/29. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TuzaBound.infinite_depth_lp_exact_699_250 (TuzaBound/FixedPoint.lean:192-199) src:884a745e646277

```
theorem infinite_depth_lp_exact_699_250 :
    (∀ n t a m b p u, IsValidInfiniteHierarchyPoint n t a m b p u →
      t ≤ (699 / 250 : ℚ) * n) ∧
    (∃ n t a m b p u, 0 < n ∧ IsValidInfiniteHierarchyPoint n t a m b p u ∧
      t = (699 / 250 : ℚ) * n)
```

Assumptions: None beyond Box 1 Stage-1 hierarchy definition (global <= 699/250 bound and attainment at n = 250). Applicability: Unconditional (closed statement; no outer premise witness needed).

TuzaBound.tuza_tight_disjointK4 (TuzaBound/ExtremalAndDuality.lean:924-926) src:1d521d8c90fd53

```
theorem tuza_tight_disjointK4 (k : ℕ) :
    tau (disjointK4 k) = 2 * nu (disjointK4 k)
```

Assumptions: Arbitrary natural number k : Nat. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TuzaBound.tuza_tight_disjointK5 (TuzaBound/ExtremalAndDuality.lean:1275-1278) src:20ea0096d26c27

```
theorem tuza_tight_disjointK5 (k : ℕ) :
    tau (disjointK5 k) = 2 * nu (disjointK5 k)
```

Assumptions: Arbitrary natural number k : Nat. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
`IsValidInfiniteHierarchyPoint`	`nat_sub`	`- a`	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
`IsValidInfiniteHierarchyPoint`	`nat_div`	`/ 2`	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
`IsValidInfiniteHierarchyPoint`	`nat_sub`	`- p`	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
`IsValidInfiniteHierarchyPoint`	`nat_div`	`/ 4`	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
`IsValidInfiniteHierarchyPoint`	`nat_sub`	`- b`	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
IsValidInfiniteHierarchyPoint	nat_sub	- u	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint	nat_div	/ 3	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
IsValidInfiniteHierarchyPoint	nat_sub	- (2 / 3 : ℚ)	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint	nat_sub	- 2	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint8129	nat_sub	- b	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint8129	nat_sub	- a	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint8129	nat_sub	- 2	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint8129	nat_div	/ 2	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
IsValidInfiniteHierarchyPoint8129	nat_sub	- p	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint8129	nat_div	/ 4	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
IsValidInfiniteHierarchyPoint8129	nat_sub	- u	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
IsValidInfiniteHierarchyPoint8129	nat_div	/ 3	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
IsValidInfiniteHierarchyPoint8129	nat_sub	- (2 / 3 : ℚ)	intended_convention	All LP hierarchy coordinates have type Q (rational field); subtraction is exact with no Nat underflow.
dual_certificate_81_29_unique	nat_div	/ 2	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
dual_certificate_81_29_unique	nat_div	/ 29	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
infinite_depth_lp_81_29_exact	nat_div	/ 29	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
infinite_depth_lp_81_29_primal_unique	nat_div	/ 29	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
infinite_depth_lp_exact_699_250	nat_div	/ 250	intended_convention	Constant rational fraction over Q (2, 3, 4, 29, or 250 nonzero); no integer floor division.
maximumPacking	classical_choose	Classical.choose	intended_convention	Classical.choose selects an existing maximum packing from exists_maximumPacking; packingNumber is invariant across choices.

[a] Covers the specified Lean 4 hazard set (Nat subtraction/division/mod, totalized real/rational division, Real.log/rpow, Finset.range, Classical.choose, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Conj. 1: Tuza conjecture tau(G) <= 2*nu(G) for all finite simple graphs	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly open conjecture (excluded from G11 denominators).
Thm. 2: Universal bound 29*tau(G) <= 81*nu(G) for every finite simple graph	Lean Kernel	`TuzaBound.tuza_bound_81_29_nat`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 graph definitions only (0 sorry, standard axioms).
Cor. 3: Strict inequality 29*tau(G) <= 81*nu(G) - 1 when nu(G) > 0 (-q over q components)	Written	Written proof	Lean: none; fidelity: n/a	Manuscript constant-coloring averaging proof (separate from pinned Lean library).
Thm. 4: Stage-1 helper-coupling bound 250*tau(G) <= 699*nu(G)	Lean Kernel	`TuzaBound.tuza_bound_nat`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 graph definitions only (0 sorry, standard axioms).
Thm. 5: Exact 81/29 infinite-depth LP barrier attained at (29,81,6,23,18,6,0)	Lean Kernel	`TuzaBound.infinite_depth_lp_81_29_exact`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel on rational hierarchy IsValidInfinite HierarchyPoint8129.
Thm. 6: Primal extreme-ray uniqueness across all coordinates and depths	Lean Kernel	`TuzaBound.infinite_depth_lp_81_29_primal_unique`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel on IsValid InfiniteHierarchyPoint8129 with t = (81/29)*n.
Thm. 7: Unique six-row dual weights (10,4,8,1,2,4)/29	Lean Kernel	`TuzaBound.dual_certificate_81_29_unique`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel on normalized six-row dual system over Q.
Prop. 8: Pre-elimination Stage-1 LP barrier 699/250 attained at n=250	Lean Kernel	`TuzaBound.infinite_depth_lp_exact_699_250`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel on Stage-1 rational hierarchy IsValidInfinite HierarchyPoint.
Prop. 9: Infinite K4 sharpness family tau(k*K4) = 2*nu(k*K4)	Lean Kernel	`TuzaBound.tuza_tight_disjointK4`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 definitions (arbitrary k : Nat).
Prop. 10: Infinite K5 sharpness family tau(k*K5) = 2*nu(k*K5)	Lean Kernel	`TuzaBound.tuza_tight_disjointK5`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 definitions via Mantel on 5 vertices (arbitrary k : Nat).
Prop. 11: Companion 32-vertex gadget H_{14,3,1}, LP ablations & partition checks	Cert Only	recompute.py (37f2a9d9d06e) [regenerable; replay pending]	External cert only; fidelity: n/a	Exact rational Python (recompute.py, regenerable_deterministic; not required for Thm. 2).

[a] Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 7fffe4b6a5d7 | Lean mode: committed_receipt | Closure SHA-256: 7f37d733e800 | Git anchor: 36f5eec0342e | Axioms SHA-256: bffac9fa3001 | Fidelity SHA-256: 290825ce1c37. External validation receipts: Pinned commit | Palomar record.

6. PRIOR WORK AND OPEN QUESTIONS

Tuza posed Conjecture 1 at the 1981 Eger colloquium (proceedings p. 888 [3]). Beyond sparse unions of K_4 and K_5, Baron and Kahn [1] constructed arbitrarily large dense graphs for any $\alpha > 0$ with $\tau > (1 - o(1))|E|/2$ and $\nu < (1 + \alpha)|E|/4$, proving that τ/ν approaches 2 even at positive density. Haxell [2] [to check] introduced the old-edge packing decomposition and proved 66/23; Yi [6] sharpened this to 63/22 (and $(162 + 4\sqrt{3})/59$) while identifying 54/19 as the barrier of Haxell's four-inequality system; Wang [4, 5] exploited completeness of the one-red residual family to reach 165/59 and released the MIT-licensed Lean 4 library `sichen-wang/tuza-lean` that `lean/TuzaBound/` directly extends (preserving copyright headers and license across 29 Lean files: 14 ported upstream modules, 8 adapted modules, and 7 added modules). Primary texts of Tuza [3], Baron–Kahn [1], Yi [6], Wang [4], and Wang's Lean repository [5] were read; for Haxell [2] [to check], the publisher abstract and bibliographic record were checked, and the full original proof remains to be read by the authors. MathSciNet and Scopus were not searched.

Corollary 3 rules out exact equality $29\tau(G) = 81\nu(G)$ on every graph with a triangle, yet leaves open whether a sequence of graphs can satisfy $\tau/\nu \to 81/29$ (any limit point would have to converge to the unique ray (5), with almost all Q pages being roots and $d/n \to 0$). Breaking 81/29 requires ruling out the proportions (5) or capturing proportional savings from repeated edges in the random cover; Conjecture 1 itself remains open.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources live in `papers/tuza-triangle-cover/draft4/` and the Lean 4 library in `lean/TuzaBound/` (Lean `v4.35.0` + Mathlib, extending Sichen Wang's MIT-licensed `sichen-wang/tuza-lean`; public Lean proof archived at `mathdiscoveries/tuza-`

`triangle-cover` and listed on Palomar as `PALOMAR-2026-11-22-000018` v2, mechanical Lean checking plus automated editorial review). Running `python3 papers/tuza-triangle-cover/draft4/recompute.py` runs the Section 1 Python box verbatim and verifies $H_{14,3,1}$, all LP certificates, and all partition checks. AI research agents built and directed by Tyler B. Hughes discovered the refinements, produced the Lean 4 formalization and figures, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 23, 2026

REFERENCES

(1) Baron, J. D.; Kahn, J. Tuza's conjecture is asymptotically tight for dense graphs. *Combin. Probab. Comput.* **2016**, *25* (5), 645–667. DOI: 10.1017/S0963548316000067.

(2) Haxell, P. E. Packing and covering triangles in graphs. *Discrete Math.* **1999**, *195* (1–3), 251–254. DOI: 10.1016/S0012-365X(98)00183-6 [to check].

(3) Tuza, Zs. Conjecture. In *Finite and Infinite Sets* (Eger, 1981), Vol. II; Hajnal, A., Lovász, L., Sós, V. T., Eds.; Colloq. Math. Soc. János Bolyai **37**; North-Holland: Amsterdam, **1984**; p 888.

(4) Wang, S. A bound below 2.8 for Tuza's conjecture. Preprint, **2026**; arXiv:2609.13831.

(5) Wang, S. `tuza-lean`: Lean 4 formalization accompanying "A bound below 2.8 for Tuza's conjecture" (MIT license). GitHub repository, **2026**; `https://github.com/sichen-wang/tuza-lean`.

(6) Yi, L. An improved upper bound for Tuza's conjecture via 2-colorable triangle families. Preprint, **2026**; arXiv:2608.23010.