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Working Draft

Two Mirrors: Pell Gates, Pronic Pairing, and Settled Neighbours for Square and Doubled-Square Block Reversals

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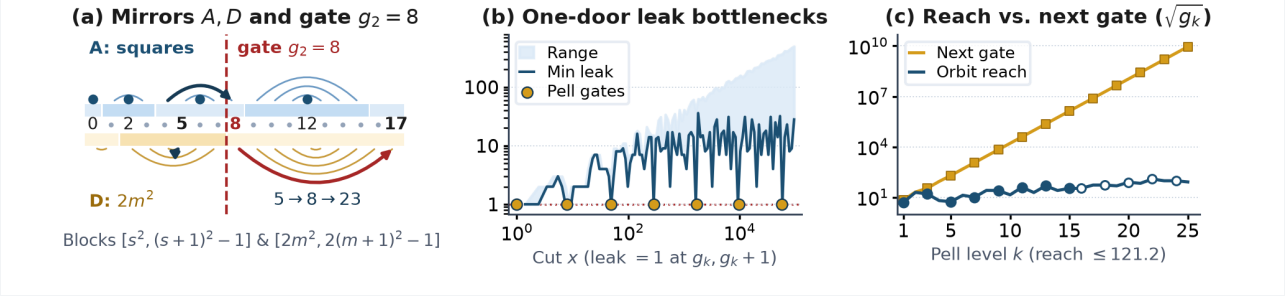
Supporting Information: `recompute.py`, audited Lean 4 library (`lean/TwoMirrors/`), and trust-boundary ledgers

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Let A and D reverse each block $[s^2,(s+1)^2-1]$ and $[2m^2,2(m+1)^2-1]$ of $\mathbb{N}$. Their bijection $f = A\circ D$ has zero drift and no exact walls ($s^2 \neq 2m^2$ for $m \geq 1$). We conjecture that every orbit of f is periodic. We prove every cycle of f has length divisible by 4. At every Pell level $g = \min(s^2, 2m^2)$ ($s^2 - 2m^2 = \pm 1$), a unique $x < g$ exits to $\geq g$ and a unique $y \geq g$ enters $< g$, both in the $\mathbb{Z}$-orbit of g; thus f has at most one runaway orbit, every $n < g$ is periodic if and only if g is periodic, and global periodicity holds if and only if every pronic $s(s+1)$ pairs with another pronic under alternating $D, A, D, \ldots$ swaps. Every Pell level spawns a 4-cycle pairing $(g-1)g$ with $(g+1)(g+2)$. Pairing A with triangular blocks makes all orbits periodic, with pronic blocks makes none periodic, and with $4m^2$ blocks makes all periodic. All ten structural and pure-kernel results (including periodicity for $n < 1{,}681$) are formalized in Lean 4 (`lean/TwoMirrors/`); deterministic Python checks close gate orbits through $g_{15} = 76{,}069{,}501{,}249$.

Verification Scope (G11): WARN: Human referee: pending
Main results: 10 of 11 in Lean Kernel (91%) · 1 Cert Only · 1 open (not counted)
Abstract claims: 10 of 11 in Lean Kernel (91%) · 1 Cert Only · 1 open (not counted)

Full Trust Boundary: Section 5.

MSC 2020: 11B83, 37E05, 11D09, 05A05 **Keywords:** block-reversal involutions, Pell equations, infinite meanders, discrete dynamical systems, Lean 4 formalization

1. INTRODUCTION AND MAIN RESULTS

Reversing consecutive blocks of integers in two incompatible ways produces an elementary, zero-drift permutation of $\mathbb{N} = \{0, 1, 2, \ldots\}$ whose global orbit structure behaves like a reversible Collatz problem. Define two involutions on $\mathbb{N}$:

- the *square mirror* A, reversing each square block $[s^2, (s+1)^2 - 1]$ ($s = \lfloor\sqrt{n}\rfloor$) via $A(n) = 2s(s+1) - n$;
- the *doubled-square mirror* D, reversing each block $[2m^2, 2(m+1)^2 - 1]$ ($m = \lfloor\sqrt{\lfloor n/2\rfloor}\rfloor$) via $D(n) = (2m+1)^2 - n$.

One forward step of the *two-mirrors map* is $f(n) = A(D(n))$. Since A and D are involutions, f is a bijection of $\mathbb{N}$ with inverse $f^{-1} = D \circ A$: every orbit is either a finite cycle or a doubly infinite trajectory escaping to infinity in both directions.

For example, $f(0) = A(D(0)) = A(1) = 3$, giving the 4-cycle $0 \to 3 \to 6 \to 1 \to 0$, while 2 initiates a 28-cycle staying within $[2, 47]$ that passes through 8 and pairs the pronic numbers 2 and 12. Because block lengths grow in the irrational ratio $\sqrt{2}$ and $s^2 \neq 2m^2$ for all $m \geq 1$, no threshold $x \geq 1$ forms an exact common wall for A and D. Yet near-coincidences at solutions of the Pell equation $s^2 - 2m^2 = \pm 1$ pinch the dynamics down to a single crossing.

Conjecture 1 (Universal Periodicity of $f = A \circ D$ — Unsettled). *Every natural number $n \in \mathbb{N}$ is periodic under $f = A \circ D$; that is, there exists $L \geq 1$ with $f^L(n) = n$.*

While Conjecture 1 remains unsettled, we prove a structural reduction cornering any counterexample to a single orbit and settling three neighbour systems. All ten proved theorems below are formalized in Lean 4 (`v4.35.0` with Mathlib, in `lean/TwoMirrors/`; Section 5).

Theorem 2 (Modulo-4 Cycle Lengths). *For all $n, L \in \mathbb{N}$, if $f^L(n) = n$, then $4 \mid L$.*

The central structural result governs every *Pell level* $g = \min(s^2, 2m^2)$ where $s, m \geq 1$ satisfy $s^2 - 2m^2 = \pm 1$ ($g_1 = 1, g_2 = 8, g_3 = 49, g_4 = 288, g_5 = 1{,}681, \ldots$, OEIS A001108).

Theorem 3 (The Pell Gate Lemma). *Let $s, m \geq 1$ satisfy $s^2 - 2m^2 = \pm 1$, and set $g = \min(s^2, 2m^2)$. Then:*

1. *there is a unique $x < g$ such that $f(x) \geq g$;*
2. *there is a unique $y \geq g$ such that $f(y) < g$;*
3. *g lies in the $\mathbb{Z}$-orbit of both x and y under f.*

Theorem 3 forces every orbit disjoint from the $\mathbb{Z}$-orbit of g to remain trapped wholly inside $[0, g)$ or wholly inside $[g, \infty)$, yielding two immediate consequences.

Corollary 4 (At Most One Runaway Orbit). *If $a, b \in \mathbb{N}$ are both non-periodic under f, then $b = f^z(a)$ for some $z \in \mathbb{Z}$. Hence f has at most one infinite orbit.*

Corollary 5 (Pell-Level Gate Reduction). *For all $s, m \geq 1$ with $s^2 - 2m^2 = \pm 1$ and $g = \min(s^2, 2m^2)$, every $n < g$ is periodic under f if and only if g is periodic under f.*

Because f is conjugated to f^{-1} by the involution A, whose fixed points are precisely the *pronic numbers* $s(s+1)$ (0, 2, 6, 12, 20, ...), periodicity also reduces to a meander pairing rule. Define the alternating walk $\alpha_0(p) = p$, with $\alpha_{k+1}(p) = D(\alpha_k(p))$ for even k and $\alpha_{k+1}(p) = A(\alpha_k(p))$ for odd k. Say a pronic p *pairs* if $\alpha_k(p)$ is a pronic distinct from p for some $k \geq 1$.

Theorem 6 (Pronic-Pairing Equivalence). *Every $n \in \mathbb{N}$ is periodic under f if and only if every pronic number $p = s(s+1)$ pairs with a second distinct pronic number under alternating $D, A, D, \ldots$ swaps.*

As a companion consequence (also formalized in `TwoMirrors.at_most_one_unpaired_pronic`), any infinite orbit contains a unique pronic, so at most one pronic can fail to pair. In addition, every Pell level generates an explicit short 4-cycle just above g^2.

Theorem 7 (Universal Pell-Level 4-Cycles). *Let $s, m \geq 1$ satisfy $s^2 - 2m^2 = \pm 1$, set $g = \min(s^2, 2m^2)$, $p_0 = (g-1)g$, and $p_1 = (g+1)(g+2)$. Then p_0 and p_1 are pronic, $\alpha_3(p_0) = p_1$, $f^2(p_0) = p_1, f^4(p_0) = p_0$, and $p_0 < D(p_0) < f(p_0) < p_1$.*

Pairing Corollary 5 with a proved-sound binary square-root check inside Lean 4 gives a pure-kernel baseline, extended by deterministic Python checks in Section 4.

Proposition 8 (Pure-Kernel Periodicity Below 1,681). *Every natural number $n < 1{,}681$ is periodic under f.*

Finally, replacing D by three natural neighbouring block mirrors settles the global dynamics completely, highlighting why the irrational-ratio, wall-free pair (A, D) sits precisely at the boundary of current proof methods. Let B reverse triangular blocks $[T_t, T_{t+1} - 1]$ ($T_t = t(t+1)/2$), C reverse pronic blocks $[s(s+1), (s+1)(s+2) - 1]$, and M_4 reverse quad-square blocks $[4m^2, 4(m+1)^2 - 1]$.

Theorem 9 (Square and Triangular Mirrors). *Every $n \in \mathbb{N}$ is periodic under $A \circ B$.*

Theorem 10 (Square and Pronic Mirrors). *No $n \in \mathbb{N}$ is periodic under $A \circ C$.*

Theorem 11 (Square and Quad-Square Mirrors). *Every $n \in \mathbb{N}$ is periodic under $A \circ M_4$.*

Figure 1 summarizes the discovery trajectory and proof boundaries. The following self-contained 9-line Python snippet illustrates the rule and verifies the $g_2 = 8$ Pell gate and its 28-cycle (replayed verbatim in `recompute.py`):

Concept in Action: Gate $g_2 = 8$ and Its 28-Cycle

```
from math import isqrt
A = lambda n: 2*isqrt(n)*(isqrt(n)+1) - n
D = lambda n: (2*isqrt(n//2)+1)**2 - n
f = lambda n: A(D(n))
x, seq = 8, []
while x not in seq: seq.append(x); x = f(x)
assert len(seq) == 28 and len(seq) % 4 == 0
assert [n for n in range(8) if f(n) >= 8] == [5]
assert [n for n in range(8, 200) if f(n) < 8] == [17]
```

Section 2 establishes notation and elementary properties matching `lean/TwoMirrors/Defs.lean`. Section 3 proves Theorems 2–11, Corollaries 4–5, and Proposition 8. Section 4 presents our deterministic computational checks (Proposition 12) alongside recorded higher-range computations and empirical behaviour. Section 5 displays the machine-generated Lean 4 Trust Boundary. Section 6 reviews prior literature and open problems, and Section 7 documents reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND SETUP

All definitions below match `lean/TwoMirrors/Defs.lean` and `Neighbours.lean` verbatim. Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

For any strictly increasing sequence $0 = b_0 < b_1 < b_2 < \ldots$, the *block mirror* associated to (b_k) reverses each finite interval $[b_k, b_{k+1} - 1]$ via $n \mapsto b_k + (b_{k+1} - 1) - n$. For A ($b_s = s^2$, $s = \lfloor \sqrt{n} \rfloor$) and D ($b_m = 2m^2$, $m = \lfloor \sqrt{\lfloor n/2 \rfloor} \rfloor$), this yields

$$A(n) = 2s(s+1) - n, \qquad D(n) = (2m+1)^2 - n,$$

with $s^2 \leq n < (s+1)^2$ and $2m^2 \leq n < 2(m+1)^2$. Since $n \leq s^2 + 2s \leq 2s(s+1)$ and $n \leq 2m^2 + 4m + 1 \leq (2m+1)^2$, neither formula underflows in $\mathbb{N}$. Both A and D preserve their respective blocks and are involutions ($A^2 = D^2 = \mathrm{id}$). Consequently, $f = A \circ D$ is a permutation of $\mathbb{N}$ with inverse $f^{-1} = D \circ A$, packaged in Lean as `fEquiv : Equiv.Perm Nat`. Conjugating by A reverses time:

$$A(f^z(n)) = f^{-z}(A(n)) \qquad \text{for all } z \in \mathbb{Z},\ n \in \mathbb{N}.$$

A square block $[s^2, (s+1)^2 - 1]$ has odd length $2s + 1$, so A fixes precisely its midpoint $s(s+1)$ (a *pronic number*, predicate `IsPronic`). A doubled-square block $[2m^2, 2(m+1)^2 - 1]$ has even length $4m + 2$, so D has no fixed points. Moreover, $s^2 \neq 2m^2$ for every $m \geq 1$ (infinite descent on 2-valuations, `no_exact_walls`), so no positive integer simultaneously opens an A-block and a D-block.

We call $n \in \mathbb{N}$ *periodic* under a map $g : \mathbb{N} \to \mathbb{N}$ (`IsPeriodic`) if there exists $L \geq 1$ with $g^L(n) = n$. For any permutation of $\mathbb{N}$, a forward orbit contained in a finite interval $[0, B)$ is necessarily periodic by the pigeonhole principle (`periodic_of_bdd_orbit`). The *leak* across a threshold $x \geq 1$ is

$$\mathrm{leak}(x) = |\{n < x : f(n) \geq x\}|.$$

Because f is a bijection on $\mathbb{N}$ and $[0, x)$ is finite, $\mathrm{leak}(x)$ also equals $|\{y \geq x : f(y) < x\}|$. Figure 2 displays the meander diagram on $[0, 24]$ and the full 28-step cycle through $g_2 = 8$.

The *Pell recurrence* $(s_1, m_1) = (1, 1)$, $(s_{k+1}, m_{k+1}) = (s_k + 2m_k, s_k + m_k)$ generates all positive integer solutions of $s^2 -$

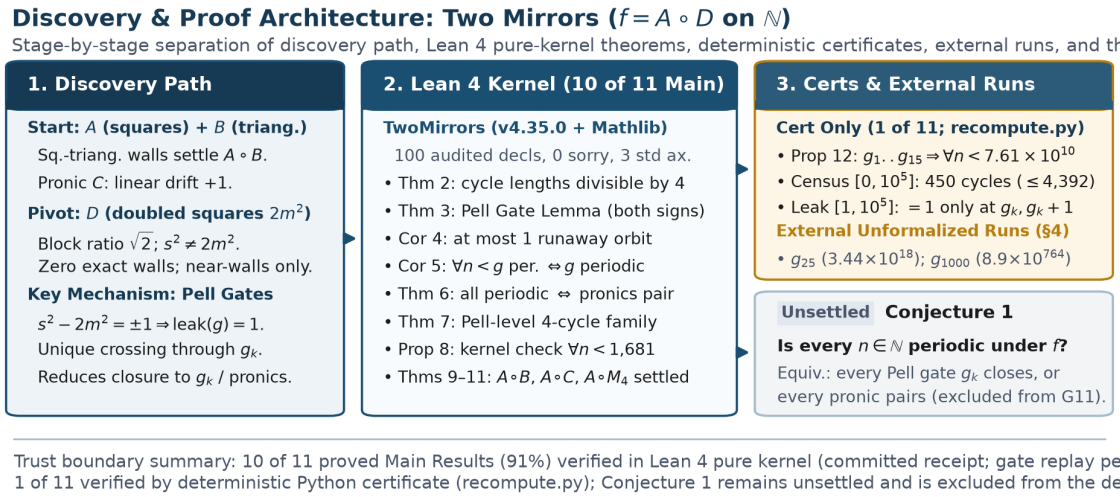


Figure 1. Discovery and proof architecture for the two-mirrors map $f = A \circ D$. Moving from the wall-confined triangular neighbour $A \circ B$ and drifting pronic neighbour $A \circ C$ to the irrational-ratio, wall-free doubled-square mirror D leads to Pell-gate and pronic-pairing reductions. Ten proved Main Results (91%) are formalized in Lean 4's pure kernel, one is verified by deterministic Python certificate (`recompute.py`), external unformalized computations extend checks further (Section 4), and Conjecture 1 remains unsettled.

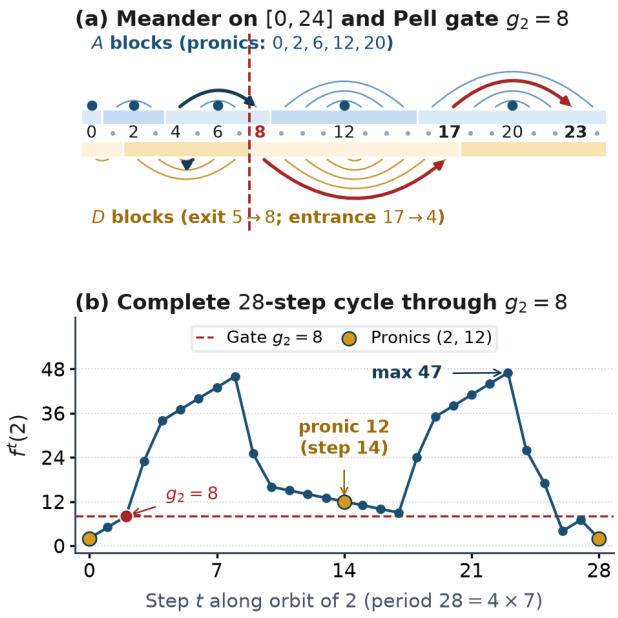


Figure 2. Panel (a): Meander representation of A (upper arcs, pronic fixed points marked by dots) and D (lower arcs) on $[0, 24]$, showing the Pell near-wall at $g_2 = 8$ where 5 is the unique exit ($5 \xrightarrow{D} 4 \xrightarrow{A} 8$) and 17 is the unique entrance ($17 \xrightarrow{D} 8 \xrightarrow{A} 4$). Panel (b): Complete 28-step cycle of 2 passing through $g_2 = 8$, staying in $[2, 47]$, and pairing pronic 2 with pronic 12 at step 14.

$2m^2 = \pm 1$, satisfying $s_k^2 - 2m_k^2 = (-1)^k$ (0-indexed in Lean as `pellPair k` with $(-1)^{k+1}$). The associated *Pell levels* are $g_k = \min(s_k^2, 2m_k^2)$, yielding $1, 8, 49, 288, 1{,}681, 9{,}800, \ldots$, which are strictly increasing and unbounded with $g_{k+1}/g_k \to (1 + \sqrt{2})^2 \approx 5.828427$.

3. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 2. Let $n \in \mathbb{N}$, $m = \lfloor \sqrt{\lfloor n/2 \rfloor} \rfloor$, $y = D(n) = (2m+1)^2 - n$, and $s = \lfloor \sqrt{y} \rfloor$, so $f(n) = A(y) = 2s(s+1) - y$. Subtracting n in $\mathbb{Z}$ gives

$$f(n) - n = 2s(s+1) - (2m+1)^2.$$

Since $s(s+1)$ is even, $2s(s+1) \equiv 0 \pmod 4$, whereas $(2m+1)^2 = 4(m^2+m) + 1 \equiv 1 \pmod 4$. Thus $f(n) - n \equiv 3 \pmod 4$ for every $n \in \mathbb{N}$ (in particular, f has no fixed points). Summing along an orbit of length L yields $f^L(n) - n \equiv 3L \pmod 4$. If $f^L(n) = n$, then $3L \equiv 0 \pmod 4$, forcing $4 \mid L$. □

Proof of Theorem 3. We treat the two sign cases $s^2 = 2m^2 + 1$ and $2m^2 = s^2 + 1$ separately.

Case 1: $s^2 = 2m^2 + 1$, so $g = 2m^2$ and $s \geq 2$. Because $g = 2m^2$ opens a D-block, D preserves $[0, g)$ and $[g, \infty)$ separately. On the A-side, g is the right endpoint of the square block $[(s-1)^2, s^2 - 1] = [(s-1)^2, g]$, so A swaps $(s-1)^2 \leftrightarrow g$ and keeps every other $z < g$ strictly below g and every other $z > g$ at or above $s^2 > g$. Writing $x_0 = D((s-1)^2) < g$ and $y_0 = D(g) \geq g$, it follows that for $x < g$, $f(x) = A(D(x)) \geq g$ holds if and only if $D(x) = (s-1)^2$, i.e., $x = x_0$, with $f(x_0) = g$. Similarly, for $y \geq g$, $f(y) < g$ holds if and only if $D(y) = g$, i.e., $y = y_0$, with $f(y_0) = (s-1)^2 < g$.

Case 2: $2m^2 = s^2 + 1$, so $g = s^2$. Here $g = s^2$ opens an A-block, so A preserves $[0, g)$ and $[g, \infty)$ separately. On the D-side, $g = 2m^2 - 1$ is the right endpoint of the doubled-square block $[2(m-1)^2, 2m^2 - 1] = [x_0, g]$ where $x_0 = 2(m-1)^2 < g$. Thus D swaps $x_0 \leftrightarrow g$ and preserves $[0, g) \setminus \{x_0\}$ and $[g, \infty) \setminus \{g\}$. Consequently, x_0 is the unique $x < g$ with $f(x) \geq g$ (having $f(x_0) = A(g) \geq g$), and $y_0 = g$ is the unique $y \geq g$ with $f(y) < g$ (having $f(g) = A(x_0) < g$).

Single-door connection. In both cases, we have a unique exit $x_0 < g$ and unique entrance $y_0 \geq g$. Consider the forward orbit $f^k(f(y_0))$ $(k \geq 0)$. Starting at $f(y_0) < g$, if this orbit never hit x_0, it would remain inside the finite interval $[0, g)$ forever

and therefore be periodic; applying f^{-1} once would place y_0 inside $[0, g)$, contradicting $y_0 \geq g$. Hence $f^k(f(y_0)) = x_0$ for some $k \geq 0$. In Case 1, $f(x_0) = g$ gives $g = f^1(x_0) = f^{k+2}(y_0)$; in Case 2, $y_0 = g$ gives $g = f^0(y_0) = f^{-(k+1)}(x_0)$. Thus g lies in the $\mathbb{Z}$-orbit of both x_0 and y_0. □

Proof of Corollaries 4 and 5. Let $g = \min(s^2, 2m^2)$ be any Pell level. If $n \in \mathbb{N}$ does not lie in the $\mathbb{Z}$-orbit of g, then by Theorem 3 no iterate $f^z(n)$ can equal the unique exit x_0 or unique entrance y_0, so by induction on $|z|$ the entire $\mathbb{Z}$-orbit of n stays inside $[0, g)$ if $n < g$ and inside $[g, \infty)$ if $n \geq g$. In particular, every non-periodic $a < g$ must lie in the $\mathbb{Z}$-orbit of g (otherwise its forward orbit would stay in $[0, g)$ and be periodic).

For Corollary 4, given non-periodic $a, b \in \mathbb{N}$, choose a Pell level $g_k > \max(a, b)$. Both a and b lie in the $\mathbb{Z}$-orbit of g_k, hence in the same $\mathbb{Z}$-orbit of f.

For Corollary 5, if every $n < g$ is periodic, then the unique exit $x_0 < g$ is periodic; since g lies in the $\mathbb{Z}$-orbit of x_0, g is also periodic. Conversely, if g is periodic, then no $n < g$ can be non-periodic (else n would share the finite orbit of g). □

Proof of Theorem 6. We first show that a pronic p ($A(p) = p$) pairs if and only if it is periodic under f. By induction, $\alpha_{2j}(p) = f^j(p)$ and $\alpha_{2j+1}(p) = D(f^j(p))$. If $\alpha_k(p) = q \neq p$ is pronic for some $k \geq 1$, then $A(q) = q$; for either $k = 2j$ or $k = 2j - 1$, this gives $A(f^j(p)) = f^j(p)$ with $j \geq 1$. Dihedral conjugation yields $A(f^j(p)) = f^{-j}(A(p))$, which equals $f^{-j}(p)$, so $f^{2j}(p) = p$ and p is periodic. Conversely, if p has least period $L > 0$, then $4 \mid L$ by Theorem 2; setting $h = L/2 > 0$, we have $f^h(p) \neq p$ and $A(f^h(p)) = f^{-h}(p) = f^h(p)$, so $\alpha_{2h}(p) = f^h(p)$ is a distinct pronic.

This immediately gives the forward implication of Theorem 6. For the converse, suppose some $a \in \mathbb{N}$ is non-periodic. By dihedral conjugation, $A(a)$ is also non-periodic, so by Corollary 4 there exists $c \in \mathbb{Z}$ with $f^c(a) = A(a)$. If $c = 2k + 1$ were odd, setting $u = f^k(a)$ would give $D(u) = A(f(u)) = A(f^{k+1}(a)) = f^{-(k+1)}(A(a)) = f^{-(k+1)+c}(a) = u$, contradicting that D has no fixed points. Hence $c = 2k$ is even, and $p = f^k(a)$ satisfies $A(p) = f^{-k}(A(a)) = f^{-k+2k}(a) = p$, so p is a pronic in the non-periodic orbit of a. Therefore p is non-periodic and fails to pair. □

Proof of Theorem 7. Let $u = sm \geq 1$. Since $\{s^2, 2m^2\} = \{g, g + 1\}$, multiplying gives the exact Pell product identity $2u^2 = g(g + 1)$. Writing $g' = g - 1 \geq 0$ and $u' = u - 1 \geq 0$, comparing squares yields $2g' + 4 \leq 4u \leq 3g' + 4$. From $2u^2 = g(g + 1)$ we check directly:

1. $p_0 = (g - 1)g$ lies in the D-block $[2(u - 1)^2, 2u^2 - 1]$, so $x = D(p_0) = (2u - 1)^2 - p_0 = g^2 + 3g + 1 - 4u$.
2. Since $g^2 \leq x < (g+1)^2$, x lies in the A-block $[g^2, (g+1)^2 - 1]$, giving $y = A(x) = 2g(g + 1) - x = g^2 + 4u - g - 1$.
3. Since $2u^2 \leq y < 2(u+1)^2$, y lies in the D-block $[2u^2, 2(u+1)^2 - 1]$, giving $D(y) = (2u+1)^2 - y = (g+1)(g+2) = p_1$.

Both p_0 and p_1 are pronic (fixed by A). Hence $\alpha_3(p_0) = D(A(D(p_0))) = p_1$, $f(p_0) = y$, $f(y) = A(D(y)) = A(p_1) = p_1$, $f(p_1) = A(D(p_1)) = A(y) = x$, and $f(x) = A(D(x)) = A(p_0) = p_0$, with strict ordering $p_0 < x < y < p_1$. At $g_2 = 8$, this gives the 4-cycle $56 \to 79 \to 90 \to 65 \to 56$. □

Proof of Proposition 8. In `lean/TwoMirrors/Defs.lean`, the binary square-root routine `isqrtAux` equals $\lfloor \sqrt{n} \rfloor$ for all $n < 4^{\text{fuel}}$, powering a proved-sound orbit verifier `checkOrbitFast`. Pure-kernel reduction (`by decide`, without `native_decide`) certifies

$$f^4(1) = 1, \quad f^{28}(8) = 8, \quad f^{28}(49) = 49, \quad f^{128}(288) = 288$$

and $f^{92}(1{,}681) = 1{,}681$. Applying Corollary 5 at $g_5 = 1{,}681 = 41^2$ (since $41^2 - 2 \cdot 29^2 = -1$) shows every $n < 1{,}681$ is periodic. □

Proof of Theorems 9–11. For $A \circ B$, the identity $2(s_k m_k)^2 = g_k(g_k + 1)$ produces square-triangular walls $(s_k m_k)^2 = T_{g_k}$ (starting 1, 36, 1,225, 41,616, 1,413,721). Each opens both an A-block and a B-block, so both mirrors keep $[0, (s_k m_k)^2)$ invariant. Since $T_{g_k} \geq g_k$ is unbounded, every $n \in \mathbb{N}$ lies below a wall and has a bounded, hence periodic, orbit under $A \circ B$.

Theorem 10 ($A \circ C$): Write $n = s^2 + d$ with $s = \lfloor \sqrt{n} \rfloor$ and $0 \leq d \leq 2s$. If $d \geq s$, then $n \in [s(s + 1), (s + 1)(s + 2) - 1]$, so $C(n) = s^2 + 4s + 1 - d \in [(s + 1)^2, (s + 2)^2 - 1]$ and $A(C(n)) = (s + 1)^2 + (d + 2)$. If $d < s$, then $s \geq 1$ and $n \in [(s-1)s, s(s+1)-1]$, so $C(n) = s^2 - 1 - d \in [(s-1)^2, s^2 - 1]$ and $A(C(n)) = (s-1)^2 + d$. In both cases the integer invariant $\Phi(n) = d - s$ increases by exactly $+1$ at every step: $\Phi((A \circ C)^L(n)) = \Phi(n) + L$. Hence $(A \circ C)^L(n) = n$ is impossible for $L \geq 1$.

Theorem 11 ($A \circ M_4$): Every quad-square $4m^2 = (2m)^2$ opens both an A-block and an M_4-block, trapping every $n < 4(n + 1)^2$ inside a finite invariant interval and forcing periodicity. □

■ 4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Our standalone exact-arithmetic Python script `recompute.py` runs in under one second and certifies the following deterministic computational summary (coupled with Corollary 5).

Proposition 12 (Deterministic Computational Checks Through g_{15} — Cert Only). *Exact-arithmetic execution of `recompute.py` verifies:*

1. *Direct return of every Pell-level gate orbit $g_1, \ldots, g_{15}$ (Table 1), with all cycle lengths divisible by 4; combined with Corollary 5 at $g_{15} = 76{,}069{,}501{,}249$, every $n < 76{,}069{,}501{,}249$ is periodic under f.*
2. *Full cycle census on $[0, 100{,}000]$: exactly 450 cycles meet $[0, 100{,}000]$, every length is divisible by 4, the longest has length 4,392 (spanning $[65{,}537, 77{,}256]$), and the largest max / min ratio is 23.5 (the 28-cycle of 2, spanning $[2, 47]$).*
3. *Exact leak profile on $[1, 100{,}000]$: $\text{leak}(x) \geq 1$ everywhere (maximum 496), and $\text{leak}(x) = 1$ occurs at exactly 14 integers, namely g_k and $g_k + 1$ for $k = 1, \ldots, 7$.*
4. *Cross-checks of involutivity, fixed points, $f(n) - n \equiv 3 \pmod 4$, and all three neighbour formulas on $[0, 100{,}000)$ (727 cycles meeting $[0, 100{,}000]$ for $A \circ B$; 0 formula discrepancies for $A \circ C$; complete block confinement for $A \circ M_4$).*

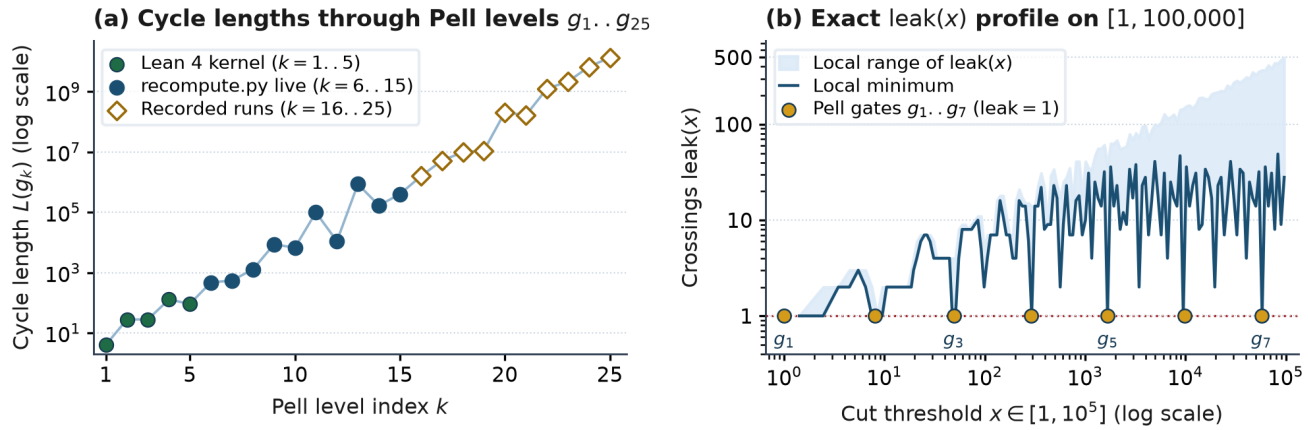


Figure 3. Panel (a): Cycle lengths $L(g_k)$ for Pell levels $k = 1 \ldots 25$ on a logarithmic scale (green circles: Lean 4 pure-kernel checks $k = 1 \ldots 5$; navy circles: live `recompute.py` checks $k = 6 \ldots 15$; open gold diamonds: recorded direct C runs $k = 16 \ldots 25$ from `gate_orbits.out`). Adjacent length ratios fluctuate between 0.111 and 78.035. Panel (b): Exact leak(x) envelope on $[1, 100{,}000]$ (computed live from f), dropping to leak(x) = 1 solely at the Pell levels $g_1 \ldots g_7$ and $g_k + 1$.

Table 1. Exact Pell levels g_k, gate-orbit cycle lengths $L(g_k)$, normalized excursion $\max|n - g_k|/\sqrt{g_k}$, and gap to the next level $(g_{k+1} - g_k)/\sqrt{g_k}$ for $k = 1 \ldots 15$ (live in `recompute.py`; $k = 1 \ldots 5$ also certified inside Lean 4's kernel).

k	g_k	$L(g_k)$	Reach/$\sqrt{g_k}$	Gap/$\sqrt{g_k}$
1	1	4	5.0	7.0
2	8	28	19.5	20.5
3	49	28	16.0	34.1
4	288	128	6.4	87.1
5	1,681	92	5.5	198.0
6	9,800	472	13.2	482.9
7	57,121	548	10.1	1,154.0
8	332,928	1,272	25.8	2,790.8
9	1,940,449	8,588	25.3	6,726.0
10	11,309,768	6,788	14.0	16,242.8
11	65,918,161	99,556	39.5	39,202.0
12	384,199,200	11,048	19.1	94,646.8
13	2,239,277,041	862,132	47.9	228,486.0
14	13,051,463,048	169,552	25.4	551,618.8
15	76,069,501,249	399,692	33.9	1,331,714.0

External computations. Beyond the live range ($k \leq 15$) of `recompute.py`, the archived logs in `docs/confirmed-strong-and-novel/two-mirrors/` record three deeper unformalized computations outside our Lean and local-certificate trust boundary:

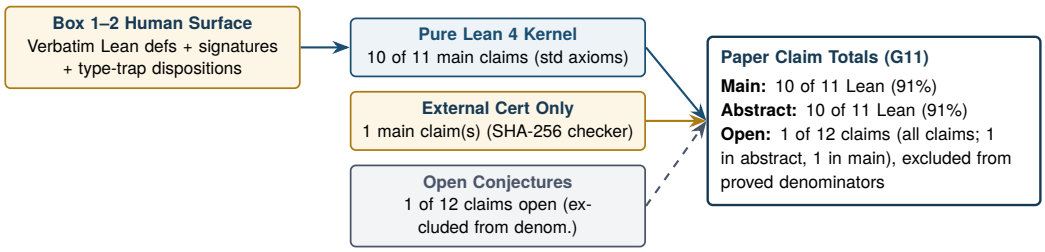
1. *Direct walks through g_{25}:* Exact C programs followed every gate orbit $g_1, \ldots, g_{25}$ step by step (`gate_orbits.out`; Figure 3(a)), closing $g_{25} = 3{,}441{,}313{,}796{,}169{,}221{,}281$ in 13,652,206,968 steps with excursion $\leq 121.2\sqrt{g_k}$ across all $k \leq 25$, while the next gate lies $8.96 \times 10^9 \sqrt{g_{25}}$ away. Across $k = 1 \ldots 25$, adjacent cycle-length ratios span 0.111 to 78.035.
2. *Extended census and leak checks:* Direct censuses closed all starts up to 10^8 (18,019 cycles), and exact leak scans through $2{\times}10^6$ (`leak.out`) found leak(x) ≥ 1 everywhere with leak(x) = 1 strictly at the 18 values g_k and $g_k + 1$ for $k = 1 \ldots 9$.
3. *Block-cancelling meander reduction to g_{1000}:* Using the meander view (preceded in finite form by [3] and mirror-maze tracing by [5]), three unformalized codes (two Python/C scripts plus a third C cross-check, `phase4_examples.out`) cancelled nested blocks in bulk to verify that the orbit through $g_{1000} \approx 8.9 \times 10^{764}$ (765 digits) ends at pronics indexed $s_{1000} - 19$ and $s_{1000} + 207$ ($s_{1000} > 2.98 \times 10^{382}$, 383 digits) after 920,869,224 and 926,391,256 reduced steps, giving a 384-digit cycle length divisible by 4. Separate Rauzy–Veech induction runs also confirmed $g_1 \ldots g_{136}$ ($g_{136} \approx 3.26 \times 10^{103}$). Because these C/Python codes are not formally verified, their higher-range outputs stay separated from Propositions 8 and 12.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel & Toolchain:** leanprover/lean4:v4.35.0; committed axiom receipt anchored to git commit 4d34d065283c (independent gate-box kernel replay pending); standard axioms only (propext, Classical.choice, Quot.sound).
2. **Core Definitions & Fidelity (Boxes 1–2):** Verify that the verbatim Lean definitions in Box 1 match Section 2 and that each Lean signature in Box 2 matches its stated theorem.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 3 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — Source-Level Approx (Gate Replay Pending)

WARN: Committed receipt (gate replay pending) | PASS: sorry/admit: 0 (per committed receipt)
PASS: native_decide: 0 (per committed receipt) | PASS: Custom axioms & meta: 0 (per committed receipt)
PASS: Coverage: 100/105 public theorems (source-level approx)
WARN: Fidelity: 10/10 exact (draft; auditor/formalizer relationship unknown) | WARN: Witnesses: pending gate replay
WARN: Human referee: pending

Box 1 — Verbatim Lean 4 Custom Definitions (Human Audit Surface)

TwoMirrors.mirrorA (TwoMirrors/Defs.lean:36-38) — Section 1 & 2 (square mirror reversing [s^2, (s+1)^2 - 1])

```
def mirrorA (n : ℕ) : ℕ :=
  let s := Nat.sqrt n
  2 * s * (s + 1) - n
```

TwoMirrors.mirrorD (TwoMirrors/Defs.lean:43-45) — Section 1 & 2 (doubled-square mirror reversing [2m^2, 2(m+1)^2 - 1])

```
def mirrorD (n : ℕ) : ℕ :=
  let m := Nat.sqrt (n / 2)
  (2 * m + 1) ^ 2 - n
```

TwoMirrors.f (TwoMirrors/Defs.lean:48-49) — Section 1 & 2 (one forward step f = A o D)

```
def f (n : ℕ) : ℕ :=
  mirrorA (mirrorD n)
```

TwoMirrors.fInv (TwoMirrors/Defs.lean:52-53) — Section 1 & 2 (backward step f^{-1} = D o A)

```
def fInv (n : ℕ) : ℕ :=
  mirrorD (mirrorA n)
```

TwoMirrors.fEquiv (TwoMirrors/Defs.lean:229-237) — Section 2 (bijection packaging f and fInv as Equiv.Perm Nat)

```
def fEquiv : Equiv.Perm ℕ where
  toFun := f
  invFun := fInv
  left_inv n := by
    change mirrorD (mirrorA (mirrorA (mirrorD n))) = n
    rw [mirrorA_involutive (mirrorD n), mirrorD_involutive n]
  right_inv n := by
    change mirrorA (mirrorD (mirrorD (mirrorA n))) = n
    rw [mirrorD_involutive (mirrorA n), mirrorA_involutive n]
```

TwoMirrors.IsPeriodic (TwoMirrors/Defs.lean:56-57) — Section 2 (existence of L > 0 with g^[L] n = n)

```
def IsPeriodic (g : ℕ → ℕ) (n : ℕ) : Prop :=
  ∃ L : ℕ, 0 < L ∧ g^[L] n = n
```

TwoMirrors.IsPronic (TwoMirrors/Defs.lean:60-61) — Section 1 & 2 (pronic numbers s*(s+1), fixed points of A)

```
def IsPronic (n : ℕ) : Prop :=
  ∃ s : ℕ, n = s * (s + 1)
```

TwoMirrors.altStep (TwoMirrors/PellGate.lean:517-519) — Section 1 (alternating D, A, D, ... swap walk)

```
def altStep : ℕ → ℕ → ℕ
  | 0, n => n
  | k + 1, n => if k % 2 = 0 then mirrorD (altStep k n) else mirrorA (altStep k n)
```

TwoMirrors.PronicPairs (TwoMirrors/PellGate.lean:540-541) — Section 1 (pairing of a pronic with a second distinct pronic)

```
def PronicPairs (p : ℕ) : Prop :=
  ∃ k : ℕ, 0 < k ∧ IsPronic (altStep k p) ∧ altStep k p ≠ p
```

TwoMirrors.mirrorB (TwoMirrors/Neighbours.lean:205-207) — Section 1 & 3 (triangular block mirror B)

```
def mirrorB (n : ℕ) : ℕ :=
  let t := triIdx n
  tri t + (tri (t + 1) - 1) - n
```

TwoMirrors.mirrorC (TwoMirrors/Neighbours.lean:343-345) — Section 1 & 3 (pronic block mirror C)

```
def mirrorC (n : ℕ) : ℕ :=
  let s := pronicIdx n
  pronic s + (pronic (s + 1) - 1) - n
```

TwoMirrors.mirror4 (TwoMirrors/Neighbours.lean:522-524) — Section 1 & 3 (quad-square block mirror M_4)

```
def mirror4 (n : ℕ) : ℕ :=
  let m := Nat.sqrt (n / 4)
  4 * m ^ 2 + (4 * (m + 1) ^ 2 - 1) - n
```

Supporting Semantic Closure Index (non-Box-1 helpers): TwoMirrors.tri (TwoMirrors/Neighbours.lean), TwoMirrors.triIdx (TwoMirrors/Neighbours.lean), TwoMirrors.pronic (TwoMirrors/Neighbours.lean), TwoMirrors.pronicIdx (TwoMirrors/Neighbours.lean).

Box 2 — Verbatim Main Theorem Signatures, Context Binders & Applicability

TwoMirrors.cycle_length_dvd_four (TwoMirrors/Defs.lean:355-358) `src:1c013a2e80fb84`

```
theorem cycle_length_dvd_four (n L : ℕ) (h : f^[L] n = n) : 4 ∣ L
```

Assumptions: Universal over all n, L in N with f^[L] n = n (instantiated e.g. by f^[4] 1 = 1; no external assumptions). Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TwoMirrors.pell_gate_lemma (TwoMirrors/PellGate.lean:324-378) `src:98f7fac2231143`

```
theorem pell_gate_lemma {s m : ℕ} (hs : 1 ≤ s) (hm : 1 ≤ m)
    (hpell : (s : ℤ) ^ 2 - 2 * (m : ℤ) ^ 2 = 1 ∨ (s : ℤ) ^ 2 - 2 * (m : ℤ) ^ 2 = -1) :
    let g
```

Assumptions: All s, m >= 1 with s^2 - 2m^2 = +-1 and g = min(s^2, 2m^2). Full Lean conclusion at PellGate.lean:324-330 asserts unique exit x < g with g <= f x, unique entrance y >= g with f y < g, and that g lies in the Z-orbit of both. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TwoMirrors.at_most_one_infinite_orbit (TwoMirrors/PellGate.lean:472-483) `src:39da8268f9dfb4`

```
theorem at_most_one_infinite_orbit {a b : ℕ}
    (ha : ¬ IsPeriodic f a) (hb : ¬ IsPeriodic f b) :
    ∃ z : ℤ, (fEquiv ^ z) a = b
```

Assumptions: Any two non-periodic a, b in N share a Z-orbit under fEquiv. Applicability: Hypothetical constraint — vacuously true if no such object exists (carrier status: ordinary_type).

TwoMirrors.all_below_periodic_iff_gate_periodic (TwoMirrors/PellGate.lean:487-500) `src:8fec13c8b066b3`

```
theorem all_below_periodic_iff_gate_periodic {s m : ℕ} (hs : 1 ≤ s) (hm : 1 ≤ m)
    (hpell : (s : ℤ) ^ 2 - 2 * (m : ℤ) ^ 2 = 1 ∨ (s : ℤ) ^ 2 - 2 * (m : ℤ) ^ 2 = -1) :
    let g
```

Assumptions: All s, m >= 1 with s^2 - 2m^2 = +-1 and g = min(s^2, 2m^2). Full Lean conclusion at PellGate.lean:487-490 asserts (forall n < g, IsPeriodic f n) <-> IsPeriodic f g. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TwoMirrors.all_periodic_iff_every_pronic_pairs (TwoMirrors/PellGate.lean:700-709) `src:b6e955ba22f937`

```
theorem all_periodic_iff_every_pronic_pairs :
    (∀ n : ℕ, IsPeriodic f n) ↔ (∀ p : ℕ, IsPronic p → PronicPairs p)
```

Assumptions: Unconditional logical equivalence between global periodicity of f on N and pairing of every pronic number under alternating D, A, D, ... swaps. Applicability: Unconditional (closed statement; no outer premise witness needed).

TwoMirrors.pell_level_four_cycle (TwoMirrors/Neighbours.lean:67-175) `src:b2aea6b0d5a176`

```
theorem pell_level_four_cycle {s m : ℕ} (hs : 1 ≤ s) (hm : 1 ≤ m)
    (hpell : (s : ℤ) ^ 2 - 2 * (m : ℤ) ^ 2 = 1 ∨ (s : ℤ) ^ 2 - 2 * (m : ℤ) ^ 2 = -1) :
    let g
```

Assumptions: All s, m >= 1 with s^2 - 2m^2 = +-1 and g = min(s^2, 2m^2). Full Lean conclusion at Neighbours.lean:67-76 asserts p_0 = (g-1)g and p_1 = (g+1)(g+2) are pronic, altStep 3 p_0 = p_1, f^[2] p_0 = p_1, f^[4] p_0 = p_0, and p_0 < D(p_0) < f(p_0) < p_1. Applicability: Instantiable — closed instantiation witness PENDING gate-box replay.

TwoMirrors.all_periodic_lt_1681 (TwoMirrors/PellGate.lean:748-753) `src:b35207725ee39b`

```
theorem all_periodic_lt_1681 : ∀ n < 1681, IsPeriodic f n
```

Assumptions: Unconditional pure-kernel proof that every n < 1681 is periodic under f (certified via checkOrbitFast_sound + by decide, zero native_decide). Applicability: Unconditional (closed statement; no outer premise witness needed).

TwoMirrors.triangular_all_periodic (TwoMirrors/Neighbours.lean:320-332) src:475d271f73965f

`theorem triangular_all_periodic : ∀ n : ℕ, IsPeriodic (mirrorA ∘ mirrorB) n`

Assumptions: Universal over all n in N for mirrorA o mirrorB. Applicability: Unconditional (closed statement; no outer premise witness needed).

TwoMirrors.pronic_no_periodic (TwoMirrors/Neighbours.lean:513-517) src:a75fafe6953902

`theorem pronic_no_periodic : ∀ n : ℕ, ¬ IsPeriodic (mirrorA ∘ mirrorC) n`

Assumptions: Universal over all n in N for mirrorA o mirrorC. Applicability: Unconditional (closed statement; no outer premise witness needed).

TwoMirrors.quadSquare_all_periodic (TwoMirrors/Neighbours.lean:577-593) src:d7f52ad2c94c2f

`theorem quadSquare_all_periodic : ∀ n : ℕ, IsPeriodic (mirrorA ∘ mirror4) n`

Assumptions: Universal over all n in N for mirrorA o mirror4. Applicability: Unconditional (closed statement; no outer premise witness needed).

Table 2. Lean 4 Type-Trap Audit Checklist — Source-Level Approximation (Gate Replay Pending).[a]

Declaration	Hazard	Operand	Category	Audited Disposition & Guard
`altStep`	`nat_mod`	`% 2`	intended_convention	Standard parity check modulo 2 > 0 (verified by altStep_even and altStep_odd).
`mirror4`	`nat_div`	`/ 4`	intended_convention	Exact floor division by 4 > 0 characterized in sqrt_div_four_eq_iff.
`mirror4`	`nat_sub`	`- 1`	intended_convention	Upper endpoint 4*(m+1)^2 - 1 never underflows since 4*(m+1)^2 >= 4.
`mirror4`	`nat_sub`	`- n`	intended_convention	Zero Nat underflow; mirror4_mem_block proves exact block invariance.
`mirrorA`	`nat_sub`	`- n`	intended_convention	Zero Nat underflow; mirrorA_mem_block proves exact square-block invariance.
`mirrorB`	`nat_sub`	`- 1`	intended_convention	Zero Nat underflow since tri (t + 1) >= 1 (tri_succ).
`mirrorB`	`nat_sub`	`- n`	intended_convention	Zero Nat underflow; mirrorB_mem_block proves exact triangular-block invariance.
`mirrorC`	`nat_sub`	`- 1`	intended_convention	Zero Nat underflow since pronic (s + 1) >= 2.
`mirrorC`	`nat_sub`	`- n`	intended_convention	Zero Nat underflow; mirrorC_mem_block proves exact pronic-block invariance.
`mirrorD`	`nat_div`	`/ 2`	intended_convention	Exact floor division by 2 > 0 characterized in sqrt_div_two_eq_iff_mem_dbl_block.
`mirrorD`	`nat_sub`	`- n`	intended_convention	Zero Nat underflow; mirrorD_mem_block proves exact doubled-square block invariance.
`pronicIdx`	`nat_sub`	`- 1`	intended_convention	Zero Nat underflow since 4*n + 1 >= 1 implies Nat.sqrt (4*n + 1) >= 1 (pronicIdx_eq_iff).
`pronicIdx`	`nat_div`	`/ 2`	intended_convention	Exact integer floor division by 2 > 0 proved equivalent to block membership in pronicIdx_eq_iff.
`tri`	`nat_div`	`/ 2`	intended_convention	Exact division by 2 > 0 proved in two_mul_tri (2 * tri t = t * (t + 1)).
`triIdx`	`nat_sub`	`- 1`	intended_convention	Zero Nat underflow since 8*n + 1 >= 1 implies Nat.sqrt (8*n + 1) >= 1 (triIdx_eq_iff).
`triIdx`	`nat_div`	`/ 2`	intended_convention	Exact integer floor division by 2 > 0 proved equivalent to block membership in triIdx_eq_iff.

[a]Covers the specified Lean 4 hazard set (`Nat` subtraction/division/mod, totalized real/rational division, `Real.log`/`rpow`, `Finset.range`, `Classical.choose`, local instances); no finite checklist certifies all semantics.

Table 3. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 2: Every cycle of f has length divisible by 4 (f^L(n) = n implies 4 | L)	Lean Kernel	`TwoMirrors.cycle_length_dvd_four`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all n, L in N (Mathlib v4.35.0; committed receipt).

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 3: Pell Gate Lemma — unique exit x < g and unique entrance y >= g, both in Z-orbit of g	Lean Kernel	`TwoMirrors.pell_gate_lemma`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all s, m >= 1 with s^2 - 2m^2 = +-1 (PellGate.lean:324-330; committed receipt).
Cor. 4: At most one runaway orbit (any two non-periodic starts share a Z-orbit)	Lean Kernel	`TwoMirrors.at_most_one_infinite_orbit`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all non-periodic a, b in N (committed receipt).
Cor. 5: Pell-level gate reduction (every n < g is periodic if and only if g is periodic)	Lean Kernel	`TwoMirrors.all_below_periodic_iff_gate_periodic`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all s, m >= 1 with s^2 - 2m^2 = +-1 (PellGate.lean:487-490; committed receipt).
Thm. 6: Pronic-pairing equivalence (all n periodic if and only if every pronic pairs under D, A, D, ...)	Lean Kernel	`TwoMirrors.all_periodic_iff_every_pronic_pairs`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all n in N and pronics s(s+1) (committed receipt).
Thm. 7: Universal Pell-level 4-cycles pairing (g-1)g and (g+1)(g+2) in 3 swaps	Lean Kernel	`TwoMirrors.pell_level_four_cycle`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all s, m >= 1 with s^2 - 2m^2 = +-1 (Neighbours.lean:67-76; committed receipt).
Prop. 8: Pure-kernel periodicity certification for every n < 1,681	Lean Kernel	`TwoMirrors.all_periodic_lt_1681`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + proved-sound binary isqrt Aux/checkOrbitFast (no native_decide; committed receipt).
Thm. 9: Square and triangular mirrors (A o B) make every n in N periodic	Lean Kernel	`TwoMirrors.triangular_all_periodic`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all n in N (square-triangular walls; committed receipt).
Thm. 10: Square and pronic mirrors (A o C) make no n in N periodic	Lean Kernel	`TwoMirrors.pronic_no_periodic`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all n in N (invariant Phi(n) = d - s; committed receipt).
Thm. 11: Square and quad-square mirrors (A o M_4) make every n in N periodic	Lean Kernel	`TwoMirrors.quadSquare_all_periodic`	Axioms: Classical.choice, Quot.sound, propext; fidelity: exact	Lean 4 pure kernel + Box 1 defs for all n in N (exact 4m^2 walls; committed receipt).
Prop. 12: Deterministic Python verification of gate orbits g_1..g_15 (all n < 76,069,501,249 periodic), census [0, 100,000], and leak [1, 100,000]	Cert Only	recompute.py (37e6ec7c1a0d) [regenerable; replay pending]	External cert only; fidelity: n/a	Deterministic exact-arithmetic Python script recompute.py + Corollary 5 (barrier: 1.56M-step direct orbits beyond practical Lean kernel reduction).
Conj. 1: Universal periodicity of f = A o D for every n in N	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled conjecture (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 687db02f46cd | Lean mode: committed_receipt | Closure SHA-256: ad24931609bf | Git anchor: 4d34d065283c | Axioms SHA-256: c31654feaa32 | Fidelity SHA-256: 7ad355339f23.

6. PRIOR WORK AND OPEN QUESTIONS

Multiple logged prior-work searches passed `litsearch.py check` (exit 0 across OEIS, arXiv, zbMATH, Math Stack Exchange, MathOverflow, Zenodo, DataCite, HAL, GitHub, and the `formal-conjectures` Lean repository, supplemented by a 6-query Google Scholar sweep in November 2026). Neither the map $f = A \circ D$, nor Conjecture 1, nor the Pell-gate and pronic-pairing reductions were found in the searches logged.

The nearest published relative found is Alwani and Vivaldi [1], whose planar lattice return map factors as a product of two block mirrors on block systems of equal growth rate, with periodicity/escape results proved for a full-density parameter set; our pair (A, D) has block-length ratio $\sqrt{2}$ and lies outside their equal-rate family. Finite block-reversal compositions appear as meander permutations of seaweed Lie algebras in Coll, Giaquinto, and Magnant [2], Coll, Magnant, and Wang [3] (block-reduction moves), and Karnauhova and Liebscher [6]. For infinite meanders, Lu [8] studies random pairs of infinite integer sets, while Curien, Kozma, Sidoravicius, and Tournier [4] prove uniqueness of any infinite path in the random infinite noodle while leaving existence open—a probabilistic counterpart to Corollary 4. Eppstein [5] gives general polynomial-time bounce-counting algorithms in mirror mazes, and Lapointe [7] `[to check]` studies orbit counts of discrete interval exchanges. Single mirrors appear in OEIS A020703 (A alone) and A038722/A061579 (B alone), without composition.

Unread and unsearched gaps. Sources not read in full include Lapointe [7] `[to check]` (abstract only), B. Fiedler (2014, Zbl 1300.37032) `[to check]` (paywalled summary only), R. K. Guy's *Unsolved Problems in Number Theory* `[to check]`, and

Lowenstein–Vivaldi lattice papers [to check] (abstracts only). Direct seqfan archive traversal, doctoral theses, and non-English literature were not searched.

Open questions. Beyond Conjecture 1 itself, three concrete questions stand open:

1. *Non-square multiples km^2:* For any non-square $k \geq 3$, Pell solutions to $s^2 - km^2 = 1$ again produce one-door gates; does $A \circ M_k$ have all orbits periodic for every non-square k?
2. *Scaled real-line model:* Near a large square U^2, rescaling by $2U$ turns A and D into reflections on intervals of constant lengths 1 and $\sqrt{2}$, where irrationality prevents exact closure; determining whether orbits of this limiting infinite interval exchange remain bounded is open.
3. *Pronic-index proximity:* Does the cycle through g_k always pair two pronics within $O(k)$ indices of s_k? (At $k = 1{,}000$, external runs give indices $s_{1000} - 19$ and $s_{1000} + 207$.)

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript files, figure scripts, and exact-arithmetic verifiers reside in papers/two-mirrors/draft1/, and the Lean 4 library resides in lean/TwoMirrors/ (Lean 4 v4.35.0 with Mathlib v4.35.0, importing Mathlib.Data.Nat.Sqrt, Mathlib.GroupTheory.Perm.Basic, Mathlib.Data.Finset.Card, and Mathlib.Tactic; 100 of 105 public theorems audited in Verify.lean, with 5 definitional := rfl unfolding equations omitted, 0 sorry, 0 native_decide, standard logical axioms only). Running python3 papers/two-mirrors/draft1/recompute.py reproduces Proposition 12 and Table 1 in under one second. AI research agents built and directed by Tyler B. Hughes generated, formalized, computationally checked, and drafted this work. No human mathematician has yet refereed the proof.

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REFERENCES

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