MATHDISCOVERIES
Mathematical Preprint

Working Draft

XOR Collatz: Structural Companion Theorems, Two-Phase Carry Drift, and Computational Evidence Through 2^{36}

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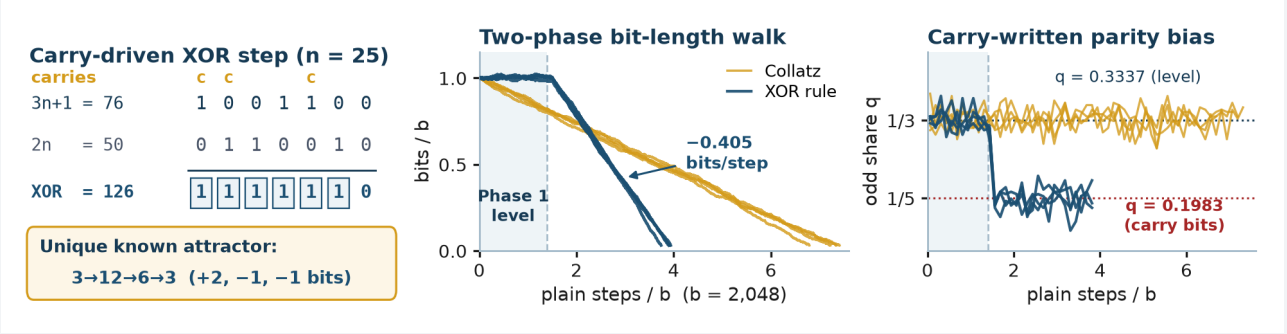
Ⓢ *Supporting Information:* `recompute.py` *(standalone exact-integer Python verifier for examples, companions, sweep, and drift)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Define $T : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ by $T(n) = n/2$ (even) and $T(n) = h(n) = (3n+1) \oplus 2n$ (odd), where $\oplus$ is bitwise XOR. Carries in $3n+1 = 2n + (n+1)$ force $n + 5 \leq h(n) \leq 5n + 1$. We conjecture every $n \geq 1$ reaches $3 \to 12 \to 6 \to 3$, and prove five companions: (1) opening k shortcut parities biject with n mod 2^k; (2) leading digits 11 lock in within $2\ell(n) - 1$ steps with $+2/-1$ bit steps, forcing 2:1 halvings-to-odds on cycles and zero fair-coin drift ($q = 1/3$); (3) $h(2^m - 1) = 3 \cdot 2^m$ reaches 3 in $m + 1$ steps; (4) $A_k = (4^k - 1)/3$ embeds $P \mapsto (x+1)P + 1$ over $\mathbb{F}_2[x]$ in base 16, so $k = 2^j$ ($j \geq 1$) peaks at $3 \cdot 2^{4k-2} \sim 6.75\, A_k^2$ ($8k - 3$ steps); and (5) $n \mapsto (n \oplus 2n) + 1$ enters $\{1, 4, 2\}$ or $\{3, 6\}$. Standalone replay verifies $n \leq 2^{18}$, excludes cycles $L \leq 12$, and measures 2,048-bit drift ($q = 0.3337$ level, then -0.4050 bits/step at $q = 0.1983$). Extended sweeps verify $n < 2^{36} = 68{,}719{,}476{,}736$ and exclude cycles $L \leq 24$. Global convergence, cycles $L > 24$, and the phase-2 carry bias remain unsettled; no Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending — **Full Trust Boundary:** Section 5.
Main results: 0 of 7 in Lean Kernel (0%) · 2 Cert Only · 5 written · 1 open (not counted)
Abstract claims: 0 of 7 in Lean Kernel (0%) · 2 Cert Only · 5 written · 1 open (not counted)

MSC 2020: 11B83, 11A63, 11Y55 **Keywords:** Collatz conjecture, bitwise XOR, binary carries, 2-adic parity, polynomial dynamics, two-phase drift

1. INTRODUCTION AND MAIN RESULTS

Replacing one ordinary arithmetic operation in the classical $3n+1$ rule by bitwise exclusive OR ($\oplus$) produces a strikingly rigid dynamical system on the positive integers. Define $T : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ by

$$T(n) := \begin{cases} n/2 & \text{if } n \text{ is even,} \\ h(n) := (3n+1) \oplus 2n & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

where $a \oplus b$ compares the binary expansions of a and b place by place and places a 1 exactly where they differ. Equivalently, on odd inputs, $h(n)$ takes $3n + 1 = 2n + (n + 1)$ and removes $2n$ without borrowing. If binary addition never carried, then $2n + (n + 1)$ would coincide with $2n \oplus (n + 1)$, so XORing $2n$ back off would return $n+1$ and every orbit would descend immediately. Instead, carries in $2n + (n+1)$ always occur and produce jumps spanning a factor between slightly above 1 and slightly above 5.

In contrast with the classical $3n + 1$ map—whose fair-coin heuristic already predicts negative logarithmic drift ($\frac{1}{3}\log_2 3 - \frac{2}{3} \approx -0.1383$ bits per plain step)—the XOR Collatz map (1) is *exactly critical* under fair-coin parities. As we prove below, once an orbit acquires leading binary digits 11, its bit length follows an exact $+2/-1$ walk whose drift $3q-1$ vanishes at the fair-coin odd-step share $q = 1/3$. While an orbit is still reading the random bits of its starting integer, its bit length indeed stays level; only after those starting bits are exhausted does the trajectory plunge steeply at roughly -0.405 bits per plain step ($q \approx 1/5$), driven entirely by a parity bias in the bits written by the map's own carries.

We state five proved structural companion theorems, two computational verification propositions, and the unsettled conjecture upfront. Throughout, $S : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ denotes the *shortcut map*

$$S(n) := \begin{cases} n/2 & \text{if } n \text{ is even,} \\ h(n)/2 & \text{if } n \text{ is odd,} \end{cases} \tag{2}$$

which pairs each odd step with the mandatory halving that follows it (as $h(n)$ is always even for odd n), and $\ell(n) := \lfloor \log_2 n \rfloor + 1$ denotes the binary bit length of $n \geq 1$.

Theorem 1 (Fair-Parity Bijection and Carry Identity). *For every odd integer $n \geq 1$, the odd branch $h(n) = (3n+1) \oplus 2n$ is even and satisfies the exact carry identity*

$$h(n) = (n+1) + 2(2n \,\&\, \neg(3n+1)), \tag{3}$$

where $\&$ and $\neg$ denote bitwise AND and bitwise complement, yielding the strict bounds $n+5 \leq h(n) \leq 5n+1$. Furthermore, for every integer $k \geq 1$, the length-k shortcut parity vector $P_k(n) \in \{0,1\}^k$, defined by

$$P_k(n) := (n \bmod 2,\ S(n) \bmod 2,\ \ldots,\ S^{k-1}(n) \bmod 2),$$

depends only on $n \bmod 2^k$ and induces a bijection $\mathbb{Z}/2^k\mathbb{Z} \to \{0,1\}^k$.

Theorem 2 (The 11-Regime and Exact Bit-Length Walk). *Call an integer $x \geq 3$ in the 11-regime when its two most significant binary digits are 11, equivalently $3 \cdot 2^{\ell(x)-2} \leq x < 2^{\ell(x)}$. Then:*

1. (Closure and exact bit-length steps.) *If x lies in the 11-regime, then $T(x)$ lies in the 11-regime, with $\ell(T(x)) = \ell(x)+2$ if x is odd and $\ell(T(x)) = \ell(x)-1$ if x is even.*
2. (Universal entry.) *Every positive integer $n \geq 1$ enters the 11-regime in at most $2\ell(n)-1$ plain steps.*
3. (Exact balance on cycles and fair-coin neutrality.) *After entering the 11-regime, N plain steps comprising O odd steps and $E = N-O$ halvings change bit length by $2O-E = (3q-1)N$, where $q = O/N$. In particular, every cycle of T lies in the 11-regime and satisfies $E = 2O$ (exact 2:1 halving-to-odd ratio, plain length $3O$, and even shortcut length $2O$), and fair-coin parities ($q = 1/3$) produce zero expected drift.*

Proposition 3 (Mersenne Collapse). *For every integer $m \geq 1$, $h(2^m-1) = 3 \cdot 2^m$. Consequently, the orbit of 2^m-1 enters 3 after exactly $m+1$ plain steps via $2^m-1 \to 3 \cdot 2^m \to 3 \cdot 2^{m-1} \to \ldots \to 6 \to 3$ (returning to 3 after 3 steps when $m = 2$).*

Theorem 4 (Alternating-Bit Climb and Hidden $\mathbb{F}_2[x]$ Dynamics). *For every integer $k \geq 1$, let $A_k := (4^k-1)/3$, whose binary representation is the alternating string $1010\ldots101_2$ with k ones and $2k-1$ bits. Define the base-16 embedding $\Phi(P) := 6 \cdot P(16)$ for $P \in \mathbb{F}_2[x]$ (identifying coefficients with $\{0,1\}$), and let $D_k(x) := ((x+1)^k+1)/x \in \mathbb{F}_2[x]$. Then:*

1. *After $2k$ shortcut steps ($4k$ plain steps, alternating odd steps and halvings), $T^{4k}(A_k) = S^{2k}(A_k) = \Phi(D_k) = 6\,D_k(16)$, governed step by step by the carry-less polynomial Collatz map $P \mapsto (x+1)P+1$ over $\mathbb{F}_2[x]$.*
2. *For every power of two $k = 2^j$ with $j \geq 1$, the orbit of A_k attains its strict global maximum $3 \cdot 2^{4k-2}$ at plain step $4k-1$—with peak-to-square ratio*

$$\frac{3 \cdot 2^{4k-2}}{A_k^2} = \frac{27}{4}(1-2^{-2k})^{-2} \longrightarrow \frac{27}{4} = 6.75 \quad (j \to \infty)$$

—and reaches 3 in exactly $8k-3$ plain steps. At the sole exception $j = 0$ ($k = 1$), $A_1 = 1 \to 6 \to 3$ reaches 3 in 2 steps with peak 6.

Theorem 5 (Convergence of the Swapped-Order Companion). *Consider the swapped-order map $\widetilde{T} : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ given by $\widetilde{T}(n) = n/2$ for even n and $\widetilde{T}(n) = (n \oplus 2n)+1$ for odd n. Along every $\widetilde{T}$-orbit, consecutive odd iterates never increase in bit length, and every positive integer $n \geq 1$ eventually enters either $1 \to 4 \to 2 \to 1$ or $3 \to 6 \to 3$.*

On the computational side, we separate checks replayed directly in our standalone script `recompute.py` from larger multi-program sweeps resting on committed verification records.

Proposition 6 (Standalone Exact Replay, Local Sweep Through 2^{18}, Short-Cycle Exclusion, and Two-Phase Drift). *Standalone exact-integer Python verification (`recompute.py`, zero floating-point arithmetic in any step predicate) certifies:*

1. *The 3-cycle $3 \to 12 \to 6 \to 3$, the odd step $25 \mapsto 126$, the full 13-step orbit of 5 to 3 (entering $\{3,6,12\}$ at step 11 with 4 odd steps, 9 halvings, and peak 192), and the carry identity (3) with $n+5 \leq h(n) \leq 5n+1$ for every odd $n < 10^5$.*
2. *Finite sanity checks of Theorems 1, 2, 4, 5, and Proposition 3: fair-parity bijection for $1 \leq k \leq 12$ (up to 4,096 residues), 11-regime invariance (32,767 numbers) and entry bound for all $n < 2^{16}$, Mersenne collapse for $1 \leq m \leq 64$, alternating-start identities for $1 \leq k \leq 64$ (all 1,024 polynomials of degree < 10 and exact $j = 1,\ldots,6$ formulas plus the $j = 0$ exception), and swapped-order convergence for $1 \leq n \leq 10^4$ (7,500 into $\{1,4,2\}$, 2,500 into $\{3,6\}$, 0 odd-length increases).*
3. *Exhaustive convergence of every $1 \leq n \leq 2^{18} = 262{,}144$ into $\{3,6,12\}$ (longest entry 143 steps at $n = 179{,}477$; mean 52.73 steps over $n \leq 200{,}000$ and 54.02 steps over $n \leq 262{,}144$; largest peak 68,719,476,732 at $n = 87{,}381 = A_9$), together with direct verification of $n = 764{,}837$ (157 steps to enter the loop, 159 to 3, peak 67,712,054,028).*
4. *Exact 2-adic finite-state column-automaton exclusion of every nontrivial cycle of length $1 \leq L \leq 12$ shortcut steps (789 nonzero necklaces, 123 balanced: only repetitions of $3 \leftrightarrow 6$ yield integer solutions).*
5. *Deterministic two-phase drift on four random 2,048-bit starts (seed 20261119): phase 1 ($[0, 1.4b)$) mean drift $+0.0006$ bits/step with odd share $q = 0.3337$; phase 2 (after $1.8b$ until below $b/4 = 512$ bits) mean drift -0.4050 bits/step with odd share $q = 0.1983$ ($3q-1 = -0.4050$), versus classical Collatz mean drifts -0.1334 ($q = 0.3353$) and -0.1373 ($q = 0.3337$).*

Proposition 7 (Committed Multi-Program Sweeps Below 2^{36} and 24-Shortcut-Step Cycle Exclusion). *Committed audit logs (`verdict.md`, `review.md`, `sweep.out`, and `drift.out`) establish:*

1. *Every positive integer $1 \leq n < 2^{36} = 68{,}719{,}476{,}736$ enters the 3-cycle $3 \to 12 \to 6 \to 3$: an exact 128-bit C sweep below $2^{35} = 34{,}359{,}738{,}368$ (longest descent 259 at $n = 2{,}908{,}181{,}973$) combined with an independent exact descent sweep over $[2^{35}, 2^{36})$ (longest descent 247 at $n = 46{,}081{,}332{,}561$, largest peak 72 bits), independently corroborated through 2^{36} in a blind re-review and through 10^8 by a third program (longest entry 236 steps at $n = 98{,}193{,}693$; peak-to-start ratio 6.06×10^8 at $n = 89{,}478{,}485 = A_{14}$).*
2. *Exact 2-adic column-system enumeration across all nonzero necklaces up to length $L = 24$ (364,723 at $L = 23$; 699,251 at $L = 24$) proves that no cycle other than $3 \to 12 \to 6 \to 3$ exists with at most 24 shortcut steps (equivalently at most 36 plain steps by Theorem 2), regardless of element magnitude.*

3. *Targeted stress tests* (2,367 *cycle-shadow starts at* 2,048 *bits,* 4,648 *special algebraic starts up to* 8,478 *bits, and random starts up to* 262,144 *bits*) *all reach* 3; 16,384*-bit random starts exhibit phase-*1 *drift* $+0.0038 \pm 0.0019$ *and phase-*2 *drift* -0.3990 ± 0.0027 *bits/step, while structured cycle-shadow families exhibit slower but still negative phase-*2 *drift* (-0.18 *for* 1^90 *shadows and* -0.26 *for* 1^30 *shadows, stable from* 1,024 *to* 16,384 *bits*).

Conjecture 8 (XOR Collatz Conjecture and Open Problems). *The following statements remain unsettled:*

1. (Global convergence.) *For every integer* $n \geq 1$*, repeated iteration of* T *eventually reaches* 3 (*equivalently,* $3 \to 12 \to 6 \to 3$ *is the sole cycle of* T *and no orbit diverges to* ∞).
2. (Long cycles.) *No cycle of* T *other than* $3 \to 12 \to 6 \to 3$ *exists with more than* 24 *shortcut steps.*
3. (Origin of carry-written parity bias.) *Explain theoretically why bits produced by the carries of* $2n + (n+1)$ *exhibit an odd-step frequency near* $q \approx 1/5$ *on random starts* (*rather than the neutral* $q = 1/3$)*, and determine whether every start family has phase-*2 *drift bounded strictly below* 0.

Figure 1 summarizes the proof and computational verification architecture across written proofs, standalone and extended certificates, and the unsettled conjecture boundary.

Concept in Action: The 3-Loop, $n = 25$, Orbit of 5, and Mersenne $2^{16} - 1$

```
def T(n): return n // 2 if n % 2 == 0 else (3 * n + 1) ^ (2 * n)
def to_three(n):
    seq = [n]
    while seq[-1] != 3: seq.append(T(seq[-1]))
    return seq
assert T(3) == 12 and T(12) == 6 and T(6) == 3
assert T(25) == 126 and len(to_three(5)) - 1 == 13
assert len(to_three((1 << 16) - 1)) - 1 == 17
```

Roadmap. Section 2 sets notation, exhibits the bit-by-bit carry mechanics for $n = 3, 5, 25$, and traces the orbit of 5 (Table 1, Figure 2). Section 3 gives complete, self-contained written proofs of Theorems 1, 2, 4, 5, and Proposition 3. Section 4 reports the standalone verification, 2-adic cycle-exclusion method, extended sweeps through 2^{36}, and two-phase drift measurements (Table 3, Table 2, Figure 3). Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 documents logged prior-work searches, relatives, unread caveats (`[to check]`), and open questions, and Section 7 details reproducibility and AI disclosure.

2. DEFINITIONS, CARRY MECHANICS, AND WORKED EXAMPLES

Every nonnegative integer n has a unique binary expansion $n = \sum_{j \geq 0} n_j 2^j$ with bits $n_j \in \{0, 1\}$. For $n \geq 1$, its bit length is $\ell(n) = \lfloor \log_2 n \rfloor + 1$, so $n_{\ell(n)-1} = 1$ and $2^{\ell(n)-1} \leq n < 2^{\ell(n)}$. Bitwise XOR ($\oplus$), bitwise AND ($\&$), and bitwise complement ($\neg$) act coordinatewise on binary digits; in particular, doubling ($2n$) shifts bits left by one position (0 at bit 0), and halving an even integer ($n/2$) deletes its trailing 0.

Why carries drive the dynamics. Write $3n+1 = 2n+(n+1)$. If binary addition of $2n$ and $n+1$ produced no carries, then $2n + (n+1)$ would equal $2n \oplus (n+1)$, and XORing $2n$ back off would yield $(2n \oplus (n+1)) \oplus 2n = n+1$. With carries included, each incoming carry flips a sum bit before $2n$ is XORed off, multiplying odd n by factors ranging from just above 1 to above 5:

- *The 3-cycle* ($n = 3$). Here $3 = 11_2$ gives $3 \cdot 3 + 1 = 10 = 1010_2$ and $2 \cdot 3 = 6 = 0110_2$ (one carry enters at bit 2); they differ in bits 3 and 2, so $h(3) = 1100_2 = 12$. Two halvings give $12 \to 6 \to 3$, closing the 3-cycle $3 \to 12 \to 6 \to 3$ (bit lengths $2 \to 4 \to 3 \to 2$).
- *One odd step from* $n = 25$. Here $25 = 11001_2$ gives $3 \cdot 25 + 1 = 76 = 1001100_2$ and $2 \cdot 25 = 50 = 0110010_2$ (carries enter at bit positions $2, 5, 6$). Every bit position except bit 0 differs, so $h(25) = 1111110_2 = 126$, jumping from $26 = n + 1$ up to 126 ($5.04\,n$).
- *The full orbit of* $n = 5$. Starting at $5 = 101_2$, the trajectory visits

$$5 \to 26 \to 13 \to 50 \to 25 \to 126 \to 63 \to 192 \to 96 \to 48 \to 24 \to 12 \to 6 \to 3,$$

entering $\{3, 6, 12\}$ at 12 after 11 steps and reaching 3 after 13 steps (4 odd steps listed in Table 1, 9 halvings, peak 192). Notice that $T(5) = 26 = 11010_2$ enters the 11-regime on step 1, after which every bit length obeys Theorem 2 ($5 \to 4 \to 6 \to 5 \to 7 \to 6 \to 8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 2$). By contrast, ordinary Collatz sends $5 \to 16 \to 8 \to 4 \to 2 \to 1$.

Table 1. The four odd steps along the 13-step orbit from $n = 5$ to 3 (recomputed verbatim in `recompute.out`).

Odd n	$3n+1$	$2n$	$h(n) = (3n+1) \oplus 2n$
5	$16 = 10000_2$	$10 = 01010_2$	$11010_2 = 26$
13	$40 = 101000_2$	$26 = 011010_2$	$110010_2 = 50$
25	$76 = 1001100_2$	$50 = 0110010_2$	$1111110_2 = 126$
63	$190 = 10111110_2$	$126 = 01111110_2$	$11000000_2 = 192$

Figure 2 displays the bit-by-bit carry addition and XOR step for $n = 3, 5, 25$ (Panel A) alongside the right-aligned binary grid of the alternating start $A_{16} = (4^{16} - 1)/3$ (Panel B), where the embedded $\mathbb{F}_2[x]$ Sierpiński pattern and locked-in 11-regime leading edge are directly visible.

3. PROOFS OF THE COMPANION THEOREMS

This section gives complete, self-contained written proofs of Theorems 1, 2, 4, 5, and Proposition 3.

Proof of Theorem 1. Let $n \geq 1$ be odd. Both $3n+1$ and $2n$ are even, so bit 0 of $h(n) = (3n+1) \oplus 2n$ is $0 \oplus 0 = 0$; hence $h(n)$ is even. For any two bits $a, b \in \{0, 1\}$, direct inspection of the four cases verifies the identity

$$a \oplus b = (a - b) + 2(1 - a)b.$$

Multiplying by 2^j and summing over all bit positions $j \geq 0$ gives, for all nonnegative integers A, B,

$$A \oplus B = (A - B) + 2(B \,\&\, \neg A).$$

Setting $A = 3n + 1$ and $B = 2n$ yields (3):

$$h(n) = (n + 1) + 2(2n \,\&\, \neg(3n + 1)).$$

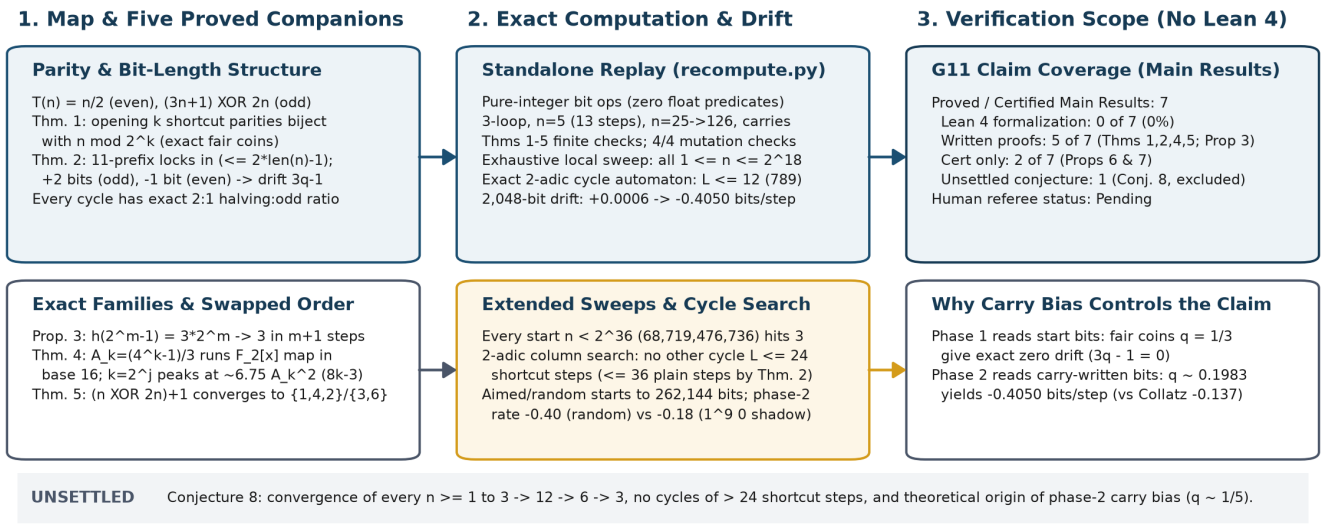


Figure 1. Discovery and verification architecture for Theorems 1, 2, 4, 5, Propositions 3, 6, 7, and Conjecture 8. No Lean 4 formalization is claimed (0 of 7 proved or certified main results in Lean 4; 5 of 7 by written proof; 2 of 7 by standalone exact Python replay and committed multi-program verification records). Conjecture 8 remains unsettled.

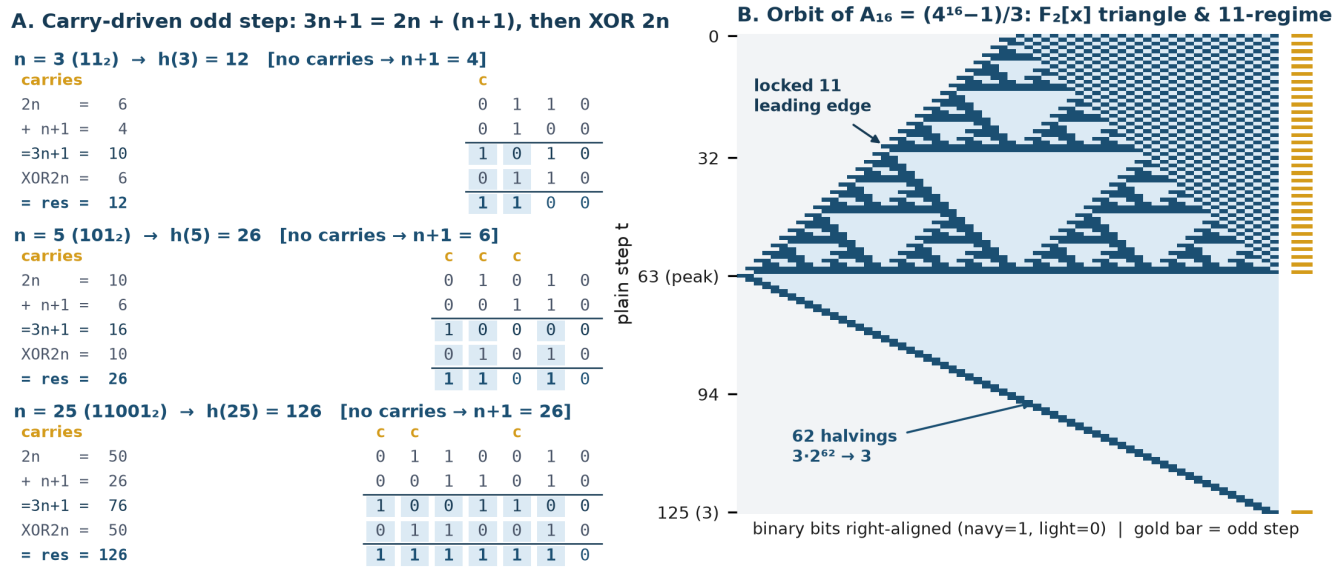


Figure 2. Bit-level mechanics of the XOR Collatz map. Panel A: schoolbook binary addition $2n+(n+1)=3n+1$ with incoming carries marked in gold (c), followed by XORing $2n$ back off (shaded columns mark bit differences) for $n=3,5,25$; dropping carries would leave $n+1$ in every case. Panel B: right-aligned binary grid of the alternating start $A_{16}=(4^{16}-1)/3$ (31 bits, 125 plain steps to 3; navy = 1, light blue = 0, gold bar = odd step), exhibiting the $\mathbb{F}_2[x]$ Sierpiński triangle on the climb to $3\cdot 2^{62}$ at step 63, the locked-in 11 leading edge, and 62 consecutive halvings down to 3.

Since $0 \le 2n \,\&\, \neg(3n+1) \le 2n$, the upper bound $h(n) \le (n+1)+4n = 5n+1$ is immediate. For the strict lower bound, let $v := v_2(n+1) \ge 1$ be the 2-adic valuation of $n+1$, so $n+1 = 2^v(2q+1)$ for some $q \ge 0$. Then $n = 2^v(2q+1)-1$ has bit $v-1$ equal to 1, so $2n$ has bit v equal to 1. Meanwhile, $n+1$ has bits $0,\dots,v-1$ all equal to 0 and bit v equal to 1. Adding $2n+(n+1)$ therefore sends no carry into bit v, and in bit v adds $1+1=2$, leaving bit v of $3n+1$ equal to 0. Consequently, bit v of $2n \,\&\, \neg(3n+1)$ equals 1, so $2n \,\&\, \neg(3n+1) \ge 2^v \ge 2$, proving $h(n) \ge (n+1)+2^{v+1} \ge n+5$.

Now establish the fair-parity bijection for S on $\mathbb{Z}_{\ge 0}$. First, for any $k \ge 2$ and any $n, m \ge 0$ with $n \equiv m \pmod{2^k}$, we have $S(n) \equiv S(m) \pmod{2^{k-1}}$: if n is even, $n/2 \equiv m/2 \pmod{2^{k-1}}$; if n is odd, both $3n+1 \equiv 3m+1 \pmod{2^k}$ and $2n \equiv 2m \pmod{2^k}$ agree in their lowest k bits, so their XOR agrees modulo 2^k, and dividing by 2 gives $S(n) \equiv S(m) \pmod{2^{k-1}}$. By induction on r, $S^r(n) \bmod 2^{k-r}$ depends only on $n \bmod 2^k$ for $0 \le r \le k-1$; hence $P_k(n)$ is well-

defined on $\mathbb{Z}/2^k\mathbb{Z}$.

Second, we claim that for every $k \geq 2$ and every $n \geq 0$,

$$S(n + 2^{k-1}) \equiv S(n) + 2^{k-2} \pmod{2^{k-1}}. \tag{4}$$

If n is even, $S(n + 2^{k-1}) = (n + 2^{k-1})/2 = S(n) + 2^{k-2}$ holds identically. If n is odd, then

$$\begin{aligned} 3(n + 2^{k-1}) + 1 &= (3n + 1) + 3 \cdot 2^{k-1} \\ &\equiv (3n + 1) + 2^{k-1} \pmod{2^k}, \\ 2(n + 2^{k-1}) &= 2n + 2^k \equiv 2n \pmod{2^k}. \end{aligned}$$

Thus adding 2^{k-1} to n flips bit $k-1$ of $3n+1$ while leaving bits $0, \ldots, k-2$ of $3n+1$ and all bits $0, \ldots, k-1$ of $2n$ untouched. XORing therefore flips bit $k-1$ of $h(n)$ and preserves bits $0, \ldots, k-2$: $h(n + 2^{k-1}) \equiv h(n) + 2^{k-1} \pmod{2^k}$. Dividing by 2 yields (4).

Finally, we prove by induction on $k \geq 1$ that $P_k : \mathbb{Z}/2^k\mathbb{Z} \to \{0,1\}^k$ is injective (hence bijective, as both finite sets have size 2^k). The base case $k = 1$ is immediate since $P_1(0) = (0)$ and $P_1(1) = (1)$. For $k \geq 2$, suppose $0 \leq n, m < 2^k$ satisfy $P_k(n) = P_k(m)$. Truncating to the first $k-1$ entries gives $P_{k-1}(n) = P_{k-1}(m)$, so $n \equiv m \pmod{2^{k-1}}$ by the induction hypothesis. If $n \neq m$, then interchanging n and m if necessary, $m = n + 2^{k-1}$. By (4), $S(m) \equiv S(n) + 2^{k-2} \not\equiv S(n) \pmod{2^{k-1}}$. Applying the induction hypothesis at level $k-1$ to the distinct residues $S(n)$ and $S(m)$ modulo 2^{k-1} gives $P_{k-1}(S(n)) \neq P_{k-1}(S(m))$, contradicting $P_k(n) = (n \bmod 2, P_{k-1}(S(n))) = P_k(m)$. Thus $n = m$, completing the induction. □

Proof of Theorem 2. Part (1). Suppose $x \geq 3$ lies in the 11-regime with bit length $L = \ell(x) \geq 2$, so $3 \cdot 2^{L-2} \leq x \leq 2^L - 1$.

- If x is even, then $L \geq 3$ (the sole 2-bit integer in the 11-regime is 3, which is odd). Halving gives $3 \cdot 2^{L-3} \leq x/2 < 2^{L-1}$, so $T(x) = x/2 \geq 3$ has bit length $L-1$ and leading digits 11.
- If x is odd, then

$$2^{L+1} < \frac{9}{4} \cdot 2^L + 1 \leq 3x + 1 \leq 3 \cdot 2^L - 2 < 3 \cdot 2^L.$$

  Hence $3x + 1$ has bit $L+1$ equal to 1, bit L equal to 0, and no nonzero bits above position $L+1$. Meanwhile, $2^L < 3 \cdot 2^{L-1} \leq 2x < 2^{L+1}$, so $2x$ has bit $L+1$ equal to 0, bit L equal to 1, and no nonzero bits above position L. XORing places $1 \oplus 0 = 1$ at bit $L+1$, $0 \oplus 1 = 1$ at bit L, and 0 at all higher bits. Therefore $3 \cdot 2^L \leq h(x) < 2^{L+2}$, meaning $\ell(T(x)) = L + 2$ and $T(x)$ lies in the 11-regime.

Part (2). If $n = 1$, then $T(1) = 6 = 110_2$ enters the 11-regime in $1 = 2\ell(1) - 1$ step. Suppose $n \geq 2$ is outside the 11-regime with $L = \ell(n) \geq 2$; then its leading two bits are 10, so $2^{L-1} \leq n < 3 \cdot 2^{L-2}$.

- If n is even, $\ell(T(n)) = \ell(n/2) = L - 1$ (bit length drops by 1 in 1 step).
- If n is odd and $3n+1 \geq 2^{L+1}$, then $3n+1 < \frac{9}{4} \cdot 2^L + 1 < 3 \cdot 2^L$ (since $L \geq 2$ implies $1 < \frac{3}{4} \cdot 2^L$). Just as in Part (1), $3n+1$ has bits $(L+1, L) = (1, 0)$ while $2n \in [2^L, 3 \cdot 2^{L-1})$ has bits $(L+1, L) = (0, 1)$, so $T(n) = h(n)$ enters the 11-regime immediately in 1 step.
- If n is odd and $3n+1 < 2^{L+1}$, then $3n+1 \geq 3 \cdot 2^{L-1} + 1 > 2^L$, so both $3n+1$ and $2n$ lie in $[2^L, 2^{L+1})$ and share bit L equal to 1. Their XOR cancels bit L, giving $h(n) < 2^L$. Because $h(n)$ is even and positive, the following halving produces $T^2(n) = h(n)/2 < 2^{L-1}$, so $\ell(T^2(n)) \leq L - 1$ (bit length drops by at least 1 in 2 steps).

Thus, while outside the 11-regime, bit length decreases by at least 1 within at most 2 steps. Reducing from $\ell(n)$ to 1 takes at most $2(\ell(n) - 1)$ steps, and $n' = 1$ enters at $T(1) = 6$ in 1 further step, giving at most $2\ell(n) - 1$ plain steps in all.

Part (3). By Parts (1) and (2), every iterate after entry satisfies $\ell(T^N(x)) - \ell(x) = 2O - E = 2O - (N - O) = (3q-1)N$. Any cycle must lie entirely in the 11-regime (else it could never return to an element outside the closed 11-regime), so periodicity forces $2O - E = 0$, i.e., $E = 2O$, plain length $O + E = 3O$, and shortcut length $O + (E - O) = 2O$ (as each of the O odd steps absorbs one halving into an odd shortcut step, leaving $E - O = O$ even shortcut steps). Under fair-coin parities in shortcut steps (1/2 odd, 1/2 even, realized on opening bits by Theorem 1), an average shortcut step uses $\frac{1}{2}(2) + \frac{1}{2}(1) = 3/2$ plain steps with 1/2 odd step, giving plain odd share $q = (1/2)/(3/2) = 1/3$ and zero expected drift $3q - 1 = 0$. □

Proof of Proposition 3. Let $m \geq 1$ and $n = 2^m - 1$ (odd). Expanding in base 2,

$$3n + 1 = 3 \cdot 2^m - 2 = 2^{m+1} + (2^m - 2) = 2^{m+1} + \sum_{j=1}^{m-1} 2^j,$$

$$2n = 2^{m+1} - 2 = 2^m + \sum_{j=1}^{m-1} 2^j$$

(with the sum empty when $m = 1$). The common terms $\sum_{j=1}^{m-1} 2^j$ cancel under bitwise XOR, while bits $m+1$ and m remain disjoint, giving

$$h(2^m - 1) = (3n + 1) \oplus 2n = 2^{m+1} + 2^m = 3 \cdot 2^m.$$

Following this single odd step by m consecutive halvings yields $3 \cdot 2^m \to 3 \cdot 2^{m-1} \to \ldots \to 6 \to 3$ in exactly $m + 1$ plain steps. □

Proof of Theorem 4. Let $R_r := (4^r - 1)/3$ for $r \geq 0$ (so $R_0 = 0$ and $R_r = 4^{r-1} + R_{r-1}$ for $r \geq 1$, with $A_k = R_k$). Define $g(b) := ((3b + 2) \oplus 2b) \oplus 1$ for $b \geq 0$.

Step 1 (Tail-stripping identity). We claim that for every $a \geq 0$ and every $r \geq 1$, the integer $x := a \cdot 4^r + R_r$ and its shortcut image $y := S(x)$ are both odd, and

$$S^2(a \cdot 4^r + R_r) = g(h(a)) \cdot 4^{r-1} + R_{r-1}. \tag{5}$$

Since $r \geq 1$, $R_r \equiv 1 \pmod 4$, so x is odd. Compute

$$3x + 1 = (3a + 1) \cdot 4^r, \qquad 2x = 2a \cdot 4^r + 2R_r.$$

Because $0 \leq 2R_r = \frac{2}{3}(4^r - 1) < 4^r$, the bit ranges above and below 4^r do not overlap in $2x$, and $3x + 1$ vanishes below 4^r. Hence

$$h(x) = ((3a + 1) \oplus 2a) \cdot 4^r + 2R_r = h(a) \cdot 4^r + 2R_r.$$

Dividing by 2 gives $y = S(x) = 2b \cdot 4^{r-1} + R_r$, where $b := h(a)$. Because R_r is odd, y is again odd. Writing $R_r = 4^{r-1} + R_{r-1}$

yields

$$3y+1 = 6b\cdot 4^{r-1} + (4^r-1) + 1 = 2(3b+2)\cdot 4^{r-1},$$
$$2y = 4b\cdot 4^{r-1} + 2R_r = (4b+2)\cdot 4^{r-1} + 2R_{r-1}.$$

Again $0 \le 2R_{r-1} < 4^{r-1}$, so the bit ranges above and below 4^{r-1} are disjoint:

$$h(y) = (2(3b+2) \oplus (4b+2))\cdot 4^{r-1} + 2R_{r-1}.$$

Dividing by 2 commutes with XOR on even integers and turns $4b+2$ into $2b+1 = (2b)\oplus 1$; thus

$$S(y) = ((3b+2)\oplus 2b\oplus 1)\cdot 4^{r-1} + R_{r-1} = g(b)\cdot 4^{r-1} + R_{r-1},$$

proving (5) (and confirming that both shortcut steps are odd, costing 4 plain steps).

Step 2 (Base-16 nibble reduction to $\mathbb{F}_2[x]$). Let $P(x) = \sum_{i\ge 0} p_i x^i \in \mathbb{F}_2[x]$ ($p_i \in \{0,1\}$, finitely many nonzero), and set $a := \Phi(P) = \sum_{i\ge 0}(6p_i)\,16^i$. Define $c_0 := 1$ and $c_i := p_{i-1}$ for $i \ge 1$. We track base-16 digits (4-bit nibbles) directly:

- In $2a = \sum_{i\ge 0}(12p_i)\,16^i$, every nibble $12p_i \in \{0,12\} \subset [0,15]$, with no carries between nibbles.
- In $3a+1 = 1 + \sum_{i\ge 0}(16p_i + 2p_i)\,16^i = \sum_{i\ge 0}(2p_i + c_i)\,16^i$, every nibble satisfies $2p_i + c_i \in \{0,1,2,3\} \subset [0,15]$, so no carry propagates beyond one nibble. Because $12p_i$ uses only the top two bits of nibble i while $2p_i + c_i$ uses only the bottom two bits, their bitwise XOR is their nibble sum:
$$b := h(a) = (3a+1)\oplus 2a = \sum_{i\ge 0} d_i\,16^i,$$
$$d_i := 14p_i + c_i \in \{0,1,14,15\}.$$
- Now form $2b\oplus 1$. At nibble 0, $2d_0 = 28p_0 + 2 = 16p_0 + (12p_0+2)$ (carry out p_0); XORing 1 turns the even nibble $12p_0+2$ into $12p_0+3 = 12p_0 + 3c_0$. At nibble $i \ge 1$, incoming carry $p_{i-1} = c_i$ gives $2d_i + c_i = 28p_i + 3c_i = 16p_i + (12p_i + 3c_i)$ (carry out p_i, nibble $12p_i + 3c_i \in \{0,3,12,15\}$). Thus uniformly for every $i \ge 0$, nibble i of $2b\oplus 1$ is $12p_i + 3c_i$.
- Next form $3b+2$. At nibble 0, $3d_0 + 2 = 42p_0 + 5 = 2\cdot 16p_0 + (10p_0 + 5c_0)$ (carry out $2p_0$). At nibble $i \ge 1$, incoming carry $2p_{i-1} = 2c_i$ gives $3d_i + 2c_i = 42p_i + 5c_i = 2\cdot 16p_i + (10p_i + 5c_i)$ (carry out $2p_i$, nibble $10p_i + 5c_i \in \{0,5,10,15\}$). Thus uniformly for every $i \ge 0$, nibble i of $3b+2$ is $10p_i + 5c_i$.

XORing nibble i of $3b+2$ with nibble i of $2b\oplus 1$ over the four states $(p_i, c_i) \in \{0,1\}^2$ gives

$$(0,0): 0\oplus 0 = 0, \qquad (1,0): 10\oplus 12 = 6,$$
$$(0,1): 5\oplus 3 = 6, \qquad (1,1): 15\oplus 15 = 0,$$

which equals $6(p_i \oplus c_i)$ in every case. Since $\sum_{i\ge 0}(p_i\oplus c_i)x^i = (x+1)P(x) + 1$ in $\mathbb{F}_2[x]$, we obtain the exact identity

$$g(h(\Phi(P))) = \Phi((x+1)P + 1). \tag{6}$$

Step 3 (Orbit of A_k and specialization to $k = 2^j$, $j \ge 1$). Writing $A_k = 0\cdot 4^k + R_k$ with $0 = \Phi(0)$ and iterating (5) and (6) k times yields, after $2k$ odd shortcut steps ($4k$ plain steps),

$$T^{4k}(A_k) = S^{2k}(A_k) = \Phi(P_k),$$

where $P_0 = 0$ and $P_{m+1} = (x+1)P_m + 1$ in $\mathbb{F}_2[x]$. Induction gives $P_m(x) = \sum_{t=0}^{m-1}(x+1)^t = ((x+1)^m + 1)/x = D_m(x)$, proving Part (1).

Now let $k = 2^j$ with $j \ge 1$. In $\mathbb{F}_2[x]$, $(x+1)^{2^j} = x^{2^j} + 1$, so $D_k(x) = x^{k-1}$ and therefore

$$T^{4k}(A_k) = 6\cdot 16^{k-1} = 3\cdot 2^{4k-3}.$$

Because plain step $4k$ is the halving half of the $2k$-th odd shortcut step, the preceding iterate $T^{4k-1}(A_k) = 2\,T^{4k}(A_k)$ equals $3\cdot 2^{4k-2}$ and has bit length $4k$. From $T^{4k-1}(A_k) = 3\cdot 2^{4k-2}$, $4k-2$ consecutive halvings descend directly to 3 at plain step $(4k-1)+(4k-2) = 8k-3$. To confirm that 3 is not hit earlier and that $3\cdot 2^{4k-2}$ is the strict global peak, note that for $k \ge 2$, A_k has bit length $2k-1 \ge 3$ and $3A_k + 1 = 4^k = 2^{(2k-1)+1}$, so by the proof of Theorem 2 (Part 2), $T(A_k)$ enters the 11-regime immediately at step 1 with bit length $2k+1$. Alternating halvings (-1) and odd steps $(+2)$ through step $4k-1$ keeps bit lengths strictly within $[2k, 4k-1]$ for all $1 \le t < 4k-1$ (always exceeding $\ell(3) = 2$ and strictly below $4k = \ell(T^{4k-1}(A_k))$). Finally, dividing $3\cdot 2^{4k-2}$ by $A_k^2 = (2^{2k}-1)^2/9$ yields the ratio $\frac{27}{4}(1-2^{-2k})^{-2}$, which converges to $27/4 = 6.75$. At $k = 1$ ($j = 0$), $A_1 = 1 \to 6 \to 3$ terminates early at step 2 with peak 6. □

Proof of Theorem 5. Let $n \ge 1$ be odd with bit length $L = \ell(n) \ge 1$, so $n_{L-1} = 1$, $n_0 = 1$, and $n_j = 0$ for $j \ge L$. Writing $m := n \oplus 2n$, bit L of m is $0\oplus 1 = 1$, bit 0 is $1\oplus 0 = 1$, and all bits above L vanish; thus m is odd with $2^L \le m \le 2^{L+1} - 1$, and $\widetilde{T}(n) = m + 1$ is even with $2^L + 1 \le \widetilde{T}(n) \le 2^{L+1}$.

Two cases arise. If $m = 2^{L+1} - 1$, then $\widetilde{T}(n) = 2^{L+1}$, and $L+1$ consecutive halvings send the orbit straight to 1 (bit length $1 \le L$), entering $1 \to 4 \to 2 \to 1$. Otherwise $m \le 2^{L+1} - 3$, so $\widetilde{T}(n) \le 2^{L+1} - 2 < 2^{L+1}$; halving once gives $\widetilde{T}^2(n) = \widetilde{T}(n)/2 < 2^L$, hence $\ell(\widetilde{T}^2(n)) \le L$, and any further halvings only decrease bit length. Consequently, the bit length of consecutive odd iterates along any $\widetilde{T}$-orbit is non-increasing. Every orbit is therefore bounded and eventually periodic.

Let C be any cycle of $\widetilde{T}$. Because odd bit length cannot increase, all odd elements of C share a common bit length $L \ge 1$. If $L = 1$, the only odd element is 1, yielding $1 \to 4 \to 2 \to 1$. Suppose now that $L \ge 2$. For every odd $n \in C$, we cannot have $m = 2^{L+1} - 1$ (that reaches 1 of length $1 < L$), so $\widetilde{T}(n) < 2^{L+1}$. Moreover, $\widetilde{T}(n)/2$ must be odd—otherwise a second halving would drop its bit length to at most $L-1$, contradicting constancy of L. Thus $\widetilde{T}(n) = (n\oplus 2n) + 1 \equiv 2 \pmod 4$, so $n \oplus 2n \equiv 1 \pmod 4$. Since bit 1 of $n\oplus 2n$ is $n_1 \oplus n_0 = n_1 \oplus 1$, this forces $n_1 = 1$ (i.e., $n \equiv 3 \pmod 4$).

When $n \equiv 3 \pmod 4$, adding 1 to $n \oplus 2n$ flips bit 0 ($1 \to 0$) and bit 1 ($0 \to 1$) with zero carries into bit 2 or above. Halving therefore maps the odd iterate n to $\tilde{S}(n) := \widetilde{T}(n)/2$ whose bits satisfy $(\tilde{S}(n))_0 = 1$ and $(\tilde{S}(n))_r = n_r \oplus n_{r+1}$ for all $r \ge 1$. Restricting to the vector $v = (v_0, \dots, v_{L-2}) := (n_1, \dots, n_{L-1}) \in \mathbb{F}_2^{L-1}$ (with $v_j = 0$ for $j \ge L-1$), the transition is the linear map $v \mapsto Dv := (I+N)v$, where $(Nv)_k = v_{k+1}$ is the nilpotent left shift.

Because every odd iterate $\tilde{S}^t(n)$ ($t \ge 0$) remains in C, we must have $(D^t v)_0 = (\tilde{S}^t(n))_1 = 1$ for every $t \ge 0$. Taking successive differences over $\mathbb{F}_2$ gives, for every $t \ge 0$,

$$0 = (D^{t+1}v)_0 \oplus (D^t v)_0 = (D^t(D\oplus I)v)_0 = (D^t w)_0,$$

where $w := Nv = (v_1, \dots, v_{L-2}, 0)$. Expanding $(D^t w)_0 = ((I+N)^t w)_0 = \sum_{r=0}^{t} \binom{t}{r} w_r = 0$ over $\mathbb{F}_2$ for $t = 0, 1, \dots, L-3$ yields $w_0 = 0$, then $w_1 = 0$, and inductively $w_r = 0$ for all $0 \le r \le L-3$. Hence $n_2 = n_3 = \cdots = n_{L-1} = 0$. If $L \ge 3$, this contradicts $n_{L-1} = 1$. Therefore $L = 2$, forcing $n = 3$ and the cycle $3 \to 6 \to 3$. □

4. COMPUTATION AND TWO-PHASE DRIFT

Standalone Exact Replay. All finite claims of Proposition 6 are verified in under 2 seconds by `recompute.py` using exact Python integers (no floating-point step predicates; output in `recompute.out`):

- *Worked examples and carry identities.* Verifies $3 \to 12 \to 6 \to 3$, $h(25) = 126$, the 13-step orbit of 5 (Table 1), and $h(n) = (n+1) + 2(2n \,\&\, \neg(3n+1))$ with $n + 5 \le h(n) \le 5n + 1$ for every odd $n < 10^5$.
- *Companion checks.* Verifies Theorem 1 for $1 \le k \le 12$ (4,096 residues at $k = 12$); Theorem 2 across all 32,767 11-regime integers below 2^{16} and all $n < 2^{16}$ (maximum entry time 19 steps $\le 2\ell(n) - 1$); Proposition 3 for $1 \le m \le 64$; Theorem 4 for $1 \le k \le 64$, all 1,024 polynomials of degree < 10, and exact peaks/step counts for $k = 2^j$ ($1 \le j \le 6$: $13, 29, 61, 125, 253, 509$ steps; peak/A_k^2 converging from 7.680000 at $k = 2$ to 6.750000 by $k = 16$) plus the $j = 0$ exception; and Theorem 5 over $1 \le n \le 10^4$ (7,500 into $\{1, 4, 2\}$, 2,500 into $\{3, 6\}$, 0 odd-length increases).
- *Local exhaustive sweep* ($n \le 2^{18} = 262{,}144$). Every start $1 \le n \le 262{,}144$ enters $\{3, 6, 12\}$: up to 200,000, the longest entry is 143 steps at $n = 179{,}477$ (mean 52.73 steps, versus classical Collatz longest 382 at 156,159 and mean 114.7 in `sweep.out`); up to 262,144, the longest remains 143 at 179,477 (mean 54.02 steps) and the largest peak is 68,719,476,732 at $n = 87{,}381 = A_9$. The 10^6-record start $n = 764{,}837$ is also checked directly (157 steps to enter the loop, 159 to 3, 47 odd steps, peak 67,712,054,028).
- *Controls.* Four negative-mutation checks (carry-less $n+1$ failure, $k = 1$ exception, failure of $+2$ bit walk outside the 11-regime at $h(9) = 14$, and two-loop split of $\widetilde{T}$) are all caught (4/4), and the Section 1 Python box runs verbatim.

Exact 2-Adic Column Automaton for Cycle Exclusion. Unlike classical $3n + 1$, the odd branch $h(n)$ is not affine, so cycles cannot be parameterized by a single rational linear equation. Instead, every shortcut parity pattern $p = (p_0, \dots, p_{L-1}) \in \{0,1\}^L$ determines a unique 2-adic solution $(x_0, \dots, x_{L-1}) \in \mathbb{Z}_2^L$ column by column from the least significant bit upward. For each odd index i ($p_i = 1$), bit k of $3x_i + 1 = x_i + 2x_i + 1$ is $x_{i,k} \oplus x_{i,k-1} \oplus c_{i,k}$, where $c_{i,k} \in \{0,1\}$ is the carry into column k; XORing $2x_i$ cancels $x_{i,k-1}$ identically, leaving bit k of $h(x_i)$ equal to $x_{i,k} \oplus c_{i,k}$. Because $x_{i+1} = S(x_i)$ shifts right by 1, bit $k \ge 1$ satisfies

$$\begin{aligned} x_{i,k} &= x_{i+1,k-1} \oplus (p_i \,\&\, c_{i,k}), \\ c_{i,k+1} &= p_i \,\&\, \mathrm{maj}(x_{i,k}, x_{i,k-1}, c_{i,k}), \end{aligned} \tag{7}$$

initialized at $x_{i,0} = p_i$ and $c_{i,1} = p_i$ (indices modulo L). The finite state $(x_{\cdot,k-1}, c_{\cdot,k}) \in \{0,1\}^{2L}$ evolves deterministically and is therefore eventually periodic. Its 2-adic solution is a nonnegative integer cycle *if and only if* the state reaches $(0, 0)$ (decided exactly by Brent cycle detection).

In `recompute.py`, running (7) across all 789 nonzero binary necklaces of length $1 \le L \le 12$ (123 balanced) confirms that the only integer solutions are the 6 repetitions of $3 \leftrightarrow 6$. In the committed C/GMP implementations (`verdict.md`, `review.md`), the same exact automaton decided every nonzero necklace through $L = 23$ (364,723 necklaces at $L = 23$) and $L = 24$ (699,251 necklaces at $L = 24$; longest decision exceeding 9.6×10^7 columns), ruling out any other cycle of at most 24 shortcut steps (hence at most 36 plain steps by Theorem 2) of any element magnitude.

Two-Phase Drift and Carry-Written Parity Bias. By Theorem 1, the opening b shortcut parity steps of a random b-bit integer n are independent fair coin flips; each shortcut step consumes 1.5 plain steps on average, so the start's own random bits supply approximately $1.5\,b$ plain steps. By Theorem 2, once the orbit enters the 11-regime, fair coins ($q = 1/3$) produce exact zero drift ($3q - 1 = 0$).

Table 3 and Figure 3 report the exact two-phase measurement reproduced in `recompute.py` (matching `drift.out` to all four decimal places) on four random 2,048-bit starts (seed 20261119): Phase 1 covers $[0, 1.4b)$ and Phase 2 runs from $1.8b$ until bit length drops below $b/4 = 512$.

Two empirical subtleties recorded in `verdict.md` and `review.md` merit explicit emphasis. First, on larger random starts (8 runs of 16,384 bits), the same two-phase split persists cleanly: $+0.0038 \pm 0.0019$ bits/step in Phase 1 and -0.3990 ± 0.0027 bits/step in Phase 2 (corroborated on 4,096-bit starts at $[-0.002, +0.022]$ and $[-0.42, -0.39]$). Second, the -0.40 Phase-2 rate is *not* universal across all structured starts: 2-adic cycle-shadow starts engineered to repeat $1^9 0$ or $1^3 0$ climb to $2k/(k+1)$ times their initial bit length during Phase 1 and then descend in Phase 2 at slower, scale-invariant rates of -0.18 and -0.26 bits/step, respectively (measured consistently at 1,024, 4,096, and 16,384 bits). Every family tested exhibited strictly negative Phase-2 drift, yet no theoretical explanation of why carry-written bits favor even parities is currently known.

Table 2. Summary of verified start sweeps and exact 2-adic cycle exclusions across standalone replay (`recompute.py`) and committed multi-program audit records.

Verification scope	Key record / outcome	Source
$1 \le n \le 2^{18}$ (262,144)	All enter {3, 6, 12}; longest entry 143 at 179,477; peak 68,719,476,732 at 87,381	`recompute.out`
$1 \le n \le 10^6$	All enter {3, 6, 12}; longest entry 157 at 764,837 (159 steps to 3)	`verdict.md`
$1 \le n \le 10^8$	All enter {3, 6, 12}; longest entry 236 at 98,193,693; max/start 6.06×10^8 at A_{14}	`review.md`
$1 \le n < 2^{35}$ (34.36×10^9)	Exact 128-bit C sweep; longest descent 259 at 2,908,181,973	`verdict.md`
$[2^{35}, 2^{36})$ (68.72×10^9)	Every odd start descends; max descent 247 at 46,081,332,561; peak 72 bits	`verdict.md`
Cycles $L \le 12$ shortcut	All 789 nonzero necklaces (123 balanced); only repetitions of $3 \leftrightarrow 6$	`recompute.out`
Cycles $L \le 24$ shortcut	All nonzero necklaces (699,251 at $L = 24$); only $3 \leftrightarrow 6$ (≤ 36 plain steps)	`review.md`
Aimed and huge starts	2,367 cycle shadows (2,048b), 4,648 special forms, random starts up to 262,144 bits	`review.md`

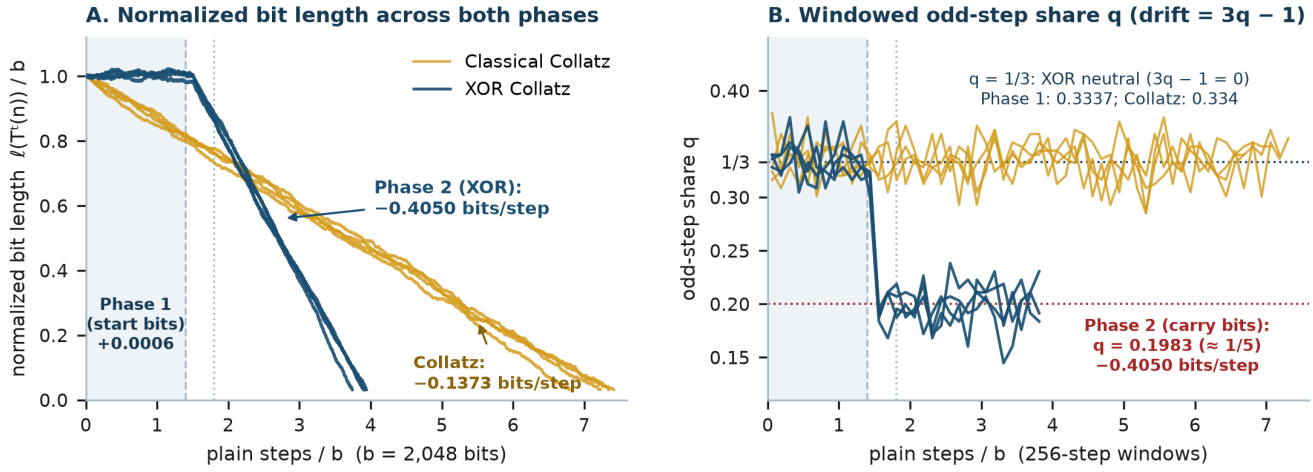


Figure 3. Two-phase drift on four random 2,048-bit starts (seed 20261119), comparing XOR Collatz (navy) and classical Collatz (gold). Panel A: normalized bit length $\ell(T^t(n))/b$ against t/b; in the shaded Phase 1 window $[0, 1.4b)$ where start bits govern parity, XOR Collatz remains level (+0.0006 bits/step) before plunging at −0.4050 bits/step in Phase 2, whereas classical Collatz descends steadily (−0.1334 and −0.1373). Panel B: odd-step share q in 256-step windows; XOR Collatz drops sharply from $q = 0.3337$ (the neutral line $q = 1/3$) to $q = 0.1983 \approx 1/5$ once carry-written bits reach bit 0.

Table 3. Two-phase bit-length drift (bits per plain step) and odd-step share q on four random 2,048-bit starts (seed 20261119; `recompute.out` and `drift.out`).

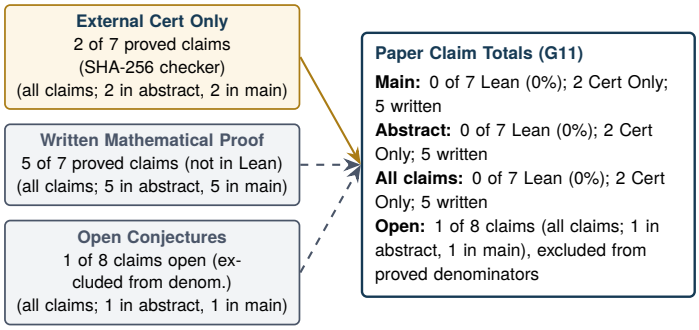
Trajectory	Phase 1 drift	q_1	Phase 2 drift	q_2
XOR start 0	+0.0028	0.3345	−0.4107	0.1964
XOR start 1	−0.0126	0.3296	−0.3880	0.2040
XOR start 2	+0.0056	0.3352	−0.4043	0.1986
XOR start 3	+0.0066	0.3355	−0.4169	0.1944
XOR mean (4 runs)	+0.0006	0.3337	−0.4050	0.1983
Collatz mean (4 runs)	−0.1334	0.3353	−0.1373	0.3337
Fair-coin ($q = 1/3$)	+0.0000	0.3333	+0.0000	0.3333

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 2 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 4 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 4. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: For odd n >= 1, h(n) = (n+1) + 2*(2n & ~(3n+1)) is even with n + 5 <= h(n) <= 5n + 1; for every k >= 1, the opening k shortcut parities P_k(n) depend only on n mod 2^k and biject Z/2^k Z -> {0,1}^k	Written	Written proof	Lean: none; fidelity: n/a	Self-contained elementary bitwise and inductive written proof (Section 3); finite check for odd n < 10^5 and 1 <= k <= 12 in recompute.py. No Lean 4 formalization.
Thm. 2: The 11-regime is closed under T with exact bit-length steps (+2 odd, -1 even), entered within <= 2*len(n) - 1 plain steps by every n >= 1; every cycle has exact 2:1 halving-to-odd ratio (even shortcut length) and fair-coin drift 3q - 1 vanishes at q = 1/3	Written	Written proof	Lean: none; fidelity: n/a	Self-contained leading-bit written proof (Section 3); finite check on all 32,767 regime numbers and all n < 2^16 in recompute.py. No Lean 4 formalization.
Prop. 3: For every integer m >= 1, h(2^m - 1) = 3 * 2^m, so 2^m - 1 reaches 3 in exactly m + 1 plain steps	Written	Written proof	Lean: none; fidelity: n/a	Elementary binary cancellation written proof (Section 3); finite check for 1 <= m <= 64 in recompute.py. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 4: For A_k = (4^k - 1)/3, T^(4k)(A_k) = S^(2k)(A_k) = 6*D_k(16) embeds P -> (x+1)P + 1 over F_2[x] in base 16; for k = 2^j (j >= 1), the orbit peaks at 3*2^(4k-2) ~ 6. 75*A_k^2 (step 4k-1) and reaches 3 in 8k - 3 plain steps (sole exception j = 0)	Written	Written proof	Lean: none; fidelity: n/a	Self-contained tail-stripping and base-16 nibble written proof (Section 3); finite check for 1 <= k <= 64, 1,024 polynomials, and j = 0..6 in recompute.py. No Lean 4 formalization.
Thm. 5: Under the swapped-order map n -> (n XOR 2n) + 1 (odd), n/2 (even), odd bit lengths never increase and every integer n >= 1 enters either 1 -> 4 -> 2 -> 1 or 3 -> 6 -> 3	Written	Written proof	Lean: none; fidelity: n/a	Self-contained F_2 linear-algebra written proof (Section 3); finite check on 1 <= n <= 10^4 in recompute.py. No Lean 4 formalization.
Prop. 6: Standalone exact Python replay certifies 3 -> 12 -> 6 -> 3, n = 5 (13 steps), n = 25 -> 126, companion finite checks, all 1 <= n <= 2^18 = 262,144, exact 2-adic cycle exclusion for L <= 12 (789 necklaces), and 2,048-bit two-phase drift (+0.0006, q = 0.3337 -> -0.4050, q = 0.1983)	Cert Only	recompute.py (d673d511eb4a) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Certified by standalone exact-integer Python script recompute.py (standard library only, zero floats in step predicates, under 2 s runtime, 4/4 negative-mutation controls).
Prop. 7: Committed multi-program exact-arithmetic records verify every start n < 2^36 = 68,719,476,736, rule out any other cycle up to L <= 24 shortcut steps (<= 36 plain steps), verify aimed/random starts up to 262,144 bits, and measure 16,384-bit and cycle-shadow phase-2 drifts	Cert Only	recompute.py (d673d511eb4a) [archived; replay pending]; drift.py (418324ed2e99) [regenerable; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Backed by committed exact-arithmetic audit records (verdict.md, review.md, sweep.out, drift.out; corroborated through 2^36 in blind re-review; raw C logs retired outside git and regenerable deterministically); recompute.py replays n <= 2^18, L <= 12, and 2,048-bit drift directly.
Conj. 8: Global convergence of every n >= 1 to 3 -> 12 -> 6 -> 3, absence of cycles beyond 24 shortcut steps, and theoretical origin of the phase-2 carry-written parity bias (q ~ 1/5 vs neutral q = 1/3) remain unsettled	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled conjecture and open problems (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: 33d7ccb2cda0 | Fidelity SHA-256: df91a53c4f11.

6. PRIOR WORK AND OPEN QUESTIONS

Nearest relatives in the literature. The map (1) sits at $a = 3$ inside the family of odd branches $n \mapsto (an + 1) \oplus (a - 1)n$, bridging integer carries and carry-less polynomial dynamics:

- *All-carry-less sibling* ($a = 2$, *OEIS A245471* [5]). Taking odd $n \mapsto n \oplus (2n + 1)$ removes all carries; as recorded in OEIS entry A245471 (read in full), every orbit is known to end in a cycle containing 1. By contrast, T retains the integer carries of $3n + 1$.
- *Polynomial Collatz analogues over* $\mathbb{F}_2[x]$ *and* $\mathbb{F}_q[x]$. The carry-less map $P \mapsto (x + 1)P + 1$ on $\mathbb{F}_2[x]$—which Theorem 4 uncovers embedded in base 16 inside the alternating orbits of T—was proved to converge for all polynomials by Hicks, Mullen, Yucas, and Zavislak [1] (`[to check]` in full text) and studied further by Snapp and Tracy [6] (`[to check]`), building on the finite-field Syracuse framework of Matthews and Leigh [4] (`[to check]`). All three treat pure polynomial rings with no integer carries.

- *Parity equidistribution and* $3x+1$ *bibliographies.* Theorem 1 is the direct analogue of Terras's parity-vector equidistribution theorem [7] (`[to check]`) for the classical $3n+1$ map, where parity equidistribution combines with multiplicative shrinkage to establish density-1 stopping times; here, by Theorem 2, fair parities yield zero drift, neutralizing the Terras density argument. General Collatz variants are surveyed in Lagarias's two annotated bibliographies [2, 3] (both searched in full text).

Logged prior-work searches. Three gate-passing literature search logs plus a 2026-11-22 Google Scholar sweep were recorded:

- *Initial and independent review logs* (2026-11-19)*:* OEIS searches by computed terms across more than 20 windows (including $h(n)$, $T(n)$, entry times to $\{3, 6, 12\}$, the orbit of 5, and record-setters: 0 hits) and by keywords (`Collatz XOR`, `Collatz bitwise`, `3n+1 XOR`: returning only A245471 [5] as a relative); arXiv, zbMATH, and web queries; full-text grep of Lagarias [2, 3]; 2 keyed Google Scholar searches (3 and 10 records) and 1 keyed CORE search (1 irrelevant obfuscation paper).
- *Follow-up CORE* (2026-11-20) *and Google Scholar* (2026-11-22) *sweeps:* 5 keyed CORE queries (1 irrelevant ordinary-Collatz record; CORE indexes titles and abstracts only) and 4 targeted Google Scholar queries (37 records covering XOR/bitwise/carry-less Collatz terminology and follow-ups to [1, 6, 4]: 0 same-rule or same-claim matches).

Consequently, the map (1), Conjecture 8, and the two-phase carry-drift phenomenon were not found in the searches logged.

Unread caveats and still-owed sources (`[to check]`). The following sources remain unread or unsearched and are flagged `[to check]` in `references-checked.md`: the paywalled full texts of Hicks et al. [1], Matthews and Leigh [4], and Terras [7], and the full text of Snapp and Tracy [6] (read via Lagarias's annotations [2, 3] and secondary summaries); R. K. Guy's *Unsolved Problems in Number Theory* (section E16, in print); direct MathOverflow and Math Stack Exchange queries (search APIs returned HTTP 429); OpenAlex and Semantic Scholar; and the `seqfan` mailing-list archive.

Open directions. Beyond the three parts of Conjecture 8, three concrete problems stand out: (i) prove that the phase-2 odd-step share q on random starts converges to a limiting constant near 0.198–0.200 by modeling the carry-propagation Markov chain on low bits in the 11-regime; (ii) extend the exact 2-adic cycle exclusion beyond $L = 24$ using Theorem 2 to restrict strictly to balanced necklaces (halving the search space at even $L \geq 26$); and (iii) formalize Theorems 1, 2, 4, 5, and Proposition 3 in Lean 4.

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure-generation scripts, and the standalone exact verifier live in `papers/xor-collatz/draft1/`. Running `python3 papers/xor-collatz/draft1/recompute.py` completes in under 2 seconds using only the Python standard library and deterministically reproduces `recompute.out` (all worked examples, carry identities for odd $n < 10^5$, finite checks of all five companion theorems, exhaustive local sweep $1 \leq n \leq 2^{18}$, exact 2-adic cycle exclusion for $L \leq 12$, the 2,048-bit two-phase drift measurement, the Section 1 Python snippet, and 4/4 negative-mutation controls). Committed summary receipts for the $n < 2^{36}$ sweep, $L \leq 24$ cycle exclusion, and extended drift runs live in the repository's `xor-collatz` documentation and audit archive (`drift.out`, `sweep.out`, `verdict.md`, `review.md`, SHA-256 pinned in Section 5); raw multi-gigabyte C sweep logs are retired outside git and are deterministically regenerable from the documented algorithms. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes discovered the XOR Collatz map and its two-phase carry-drift phenomenon, proved Theorems 1, 2, 4, 5, and Proposition 3, wrote and independently re-verified the Python and C sweep and cycle-exclusion programs, conducted the logged prior-work searches, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 24, 2026

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