MATHDISCOVERIES
Mathematical Preprint

Working Draft

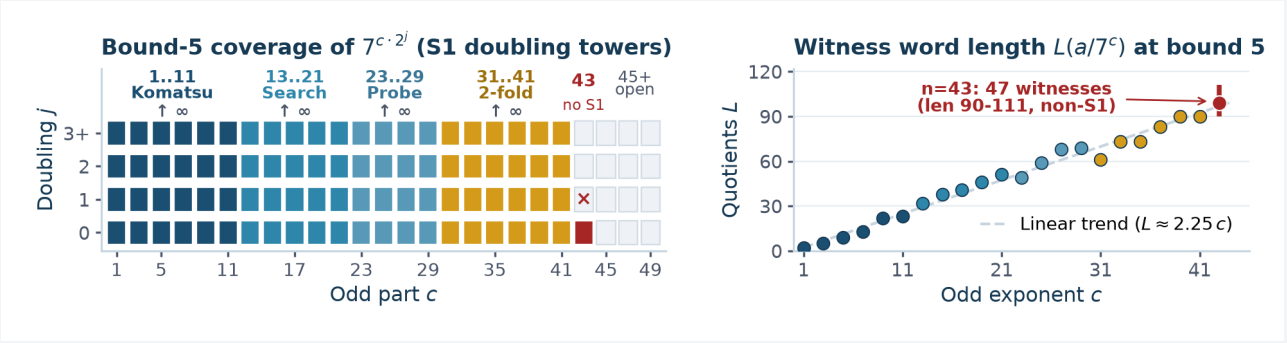
Zaremba's Conjecture at Bound 5 for Powers of 7 through Odd Part 41 and Exponent 43

Tyler B. Hughes*,†

†Independent Researcher, United States

Ⓢ *Supporting Information:* `recompute.py` *(standalone pure-Python exact Euclidean verifier)*

Disclosure: AI-assisted working draft directed by Tyler B. Hughes. Not for submission as is. No human mathematician has yet refereed the proof.



ABSTRACT: Zaremba's conjecture asserts that every integer $q \geq 2$ admits a coprime numerator a whose canonical continued fraction $a/q = [0; a_1, \ldots, a_L]$ has all partial quotients at most 5. For powers of 7, Shulga proved bound 6 for every 7^n and Komatsu settled odd parts $c \in \{1, 3, \ldots, 11\}$ at bound 3; bound 5 for all 7^n remains unsettled. Combining exact Euclidean witnesses with Dubno's S1 doubling theorem ($2 \leq a_1, a_L \leq 4$), we settle bound 5 for every 7^n whose odd part is at most 41, plus 47 non-S1 witnesses for $n = 43$ (lengths 90 to 111; $n = 86$ unsettled). The least unsettled exponent at bound 5 is $n = 45$. At smaller bounds, we extend bound 3 to odd parts up to 13 (least unsettled 15), settle bound 4 to odd parts up to 15 (least unsettled 17), and confirm exhaustively that bound 2 fails at 7^7. We give self-contained proofs of the classical and twisted folding lemmas, the unimodular odd-fold obstruction at bound 5, S1/S2 doubling, and bound-6 closure for powers of 7. No Lean 4 formalization is claimed.

Verification Scope (G11): WARN: Human referee: pending
Main results: 0 of 4 in Lean Kernel (0%) · 2 Cert Only · 2 written · 1 open (not counted)
Abstract claims: 0 of 4 in Lean Kernel (0%) · 2 Cert Only · 2 written

Full Trust Boundary: Section 5.

MSC 2020: 11A55, 11J70, 11Y99 **Keywords:** Zaremba's conjecture, continued fractions, partial quotients, powers of 7, folding lemma, doubling witnesses

1. INTRODUCTION AND MAIN RESULTS

For any reduced fraction a/q with $0 < a < q$ and $\gcd(a, q) = 1$, Euclid's algorithm produces its unique *canonical* finite simple continued fraction

$$\frac{a}{q} = [0; a_1, a_2, \ldots, a_L], \qquad a_i \geq 1, \quad a_L \geq 2.$$

Let $K(a/q) := \max_{1 \leq i \leq L} a_i$ denote the largest partial quotient of this canonical expansion. Zaremba [11] (`[to check]`) conjectured in 1971 that every integer $q \geq 2$ admits a coprime numerator a with $K(a/q) \leq 5$. Bourgain and Kontorovich [2] (`[to check]`) established the conjecture for a density-one subset of denominators at bound 50, and Huang [4] (`[to check]`) lowered the density-one bound to 5; however, density-one results do not cover sparse geometric sequences such as prime powers $q = p^n$.

For pure prime powers, Niederreiter [7] proved $K(a/q) \leq 3$ for every 2^n and 3^n, and $K(a/5^n) \leq 4$ for every 5^n, while Yodphotong and Laohakosol [10] (`[to check]`) proved bound 5 for every 6^n, and Dubno [3] proved bound 5 for every 12^n and 18^n. For powers of 7, Shulga [9] proved the general radical bound $K(a/q) \leq \mathrm{rad}(q) - 1$ for every $q \neq 2^n, 3^n$, giving $K(a/7^n) \leq 6$ for all $n \geq 1$. At smaller bounds, Komatsu [5] proved $K(a/7^{c\cdot 2^j}) \leq 3$ for every $j \geq 0$ and odd parts $c \in \{1, 3, 5, 7, 9, 11\}$, while bound 5 for all powers of 7 remains unsettled.

Two elementary folding mechanisms turn finitely many odd-exponent witnesses into infinite towers $7^{c\cdot 2^j}$ ($j \geq 0$). Call a canonical word $w = (a_1, \ldots, a_L)$ of length $L \geq 2$ *shape S1 at bound k* (following Dubno [3]) if $K(w) \leq k$ and $2 \leq a_1, a_L \leq k - 1$. Repeated application of Mendès France's classical 1-fold preserves S1 and squares the denominator, sending 7^c to $7^{c\cdot 2^j}$ for every $j \geq 0$. Combining S1 witnesses across four construction regimes with exact Euclidean verification settles every exponent with odd part at most 41, together with the isolated exponent $n = 43$.

Theorem 1 (Bound 5 for Powers of 7 through Odd Part 41 and at $n = 43$). *Let $n \geq 1$ be an integer, and write $n = c \cdot 2^j$ with c odd and $j \geq 0$. If $c \leq 41$, or if $n = 43$, then there exists an integer a with $1 \leq a < 7^n$ and $\gcd(a, 7^n) = 1$ such that the canonical continued fraction of $a/7^n$ satisfies $K(a/7^n) \leq 5$. Specifically:*

1. *For each of the 21 odd integers $c \in \{1, 3, 5, \ldots, 41\}$, Table 1 exhibits an explicit coprime numerator a_c for 7^c whose canonical expansion has shape S1 at bound 5 ($2 \leq a_1, a_L \leq 4$); repeated 1-folding (Proposition 4) then produces an S1 bound-5 witness for $7^{c \cdot 2^j}$ for every $j \geq 0$.*
2. *For $n = 43$, forty-seven explicit coprime numerators for 7^{43} (34 via twisted 7^9-folds of 7^{17} and 13 via twisted 7^7-folds of 7^{18}, lengths 90 to 111; representative rows in Table 2) satisfy $K(a/7^{43}) \leq 5$. None of these 47 witnesses is S1 (each begins with $a_1 = 5$), so $n = 86$ is not settled by doubling.*

Consequently, the least unsettled exponent at bound 5 is $n = 45$.

Applying the same exact-arithmetic and doubling framework at smaller bounds $k \in \{2, 3, 4\}$ fixes the neighboring frontiers for powers of 7.

Proposition 2 (Smaller-Bound Frontier for Powers of 7). *Under the canonical continued-fraction convention:*

1. (Bound 4.) *For every $n = c \cdot 2^j$ with odd part $c \leq 15$ and $j \geq 0$, there exists a coprime numerator a with $K(a/7^n) \leq 4$, certified by 8 explicit S1 witnesses for $c \in \{1, 3, \ldots, 15\}$ (Table 3) and Proposition 4. The least unsettled exponent at bound 4 is $n = 17$.*
2. (Bound 3.) *For every $n = c \cdot 2^j$ with odd part $c \leq 13$ and $j \geq 0$, there exists a coprime numerator a with $K(a/7^n) \leq 3$, certified by Komatsu's seeds 5/7 and 30/49 together with 6 explicit S2 witnesses for $c \in \{3, 5, \ldots, 13\}$ (Table 3, extending Komatsu's $c \leq 11$ via 59,478,693,243/7^{13}) and Proposition 4. The least unsettled exponent at bound 3 is $n = 15$.*
3. (Bound 2.) *Exhaustive evaluation of all $\varphi(7^n)$ units for $1 \leq n \leq 8$ confirms that bound 2 holds for $n \in \{1, 2, 3, 4, 5, 6, 8\}$ (with 1, 4, 4, 1, 5, 1, and 4 good numerators, respectively) and fails at $7^7 = 823{,}543$ (0 good numerators among 705,894 units, confirming Komatsu [5]).*

Two algebraic results proved in Section 3 underpin Theorem 1 and Proposition 2: Lemma 3 proves the classical Mendès France folding lemma [6] (`[to check]`), the twisted-fold characterization for denominators zq^2, and the exact obstruction explaining why classical unimodular odd folds ($t = 1$, $z = 7^j$) inevitably introduce a partial quotient $7^j - 1 \geq 6 > 5$; Proposition 4 proves S1 preservation and doubling (following Dubno [3]), S2 doubling (following Komatsu [5]), and the four-line bound-6 closure for all 7^n (reproducing Shulga [9] on powers of 7). Figure 1 summarizes the construction stages, exact checks, and verification boundary.

Instant Witness Check: Frontier $c = 41$ (S1) and $n = 43$

```
def cf(a, q):
    w = []
    while a:
        w.append(q // a); a, q = q % a, a
    return w
a41 = 158590358148713620261345666483555513
w41 = cf(a41, 7**41)
assert max(w41) <= 5 and 2 <= w41[0] <= 4 and 2 <= w41[-1] <= 4
a43 = 373620542170367167107454431396403034
w43 = cf(a43, 7**43)
assert len(w41) == 90 and len(w43) == 90
assert max(w43) <= 5 and w43[0] == 5
```

Roadmap. Section 2 fixes continued-fraction conventions, clarifies canonical versus non-canonical expansions, and walks through concrete folding examples. Section 3 proves Lemma 3 and Proposition 4. Section 4 presents the explicit witness tables, the two-step twisted-fold construction for odd parts 31–41 and $n = 43$, quotient and endpoint statistics, exhaustive small-exponent counts, and negative-mutation checks. Section 5 presents the machine-generated Reader's Trust Boundary, Section 6 details prior-work searches, unread items (`[to check]`), and unsettled problems (Conjecture 5), and Section 7 documents reproducibility and AI disclosure.

2. DEFINITIONS, NOTATION, AND EXAMPLES

For any finite sequence of positive integers $w = (a_1, \ldots, a_r)$, define the finite simple continued fraction

$$[0; w] = [0; a_1, \ldots, a_r] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_r}}} \in (0, 1].$$

Every rational a/q with $0 < a < q$ and $\gcd(a, q) = 1$ ($q \geq 2$) has exactly two expansions: its *canonical* expansion $w = (a_1, \ldots, a_L)$ produced by Euclid's algorithm, which satisfies $a_L \geq 2$, and its *non-canonical* expansion $(a_1, \ldots, a_{L-1}, a_L - 1, 1)$ ending in 1. We set $K(a/q) := \max_{1 \leq i \leq L} a_i$ using the canonical expansion, matching Komatsu [5], Dubno [3], Shulga [9], and Shkredov [8] (`[to check]`). (Niederreiter [7] uses the expansion ending in 1; the two conventions agree on whether q has a bound-k numerator for all powers of 7 checked here, though they differ on small integers such as $q = 6$, where $5/6 = [0; 1, 5] = [0; 1, 4, 1]$.)

For any integer c, write the elementary unimodular matrix

$$M(c) := \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}, \qquad J := M(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For either expansion $w = (a_1, \ldots, a_r)$ of a/q, the standard continuant identity gives

$$W := M(a_1)M(a_2)\cdots M(a_r) = \begin{pmatrix} q & q' \\ a & p' \end{pmatrix},$$

where $p'/q' = [0; a_1, \ldots, a_{r-1}]$ is the penultimate convergent (with $p' = 0, q' = 1$ when $r = 1$) and $\det W = qp' - aq' = (-1)^r$. Defining $e := aq' - p'q = (-1)^{r-1} \in \{\pm 1\}$, we have $\det W = -e$. Because each $M(c)$ is symmetric, transposing gives $M(a_r) \cdots M(a_1) = W^{\mathsf{T}}$, so the reversed word $(a_r, \ldots, a_1)$

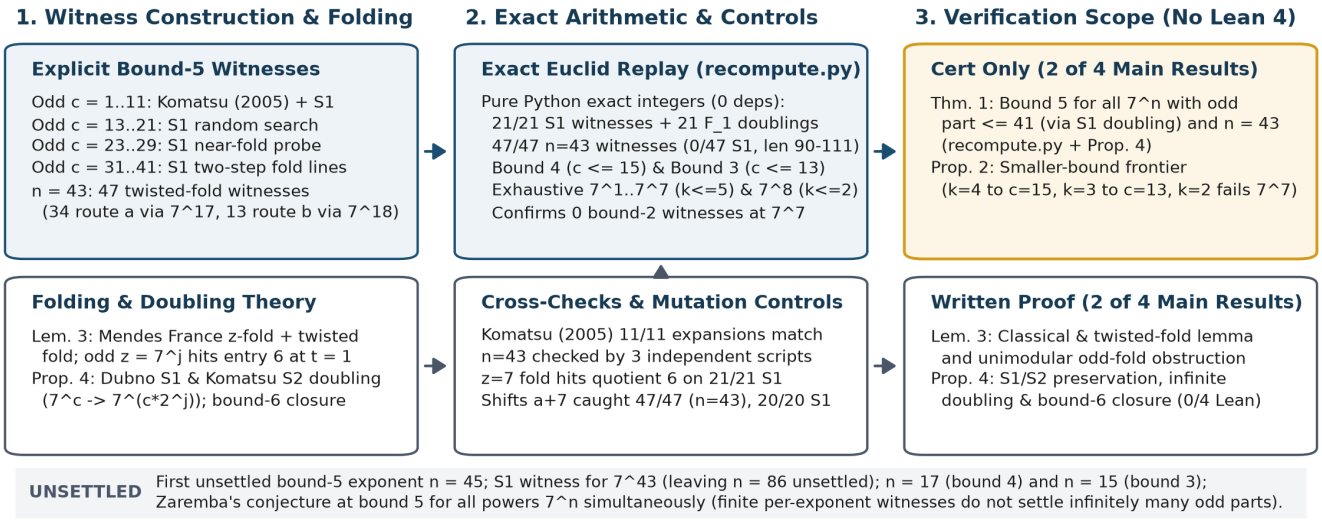


Figure 1. Discovery and verification architecture for Theorem 1, Proposition 2, Lemma 3, and Proposition 4. No Lean 4 formalization is claimed (0 of 4 proved main results in Lean 4; 2 of 4 by archived pure-Python certificate script `recompute.py`; 2 of 4 by written proof). All 21 S1 witnesses, all 47 witnesses for $n = 43$, Komatsu's 11 expansions, smaller-bound tables, exhaustive 7^1–7^7 and 7^8 counts, and negative-mutation controls are replayed deterministically. Exponents $n = 45$ and $n = 86$, and bound 5 for all 7^n, remain unsettled.

evaluates to q'/q. We write $K(w^-) := q'$ for the penultimate convergent denominator, and we allow the algebraic *zero-merging rule*

$$(\dots, x, 0, y, \dots) \longmapsto (\dots, x + y, \dots),$$

which preserves matrix products because $M(x)M(0)M(y) = M(x + y)$ holds for all integers x, y.

Boundary shapes S1 and S2. Fix a bound $k \geq 3$.

- A canonical word $w = (a_1, \dots, a_L)$ has *shape S1 at bound k* (Dubno [3]) if $L \geq 2$, $\max_i a_i \leq k$, and $2 \leq a_1, a_L \leq k - 1$.
- A canonical word $w = (a_1, \dots, a_L)$ has *shape S2 at bound k* (Komatsu [5]) if $L \geq 3$, $a_1 = a_2 = 1$, $a_L = 2$, and $\max_i a_i \leq k$.

Notice why S1 requires $k \geq 4$ to start from $2/7 = [0; 3, 2]$: at $k = 3$, S1 requires $a_1, a_L \in \{2\}$, which $[0; 3, 2]$ fails ($a_1 = 3$). For $k = 3$, Komatsu [5] instead used $5/7 = [0; 1, 2, 2]$ directly for 7^1 and $30/49 = [0; 1, 1, 1, 1, 2, 1, 2]$ (shape S2) to cover 7^{2^j} for all $j \geq 1$.

Worked example: doubling versus the $z = 7$ obstruction. Start from the base S1 witness $2/7 = [0; 3, 2]$ ($r = 2$, $a_1 = 3$, $a_2 = 2 \in \{2, 3, 4\}$). Applying the classical 1-fold gives

$$F_1(3, 2) = (3, 2, 0, 1, 1, 3) \longmapsto (3, 3, 1, 3),$$

whose value is $[0; 3, 3, 1, 3] = 15/49 = 2/7 + 1/49$: canonical, bound $3 \leq 5$, and still S1 with endpoints $(3, 3)$. Repeating F_1 yields $736/7^4 = [0; 3, 3, 1, 4, 2, 1, 3, 3]$ and settles 7^{2^j} for every $j \geq 0$. By contrast, applying the classical 7-fold to reach the odd power 7^3 yields

$$F_7(3, 2) = (3, 2, 6, 1, 1, 3), \qquad [0; 3, 2, 6, 1, 1, 3] = \frac{99}{343},$$

which stays at bound 6 (and remains S1 at bound 6!) but introduces the forbidden partial quotient $z - 1 = 6$ at the junction. Reaching 7^3 at bound 5 therefore requires leaving the unimodular $t = 1$ fold, for example via $72/7^3 = [0; 4, 1, 3, 4, 4]$, which is S1 at bound 5 and doubles to all $7^{3 \cdot 2^j}$.

3. FOLDING LEMMAS, OBSTRUCTION, AND DOUBLING

This section gives self-contained proofs of the two written results: Lemma 3 (classical folding, twisted-fold characterization, and the unimodular odd-fold obstruction) and Proposition 4 (preservation of S1 and S2 under folds, infinite doubling, and bound-6 closure for powers of 7).

Lemma 3 (Classical Folding Lemma and Twisted-Fold Obstruction). *Let a/q be a reduced fraction with $0 < a < q$ and $q \geq 2$, and let $w = (a_1, \dots, a_r)$ be either of its two continued-fraction expansions ($a_i \geq 1$). Write $q' = K(w^-)$, $e = aq' - p'q = (-1)^{r-1} \in \{\pm 1\}$, and $r_{\mathrm{rev}} = q'/q = [0; a_r, \dots, a_1] \in (0, 1)$.*

1. (Classical z-fold, Mendès France [6] `[to check]`.) *For every integer $z \geq 1$, the word*

$$F_z(w) := (a_1, \dots, a_r, z - 1, 1, a_r - 1, a_{r-1}, \dots, a_1),$$

normalized by the zero-merging rule, has positive integer entries and evaluates to the reduced fraction

$$[0; F_z(w)] = \frac{a}{q} + \frac{(-1)^r}{zq^2}$$

with denominator zq^2. When $a_r \geq 2$, its largest entry is $\max(K(w), z - 1)$ for $z \geq 2$, and $\max(a_1, \dots, a_{r-1}, a_r + 1)$ for $z = 1$.

2. (Twisted-fold characterization and bound-5 odd-fold obstruction.) *Let $z \geq 2$. A reduced fraction $A/(zq^2)$ admits a continued-fraction expansion (w, u) with nonempty tail u if and only if $A = azq - te$ for an integer t satisfying $1 \leq t < z/(1 + r_{\mathrm{rev}})$ and $\gcd(t, zq) = 1$. In that case u is an expansion of $tq/(zq - tq')$ with first entry $u_1 = \lfloor z/t - r_{\mathrm{rev}} \rfloor$. In particular, for $z = 7^j$ ($j \geq 1$):*

- *the unimodular fold $t = 1$ always has $u_1 = 7^j - 1 \geq 6$;*
- *every bound-5 twisted fold (w, u) over 7^{2m+j} satisfies $t > 7^j/7$ (hence $t \geq 2$); and*
- *no composition of classical z-folds reaches any odd power 7^n (n odd) at bound 5.*

Proof. For part (1), let $W = M(a_1)\cdots M(a_r) = \left(\begin{smallmatrix} q & q' \\ a & p' \end{smallmatrix}\right)$, so $M(a_r)\cdots M(a_1) = W^{\mathsf{T}}$ and $qp' - aq' = (-1)^r$. Direct matrix multiplication gives $M(a_r - 1)M(a_r)^{-1} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, whence

$$M(a_r - 1)M(a_{r-1})\cdots M(a_1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W^{\mathsf{T}},$$

and likewise

$$M(z-1)M(1)\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}.$$

Because the zero-merging rule preserves matrix products, the matrix product of $F_z(w)$ equals

$$W\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} W^{\mathsf{T}} = \begin{pmatrix} zq^2 & zaq - (-1)^r \\ zaq + (-1)^r & za^2 \end{pmatrix},$$

where the off-diagonal entries use $qp' - aq' = (-1)^r$. Reading the first column gives $[0; F_z(w)] = (zaq + (-1)^r)/(zq^2) = a/q + (-1)^r/(zq^2)$. Since $\det W = \pm 1$ and $\det\left(\begin{smallmatrix} z & -1 \\ 1 & 0 \end{smallmatrix}\right) = 1$, the product has determinant 1, so numerator and denominator are coprime and the exact denominator is zq^2. The only possible zeros before merging are $z - 1$ (if $z = 1$) and $a_r - 1$ (if $a_r = 1$); each sits strictly between positive entries unless $r = 1$ and $a_1 = 1$, which would force $a/q = 1$, excluded by $q \geq 2$. When $a_r \geq 2$ and $z \geq 2$, no zeros occur and the entries are $a_1, \ldots, a_r, z-1, 1, a_r - 1, a_{r-1}, \ldots, a_1$, giving maximum $\max(K(w), z - 1)$. When $a_r \geq 2$ and $z = 1$, merging $(a_r, 0, 1)$ gives $(a_1, \ldots, a_{r-1}, a_r + 1, a_r - 1, a_{r-1}, \ldots, a_1)$, with maximum $\max(a_1, \ldots, a_{r-1}, a_r + 1)$.

For part (2), if (w, u) expands $A/(zq^2)$ with nonempty tail u, then the first column of $M(w)M(u)$ is $(zq^2, A)^{\mathsf{T}}$. Write $(K(u), K(u^+))^{\mathsf{T}}$ for the first column of $M(u)$, so that $[0; u] = K(u^+)/K(u) \in (0, 1)$. Left multiplication by W^{-1} gives

$$\begin{pmatrix} K(u) \\ K(u^+) \end{pmatrix} = -e\begin{pmatrix} p' & -q' \\ -a & q \end{pmatrix}\begin{pmatrix} zq^2 \\ A \end{pmatrix},$$

whence $K(u^+) = -e(qA - azq^2) = qt$ with $t := e(azq - A)$. Since $K(u^+) \geq 1$ and $q \geq 2$, we have $t \geq 1$ and $A = azq - te$. Substituting A into the top coordinate and using $p'q - aq' = -e$ with $e^2 = 1$ yields

$$K(u) = -e(zq(p'q - aq') + q'te) = zq - tq'.$$

Thus $[0; u] = tq/(zq - tq') = t/(z - tr_{\mathrm{rev}})$. The inequality $[0; u] < 1$ is equivalent to $t < z/(1 + r_{\mathrm{rev}})$. Since $\det M(u) = \pm 1$, we have $\gcd(zq - tq', tq) = 1$, which reduces to $\gcd(t, zq) = 1$ because $\gcd(q', q) = 1$. Conversely, for any integer t with $1 \leq t < z/(1 + r_{\mathrm{rev}})$ and $\gcd(t, zq) = 1$, taking u to be an expansion of $tq/(zq - tq')$ reverses the matrix calculation and yields $M(w)M(u)$ with first column $(zq^2, azq - te)^{\mathsf{T}}$, and $u_1 = \lfloor (zq - tq')/(tq) \rfloor = \lfloor z/t - r_{\mathrm{rev}} \rfloor$.

Finally, specialize to $z = 7^j$ ($j \geq 1$). At $t = 1$, $0 < r_{\mathrm{rev}} < 1$ forces $u_1 = \lfloor 7^j - r_{\mathrm{rev}} \rfloor = 7^j - 1 \geq 6$. If both w and u have entries at most 5, then $a_r \leq 5$ implies $r_{\mathrm{rev}} > 1/6$, and $u_1 = \lfloor z/t - r_{\mathrm{rev}} \rfloor \leq 5$ implies $z/t - r_{\mathrm{rev}} < 6$, so $t > z/(6 + r_{\mathrm{rev}}) > z/7 = 7^{j-1}$ (in particular $t \geq 2$). Moreover, in any sequence of classical z-folds ending at denominator 7^n with n odd, the final fold has denominator $zq^2 = 7^n$, so q is a power of 7 and $v_7(z) = n - 2v_7(q)$ is odd, forcing $z \geq 7$. The entry $z - 1 \geq 6$ in $F_z(w)$ is flanked by $a_r \geq 1$ and 1, so it is never merged with a zero, and canonicalizing changes only the final entry of $F_z(w)$ (which lies at least two positions after $z - 1$). Thus every classical fold reaching an odd power of 7 contains an entry at least 6. □

Proposition 4 (Preservation of Shapes S1 and S2, Infinite Doubling, and Bound-6 Closure). *Let $k \geq 3$ be an integer, and let a/q ($q \geq 2$) have canonical word $w = (a_1, \ldots, a_r)$.*

1. (S1 preservation and doubling, Dubno [3].) *If w has shape S1 at bound k ($r \geq 2$, $\max_i a_i \leq k$, $2 \leq a_1, a_r \leq k - 1$), then for $z = 1$ and for every integer z with $2 \leq z \leq k + 1$, the folded word $F_z(w)$ is canonical and has shape S1 at bound k, with first and last entries both equal to a_1 and denominator zq^2. In particular, if 7^c admits an S1 witness at bound $k \in \{3, 4, 5\}$, then $7^{c \cdot 2^j}$ admits an S1 witness at bound k for every $j \geq 0$.*
2. (S2 doubling, Komatsu [5].) *If $w = (1, 1, a_3, \ldots, a_{r-1}, 2)$ has shape S2 at bound $k \geq 3$, then the canonical form of $F_1(w)$ has shape S2 at bound k and denominator q^2; hence any S2 witness for 7^c yields an S2 witness for $7^{c \cdot 2^j}$ for every $j \geq 0$.*
3. (Bound-6 closure for all 7^n, reproducing Shulga [9] on powers of 7.) *Every integer $n \geq 1$ admits a coprime numerator a with $K(a/7^n) \leq 6$.*

Proof. For part (1), since w is S1 at bound k, we have $r \geq 2$, $2 \leq a_1 \leq k - 1$, and $2 \leq a_r \leq k - 1$. By Lemma 3(1), $F_z(w)$ has denominator zq^2 and largest entry $\max(K(w), z - 1) \leq k$ when $2 \leq z \leq k + 1$, or $\max(a_1, \ldots, a_{r-1}, a_r + 1) \leq k$ when $z = 1$ (using $a_r \leq k - 1$). Because $a_r \geq 2$, no zero arises except $z - 1 = 0$ at $z = 1$, which merges into $a_r + 1$; and since $r \geq 2$, the first entry of $F_z(w)$ remains a_1 and the last entry is $a_1 \geq 2$. Therefore $F_z(w)$ is already canonical, has length at least $2r \geq 4$, and has both endpoints equal to $a_1 \in \{2, \ldots, k - 1\}$, so $F_z(w)$ is S1 at bound k. Induction on $j \geq 0$ using $z = 1$ sends denominator $7^{c \cdot 2^j}$ to $7^{c \cdot 2^{j+1}}$ while keeping shape S1 at bound k. (Note that $r \geq 2$ is necessary: for $r = 1$, $F_1((2)) = (4)$ has denominator 4 and length 1.)

For part (2), applying Lemma 3(1) with $z = 1$ to $w = (1, 1, a_3, \ldots, a_{r-1}, 2)$ produces the uncanonicalized word

$$(1, 1, a_3, \ldots, a_{r-1}, 3, 1, a_{r-1}, \ldots, a_3, 1, 1),$$

whose canonical form merges the trailing $(1, 1)$ into 2, yielding

$$(1, 1, a_3, \ldots, a_{r-1}, 3, 1, a_{r-1}, \ldots, a_3, 2).$$

Since $r \geq 3$ and $k \geq 3$, every entry is at most k, the word begins with $(1, 1)$ and ends with 2, and its denominator is q^2. Induction on j settles $7^{c \cdot 2^j}$ for all $j \geq 0$.

For part (3), apply part (1) at $k = 6$. Both $z = 1$ and $z = 7 = k + 1$ satisfy the hypothesis of part (1), so F_1 sends any S1 bound-6 witness for 7^m to one for 7^{2m} and F_7 sends it to one for 7^{2m+1}. Starting from the S1 bound-6 base witness $2/7 = [0; 3, 2]$ ($2 \leq 3, 2 \leq 5$), strong induction on $n \geq 1$ via $m \mapsto 2m$ and $m \mapsto 2m + 1$ yields an S1 bound-6 witness for every 7^n. □

4. COMPUTATIONAL VERIFICATION AND EMPIRICAL PICTURE

Explicit Bound-5 S1 Witnesses for Odd Parts up to 41. Table 1 lists all 21 explicit S1 numerators a_c for 7^c ($c \in \{1,3,\dots,41\}$) verified in exact integer arithmetic by `recompute.py`. They arise from four progressively structured construction methods:

1. *Odd parts* 1–11 (*Komatsu's range*). Komatsu [5] settled these six odd parts at bound 3 via S2 words; Table 1 provides compact S1 witnesses at bound 5 (and `recompute.py` independently verifies all 11 printed expansions in Komatsu [5]).
2. *Odd parts* 13–21 (*targeted random search*). Direct sampling of $a \in (\lfloor 7^c/5 \rfloor, \lfloor 7^c/2 \rfloor]$ with early-abort Euclid testing finds S1 bound-5 witnesses up to $c = 21$ (requiring between 1,893 draws at $c = 13$ and 98,208 draws at $c = 21$).
3. *Odd parts* 23–29 (*near-fold probe*). Searching twisted near-folds of smaller good powers of 7 yields S1 witnesses for $c \in \{23, 25, 27, 29\}$ (lengths 49 to 69).
4. *Odd parts* 31–41 (*two-step twisted-fold lines*). Applying Lemma 3(2) twice from exhaustive good parents at 7^m ($m \in \{6,7,8\}$) with steps $(z_1, z_2) \in \{7, 49, 343\}^2$ yields 190 exact bound-5 numerators across $31 \le n \le 41$, including multiple S1 witnesses at every odd exponent (14 at $c = 31$, 15 at $c = 33$, 6 at $c = 35$, 11 at $c = 37$, 2 at $c = 39$, and 8 at $c = 41$). Table 1 lists the shortest S1 witness at each odd exponent; `recompute.py` verifies exact coprimality, $K \le 5$, S1 endpoints, the intermediate convergent denominators 7^m and $7^{2m+v_7(z_1)}$, and one-step doubling F_1 to 7^{2c} across all 21 rows.

Forty-Seven Explicit Witnesses for $n = 43$ and Why Doubling Stops. To reach $7^{43} \approx 2.17 \times 10^{36}$, Lemma 3(2) was applied with widening fold factors along two independent routes starting from intermediate middle words built from 7^8:

- *Route a* ($7^{17} \to 7^{43}$ *via* $z = 7^9$)*:* folding 16 word forms from 8 middle words of 7^{17} by 7^9 yielded 34 distinct bound-5 numerators of 7^{43} (lengths 90 to 102).
- *Route b* ($7^{18} \to 7^{43}$ *via* $z = 7^7$)*:* folding 185 word forms from 100 middle words of 7^{18} by 7^7 yielded 13 distinct bound-5 numerators of 7^{43} (lengths 97 to 111).

All 47 numerators were checked independently by the generating program, by a separate standalone checker, by an adversarial falsifier's independent script, and again verbatim inside `recompute.py` (confirming $0 < A < 7^{43}$, $\gcd(A, 7^{43}) = 1$, $A = a_{\text{par}}7^{m+j} - et$, $K(A/7^{43}) \le 5$, and presence of 7^8 and 7^{17} or 7^{18} among the convergent denominators). Table 2 displays the shortest witness from each route; all 47 numerators are embedded in `recompute.py`.

Because parent middle words were ordered by increasing numerator (so $a_{\text{par}}/7^m < 1/5$), every one of the 47 witnesses begins with $a_1 = 5$. Their endpoint pairs (a_1, a_L) distribute as $(5,2)$ (20 witnesses), $(5,3)$ (10), $(5,4)$ (10), and $(5,5)$ (7). Consequently 0 of 47 satisfy the S1 condition $2 \le a_1, a_L \le 4$: applying F_1 produces trailing entry $a_1 = 5$, and a second fold or $a_L = 5$ increases $a_L + 1$ to 6. Thus $n = 43$ is settled, while $n = 86 = 2 \cdot 43$ remains unsettled until an S1 witness for 7^{43} is found.

Figure 2 summarizes the empirical quotient and endpoint statistics across all 21 S1 witnesses (997 total partial quotients: 422, 224, 153, 126, 72 for digits 1–5, or 42.3%, 22.5%, 15.3%, 12.6%, 7.2%) and all 47 witnesses for $n = 43$ (4,632 total quotients: 2,032, 1,042, 694, 465, 399, or 43.9%, 22.5%, 15.0%, 10.0%, 8.6%). Compared with the classical Gauss–Kuzmin distribution conditioned on $\{1,\dots,5\}$, both witness families put slightly less weight on 1 (42.3%–43.9% versus 53.3%) and slightly more weight on 3, 4, 5, matching the known shift of the stationary measure on bounded-quotient Cantor sets. Panel B of Figure 2 highlights the sharp separation between the 3×3 S1 interior square (21 witnesses) and the $a_1 = 5$ boundary row (47 witnesses for $n = 43$).

Smaller Bounds, Exhaustive Census, and Mutation Controls. Table 3 lists the explicit S1 witnesses for bound 4 ($c \in \{1,3,\dots,15\}$) and S2 witnesses for bound 3 ($c \in \{3,5,\dots,13\}$, paired with Komatsu's seeds $5/7 = [0;1,2,2]$ and $30/49 = [0;1,1,1,1,2,1,2]$) establishing Proposition 2(1)–(2). Table 4 gives the exact exhaustive counts of coprime numerators with $K(a/7^n) \le k$ ($k = 2,3,4,5$) for $1 \le n \le 7$, reproduced by `recompute.py` together with the exhaustive bound-2 scan at 7^8 (where 4 of 4,941,258 units succeed: 2,119,048, 2,202,689, 3,562,112, and 3,645,753). At $7^7 = 823{,}543$, all 705,894 units have $K(a/7^7) \ge 3$, proving that bound 2 fails at 7^7. Figure 3 plots the exponential decay of the good-numerator share $g_k(7^n)$ alongside the settled odd-part frontier across $k \in \{2,3,4,5,6\}$.

Self-checks and negative-mutation controls. In addition to verifying every witness directly, `recompute.py` runs finite verification sweeps for Lemma 3 (8,802 cases of classical folding across $2 \le q \le 40$, both forms, $z = 1$–9; 4,158 cases of twisted folding across $2 \le q \le 25$, $z = 2$–7) and Proposition 4 (900 cases of S1/S2 preservation across $2 \le q \le 40$, $k \in \{3,4,5\}$; all 64/64 exponents $1 \le n \le 64$ in the bound-6 closure from 2/7, reaching length 128 at $n = 64$). Four negative-mutation suites verify that the checker rejects faults cleanly: multiplying each S1 numerator by 7 mod 7^c fails coprimality (21/21 caught); applying the classical 7-fold F_7 to each S1 witness produces partial quotient $6 > 5$ (21/21 caught, illustrating Lemma 3(2)); shifting $a \mapsto a + 7$ breaks bound 5 on all 47/47 witnesses for $n = 43$; and shifting $a \mapsto a + 7$ breaks bound 5 or S1 on all 20/20 S1 witnesses with $c \ge 3$.

Table 1. All 21 explicit S1 bound-5 witnesses $a_c/7^c$ for odd parts $1 \le c \le 41$ (Theorem 1(1)). Every row satisfies $\gcd(a_c, 7^c) = 1$, $K(a_c/7^c) \le 5$, and $2 \le a_1, a_L \le 4$, so Proposition 4(1) settles $7^{c \cdot 2^j}$ for all $j \ge 0$. All 21 witnesses and their F_1 doublings to 7^{2c} are verified by `recompute.py`.

c	Construction method	L	(a_1, a_L)	K	Explicit S1 numerator a_c
1	Komatsu range (S1)	2	(3, 2)	3	2
3	Komatsu range (S1)	5	(4, 4)	4	72
5	Komatsu range (S1)	9	(4, 2)	5	3469
7	Komatsu range (S1)	13	(4, 3)	5	187440
9	Komatsu range (S1)	22	(4, 3)	5	8453376
11	Komatsu range (S1)	23	(4, 2)	5	410505146
13	Random search	32	(3, 3)	5	25426309987
15	Random search	38	(4, 4)	4	995852075430
17	Random search	41	(3, 2)	5	73173037382640
19	Random search	46	(2, 2)	5	4402150398387787
21	Random search	51	(2, 2)	5	207554712365340118
23	Near-fold probe	49	(4, 2)	5	6170059067117542465
25	Near-fold probe	59	(4, 2)	5	288665623902836712685
27	Near-fold probe	68	(2, 3)	5	254747183332865232242163
29	Near-fold probe	69	(3, 2)	5	872922286595579805281859
31	Two-step fold ($m = 6; 343, 7$)	61	(3, 2)	5	41397447227219683289779486
33	Two-step fold ($m = 6; 343, 343$)	73	(4, 4)	5	16654141116371943255903089 39
35	Two-step fold ($m = 7; 343, 7$)	73	(3, 2)	5	11890329979124673429653592 2530
37	Two-step fold ($m = 7; 343, 343$)	83	(4, 4)	5	38972773296549009889459946765 48
39	Two-step fold ($m = 8; 49, 343$)	90	(3, 2)	5	26348741240431566695144480388 9837
41	Two-step fold ($m = 8; 343, 343$)	90	(2, 2)	5	15859035814871362026134566648355513

Table 2. Representative explicit bound-5 witnesses for 7^{43} (Theorem 1(2)): the shortest witness from route a (34 total, $m = 17$, $z = 7^9$) and route b (13 total, $m = 18$, $z = 7^7$). All 47 numerators (lengths 90–111, median 99, all beginning with $a_1 = 5$, 0 of 47 S1) are embedded and verified in `recompute.py`.

Route	Parent 7^m	Fold 7^j	t	L	(a_1, a_L)	Explicit numerator A for 7^{43}
Route a (shortest of 34)	7^{17}	7^9	20,490,173	90	(5, 2)	373620542170367167107454431396403034
Route b (shortest of 13)	7^{18}	7^7	587,998	97	(5, 4)	378637616930736313613299671584095915

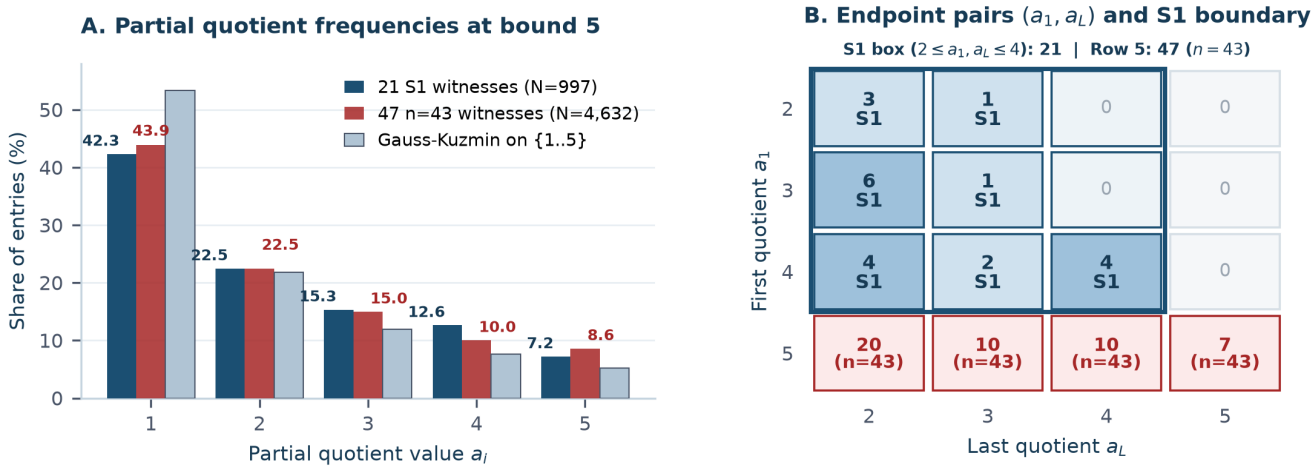


Figure 2. Partial quotient frequencies and endpoint structure across bound-5 witnesses verified in `recompute.py`. Panel A: empirical shares of digits 1–5 across the 21 S1 witnesses (997 total entries) and the 47 witnesses for $n = 43$ (4,632 total entries), compared with the Gauss–Kuzmin distribution conditioned on $\{1, \ldots, 5\}$. Panel B: occupancy of endpoint pairs $(a_1, a_L) \in \{2, 3, 4, 5\}^2$, showing all 21 odd-part witnesses ($c \le 41$) strictly inside the S1 doubling square $2 \le a_1, a_L \le 4$ and all 47 witnesses for $n = 43$ on the non-S1 boundary row $a_1 = 5$.

Table 3. Explicit witnesses settling odd parts $c \le 15$ at bound 4 (S1) and $c \le 13$ at bound 3 (S2, with Komatsu's 5/7 and 30/49 for $c = 1$), verified in `recompute.py`.

Bound 4 (S1, $c \le 15$)			Bound 3 (S2, $c \le 13$)		
c	L	Numerator a	c	L	Numerator a
1	2	2	1*	3/7	5 (7^1), 30 (7^2)
3	6	90	3	9	198
5	12	4397	5	15	9423
7	20	229435	7	20	459782
9	21	15353104	9	26	23244491
11	29	895227286	11	31	1205149989
13	37	42583757710	13	44	59478693243
15	39	1271091970240	–	–	(least unsettled: 15)

Table 4. Exhaustive count of coprime numerators $a \in [1, 7^n)$ with $K(a/7^n) \le k$ for $1 \le n \le 7$ (verified in `recompute.py`; bound-2 count at 7^8 is 4 among 4,941,258 units).

n	$\varphi(7^n)$	$k \le 2$	$k \le 3$	$k \le 4$	$k \le 5$	Share ($k \le 5$)
1	6	1	4	4	4	0.66667
2	42	4	8	13	16	0.38095
3	294	4	12	31	55	0.18707
4	2,058	1	36	102	216	0.10496
5	14,406	5	52	271	784	0.05442
6	100,842	1	162	968	3,054	0.03029
7	705,894	**0**	278	3,337	11,693	0.01656

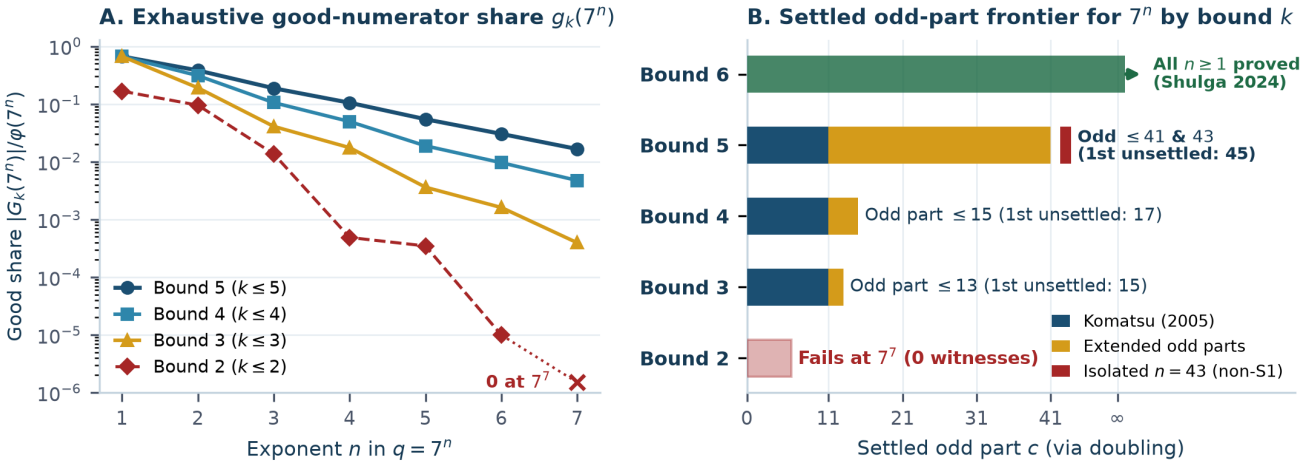


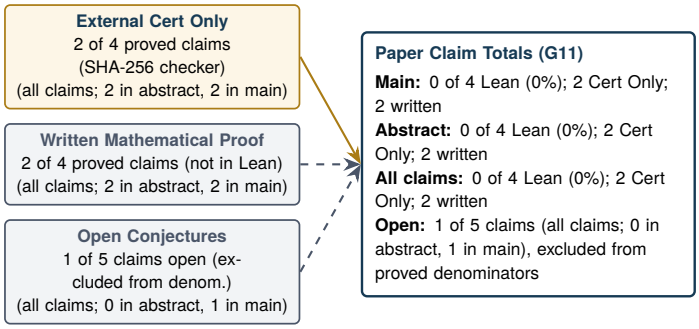
Figure 3. Good-numerator decay and bound-k frontier for powers of 7. Panel A: exact exhaustive share $g_k(7^n) = |\{a : \gcd(a, 7^n) = 1,\ K(a/7^n) \le k\}|/\varphi(7^n)$ for $1 \le n \le 7$ and $k \in \{2, 3, 4, 5\}$, showing the exact zero at 7^7 for $k = 2$ and exponential decay at $k = 3, 4, 5$. Panel B: settled odd-part frontier across $k \in \{2, 3, 4, 5, 6\}$, contrasting Komatsu's [5] baseline ($c \le 11$) with the extensions in Theorem 1 and Proposition 2 and Shulga's [9] bound-6 theorem for all $n \ge 1$.

5. FORMAL VERIFICATION & READER'S TRUST BOUNDARY

What You Must Trust to Accept This Paper:

1. **Lean 4 Kernel:** No Lean 4 formalization is claimed in this manuscript.
2. **Written Proofs:** All mathematical arguments rest on standard human-readable proofs.
3. **External Certificates & Written Proofs:** 1 SHA-256-pinned external artifact(s) plus any claims marked Written or Unsettled in Table 5 below.

Reader's Trust-Flow Architecture (Distinct Evidence Paths to Claim Totals)



Anti-Shortcut Audit Strip — No-Lean Disclosure

N/A: Lean 4: Not claimed | N/A: sorry / admit: N/A | N/A: native_decide: N/A | N/A: Escape hatches: N/A | N/A: Library coverage: N/A | N/A: Lean fidelity: N/A | WARN: Human referee: pending

Table 5. Per-Claim Reader's Trust Boundary Ledger (G11 Denominators Derived Mechanically).[a]

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Thm. 1: Bound 5 holds for every 7^n with odd part c <= 41 (21 explicit S1 witnesses + S1 doubling) and for n = 43 (47 explicit non-S1 witnesses, lengths 90 to 111); least unsettled exponent is n = 45 (and n = 86 is unsettled)	Cert Only	recompute.py (aeb7a52f66cc) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. All 21 S1 bound-5 witnesses (c = 1.. 41), their F_1 doublings, and all 47 witnesses for n = 43 are verified in exact integer arithmetic by recompute.py (with n = 43 independently rechecked by three separate programs); infinite doubling 7^c -> 7^(c*2^j) relies on the written proof of Proposition 4 (following Dubno 2024).
Prop. 2: Smaller-bound frontier for 7^n: bound 4 holds for odd part <= 15 (least unsettled 17), bound 3 holds for odd part <= 13 (least unsettled 15, extending Komatsu's 11), and bound 2 fails exhaustively at 7^7 (0 of 705,894 units) while holding for n in {1, 2, 3, 4, 5, 6, 8}	Cert Only	recompute.py (aeb7a52f66cc) [archived; replay pending]	External cert only; fidelity: n/a	No Lean 4 formalization. Certified in exact integer arithmetic by recompute.py (8 S1 witnesses for bound 4, Komatsu seeds plus 6 S2 witnesses for bound 3, 11/11 Komatsu 2005 expansions, and exhaustive unit scans of 7^1..7^7 and 7^8 at bound 2), combined with Proposition 4 for infinite doubling.
Lem. 3: Classical Mendes France z-fold identity, twisted-fold characterization over z*q^2, and unimodular odd-fold obstruction (t = 1 gives quotient 7^j - 1 >= 6; bound 5 forces t > 7^j/7 >= 2)	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof via 2x2 continuant matrix identities (Section 3; classical z-fold credited to Mendes France 1973 [to check]); finite sweeps over 8,802 folding and 4,158 twisted-fold cases also checked in recompute.py. No Lean 4 formalization.

Paper Claim	Proof Tier	Lean Decl / Artifact	Actual Axioms & Fidelity	Assumptions & What You Must Trust
Prop. 4: Preservation of shape S1 under z-folds (z = 1 and 2 <= z <= k + 1; Dubno 2024), Komatsu S2 doubling under F_1, infinite doubling 7^c -> 7^(c*2^j), and bound-6 closure for all 7^n (reproducing Shulga 2024 on powers of 7)	Written	Written proof	Lean: none; fidelity: n/a	Self-contained written proof from Lemma 3 (Section 3, crediting Dubno 2024, Komatsu 2005, and Shulga 2024); 900 S1/S2 fold cases and 64/64 bound-6 closure exponents also checked in recompute.py. No Lean 4 formalization.
Conj. 5: Unsettled problems: S1 bound-5 witness for 7^43 (for n = 86), settling n = 45 (and n = 17 at bound 4, n = 15 at bound 3), and Zaremba's conjecture at bound 5 for all powers 7^n simultaneously	Unsettled	Open conjecture	Lean: none; fidelity: n/a	Explicitly unsettled open questions (excluded from G11 denominators).

[a]Allowed pure-kernel axioms: `propext`, `Classical.choice`, `Quot.sound`. Actual axioms printed above are read directly from verified outputs. Provenance nonce: `e1f34a49c767` | Fidelity SHA-256: `94224bffd45f`.

6. PRIOR WORK AND OPEN QUESTIONS

Zaremba [11] (`[to check]`) introduced the bounded-partial-quotient conjecture in the context of good lattice points for numerical integration, and Mendès France [6] (`[to check]`) established the folding lemma relating continued fractions of a/q and $a/q \pm 1/(zq^2)$. For density-one sets of denominators, Bourgain and Kontorovich [2] (`[to check]`) proved bound 50 and Huang [4] (`[to check]`) proved bound 5; these metric results leave every zero-density sequence such as $\{7^n\}$ untouched.

For geometric sequences d^n, Niederreiter [7] proved bound 3 for 2^n and 3^n and bound 4 for 5^n; Yodphotong and Laohakosol [10] (`[to check]`) proved bound 5 for 6^n; Komatsu [5] proved bound 3 for $7^{c\cdot 2^j}$ with $c \in \{1,3,5,7,9,11\}$ via shape S2 doubling; Dubno [3] formalized the S1 boundary condition ($1 < a_1, a_L < k$), S1 preservation under z-folds, and repeated doubling, settling bound 5 for powers of 12 and 18 (for powers of 7, every split $d = x^2y$ has $xy \geq 7$, so Dubno's algorithm yields bound 6 at best); and Shulga [9] proved $K(a/q) \leq \mathrm{rad}(q) - 1$ for every $q \neq 2^n, 3^n$, yielding bound 6 for every 7^n. Ayadi and Komatsu [1] (`[to check]`) treat q^{2^j} for $2 \leq q \leq 100$ (which intersects powers of 7 only at odd part $c = 1$ in the abstract and secondary summaries inspected) alongside polynomial recurrence families. Two recent preprints prove large absolute constants without settling bound 5: Shkredov [8] (`[to check]`) gives an effective large constant for sufficiently large prime powers, and Zhang [12] (`[to check]`, unrefereed) claims a large absolute bound for all q while explicitly noting that lowering the bound to 5 requires an additional idea.

Prior-work searches logged. Two systematic literature searches were conducted: an initial web and database search across arXiv (396 records on `all:Zaremba`, 38+ directly on Zaremba's conjecture, plus three targeted abstract queries), zbMATH Open (23 records), Crossref (60 listed records), OEIS (10 records), Semantic Scholar, and OpenAlex (tracing citations of Komatsu [5]), followed by a 5-query Google Scholar sweep on 2026-11-22 (50 records inspected). No source settling bound 5 for powers of 7 beyond Komatsu's odd parts 1–11 (or extending bound 4 to $c \leq 15$ and bound 3 to $c = 13$) was found in the searches logged.

Unread and unsearched items (`[to check]`). The following sources remain unread in full or unsearched and are flagged `[to check]` in `references-checked.md`: (1) paywalled full text of Ayadi and Komatsu [1] (checked via abstract, Scholar snippets, and secondary summary only); (2) unrefereed preprint of Zhang [12] beyond its abstract and introduction, and body proofs of Shkredov [8]; (3) full texts of Yodphotong and Laohakosol [10] (zbMATH summary only), Zaremba [11], Mendès France [6] (proof of Lemma 3 is self-contained), Bourgain and Kontorovich [2], and Huang [4] (introductions inspected via arXiv); (4) page 310 of Niederreiter [7]; and (5) MathSciNet, theses outside open indexes, and non-English-language journals not indexed by zbMATH or Crossref.

Conjecture 5 (Unsettled Exponents and Bound 5 for All Powers of 7). *The following problems remain unsettled:*

1. *Exhibit an S1 bound-5 witness for 7^{43} (settling $7^{43\cdot 2^j}$ for all $j \geq 0$, starting with $n = 86$).*
2. *Settle $n = 45$ (the least unsettled exponent at bound 5), as well as $n = 17$ at bound 4 and $n = 15$ at bound 3.*
3. *Prove (or refute) Zaremba's conjecture at bound 5 for every 7^n ($n \geq 1$) simultaneously. By Lemma 3(2), any inductive proof for all odd exponents must go beyond classical unimodular folding.*

7. DATA AVAILABILITY, REPRODUCIBILITY, AND USE OF AI

All manuscript sources, figure generation scripts, and the standalone exact-arithmetic verifier live in `papers/zaremba-powers-of-7/draft1/`. Running `python3 papers/zaremba-powers-of-7/draft1/recompute.py` requires only the Python 3 standard library, completes in under 3 seconds, and deterministically verifies all 21 S1 bound-5 witnesses ($c \in \{1, 3, \ldots, 41\}$) and their F_1 doublings, all 47 explicit witnesses for $n = 43$, all 8 bound-4 and 6 bound-3 odd-part witnesses plus Komatsu's 11 printed expansions, exhaustive counts on 7^1–7^7 (bounds 2–5) and 7^8 (bound 2), finite verification sweeps for Lemma 3 and Proposition 4, all four negative-mutation suites, and the Section 1 Python snippet. No Lean 4 formalization is claimed in this draft.

AI research agents built and directed by Tyler B. Hughes discovered the bound-5 witnesses beyond Komatsu's range and the smaller-bound extensions, wrote and independently rechecked the exact Euclidean verification scripts, and drafted this manuscript. No human mathematician has yet refereed the proof.

Drafted: November 2026 **Updated:** November 24, 2026

REFERENCES

(1) Ayadi, K.; Komatsu, T. Continued fraction expansions towards Zaremba's conjecture. *Experimental Mathematics* **2024**, *33* (4), 755–767. DOI: 10.1080/10586458.2023.2293285.

(2) Bourgain, J.; Kontorovich, A. On Zaremba's conjecture. *Annals of Mathematics* **2014**, *180* (1), 137–196. DOI: 10.4007/annals.2014.180.1.3. arXiv:1107.3776.

(3) Dubno, E. Zaremba's conjecture for geometric sequences: an algorithm. *The American Mathematical Monthly* **2024**, *131* (6), 530–535. DOI: 10.1080/00029890.2024.2323902. arXiv:2310.11279.

(4) Huang, S. An improvement to Zaremba's conjecture. *Geometric and Functional Analysis* **2015**, *25* (3), 860–914. DOI: 10.1007/s00039-015-0327-6. arXiv:1310.3772.

(5) Komatsu, T. On a Zaremba's conjecture for powers. *Sarajevo Journal of Mathematics* **2005**, *1* (1), 9–13. Zbl 1068.11004.

(6) Mendès France, M. Sur les fractions continues limitées. *Acta Arithmetica* **1973**, *23* (2), 207–215. DOI: 10.4064/aa-23-2-207-215.

(7) Niederreiter, H. Dyadic fractions with small partial quotients. *Monatshefte für Mathematik* **1986**, *101* (4), 309–315. DOI: 10.1007/BF01559394.

(8) Shkredov, I. D. On some results of Korobov and Larcher and Zaremba's conjecture. Preprint, **2026**, arXiv:2603.14116.

(9) Shulga, N. Radical bound for Zaremba's conjecture. *Bulletin of the London Mathematical Society* **2024**, *56* (8), 2615–2624. DOI: 10.1112/blms.13087. arXiv:2310.09801.

(10) Yodphotong, M.; Laohakosol, V. Proofs of Zaremba's conjecture for powers of 6. In *Proceedings of the International Conference on Algebra and Its Applications* (*ICAA 2002*); Chulalongkorn University: Bangkok, **2002**; pp 278–282. Zbl 1070.11003.

(11) Zaremba, S. K. La méthode des "bons treillis" pour le calcul des intégrales multiples. In *Applications of Number Theory to Numerical Analysis*; Zaremba, S. K., Ed.; Academic Press: New York, **1972**; pp 39–119. DOI: 10.1016/B978-0-12-775950-0.50009-1.

(12) Zhang, X. Expansion in $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ and Zaremba's conjecture. Preprint, **2026**, arXiv:2605.02518.